


Lecture 17

03/27/23

Today:

- Lyapunov functions (Examples; cont'd)
- LaSalle's Invariance Principle
- LTI Systems

Recall: $\dot{x} = -g(x)$; $x(t) \in \mathbb{R}$

$$V(x) = \frac{1}{2}x^2 \Rightarrow \dot{V} = -x \cdot g(x)$$

If $x \cdot g(x) > 0$, $\forall x \in (-a_1, a_2) \Rightarrow$
LAS of $\dot{x} = 0$

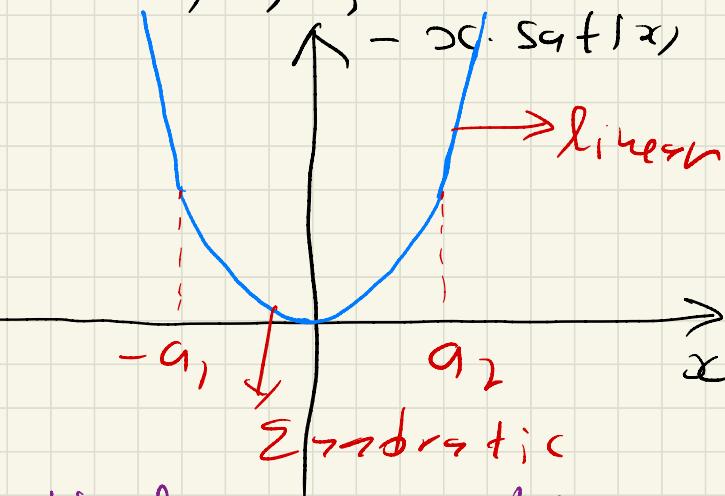
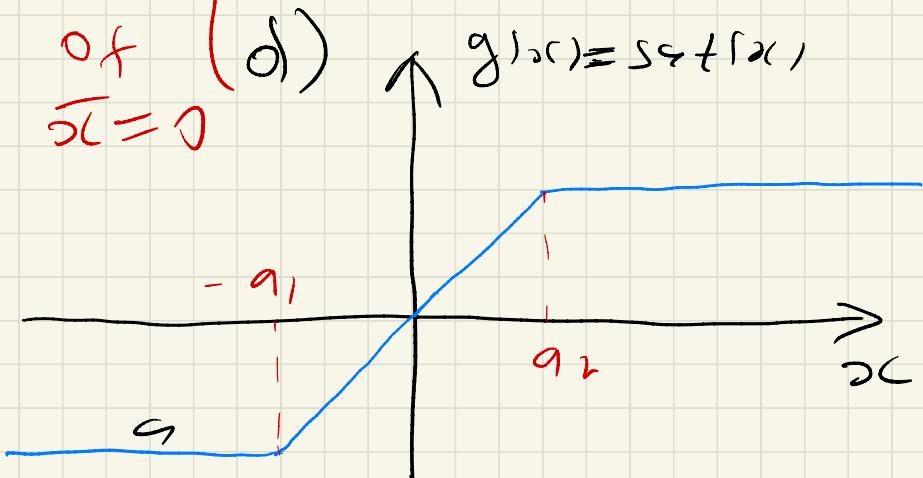
$$x \cdot g(x) > 0, \forall x \in \mathbb{R} \Rightarrow \frac{\partial A_S}{\partial x} = 0 \quad \textcircled{1}$$

$$Ex: \begin{cases} a) g(x) = \sin(x); & a_1 = a_2 = \infty \\ b) g(x) = x^3; & a_1 = a_2 = +\infty \\ c) g(x) = x^{2k-1}; & a_1 = a_2 = +\infty \\ d) g(x) = \sin x; & k = 1, 2, 3, \dots \end{cases}$$

LAS
of
 $\frac{\partial}{\partial x} \Rightarrow$

GAS

of
 $\frac{\partial}{\partial x} = 0$



Hyper function

Useful for eliminating the influence of outlier in statistical analysis (as opposed to LS fit)

(2)

Another Lyapunov function candidate

$$V(x) = \int_0^x g(\tilde{x}) d\tilde{x} \quad \left[g(x) = a x \Rightarrow V(x) = \frac{a}{2} x^2 \right]$$

$$\dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} = g(x) \cdot [-g(x)] = -g^2(x) < 0$$

$$\forall x \in (-a_1, a_2) \setminus \{0\}$$

More about rate of convergence
later: The best we can hope

for is exponential convergence
rate [Exponential stability
definition of stability for $\dot{x} = f(x)$] ③

Note: We already exploited structural properties of $\dot{g}(x) = \sin(x)$ is periodic. In fact, for Pendulum-like systems:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - b x_2 \end{cases} \quad \left. \begin{array}{l} x_1 \cdot g(x_1) > 0, \forall x_1 \in (-\pi_1, \pi_2) \setminus \{0\} \\ b > 0 \end{array} \right\} \text{ok}$$

$$\gamma(x_1, x_2) = \int_0^{x_1} g(-\xi) d\xi + \frac{1}{2} x_2^2$$

$$\dot{\gamma} = g(x_1) \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2 =$$

$$= \boxed{g(x_1) x_2} + \boxed{x_2 [-g(x_1) - b x_2]} =$$

$$= \boxed{0 \cdot x_2^2} - b \cdot x_2^2 \leq 0 \Rightarrow \overline{x} = 0 \text{ is stable.} \quad (4)$$

Note: Additional work required to show LAS of $\bar{x} = 0$.

One approach:

- LaSalle's invariance principle

Another approach:

- Try different Lyapunov functions

→ Would $\gamma(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ work?

$$\begin{aligned}\dot{\gamma} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1 x_2 + x_2 \left\{ -g(x_1) - h x_2 \right\} \\ &= x_1 x_2 - x_2 g(x_1) - h x_2\end{aligned}$$

 
sign-indefinite

(\Rightarrow not \oplus or \ominus)

 nice; negative

(5)

IF $(a \pm b)^2 \geq 0 \Rightarrow a^2 + b^2 \geq \mp 2ab$

$\mp ab \leq \frac{1}{2}(a^2 + b^2)$: Young's \neq
(simplest instance),

For our example:

$$V \leq \frac{1}{2} [x_1^2 + g^2(x_1)] - \underbrace{(b-1)x_2^2}_{) + b > 0} + b \quad \text{😊}$$

deal breaker

\therefore This choice $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

of Lyapunov function contradicts
DOES NOT Work!!!

Tool that allows us to prove asymptotic stability of $\bar{x} = 0$ for $\dot{x} = f(x)$ where
 $\dot{V} \leq 0$ [negative semi-definite]

as opposed to

$\dot{V} < 0$ [negative definite]

Note: for $\ddot{x} + b\dot{x}_2 - d_2 x_2^2$ -like system $\dot{x}_2 = 0$ \Rightarrow

$$x_1 \neq 0; x_2 = 0 \Rightarrow \dot{V} = 0 \Rightarrow$$

For

$$\Rightarrow \dot{V} \leq 0$$

$$\dot{V} = \int_0^{x_1} g(-\bar{x}) dx + \frac{1}{2} x_2^2$$

Then [LaSalle's Invariance Principle]:

Let $\bar{x} = 0$ be an eq. point of

$\dot{x} = f(x)$ Time-invariant system

and let $V(x) \leq 0$ for all

$$x \in \Omega_c := \{x ; V(x) \leq c\}.$$

Lyapunov function candidate

Define $S := \{x \in \Omega_c ; V(x) = 0\}$ and let

M be the largest invariant set in S .

Then for every $x(0) \in \Omega_c \Rightarrow$

$$\lim_{t \rightarrow \infty} x(t) \in M. \quad \blacksquare$$

Back to pendulum-like system:

$$\ddot{x}_1 = -b \cdot x_2^2 = 0 \Rightarrow x_2 = 0 \Rightarrow$$
$$\Rightarrow S = \{ (x_1, 0) ; x_1 \in [-a, a] \}$$

Q: What is the largest invariant set in S under one dynamics?

i.e., for $\dot{x}_1 = x_2$
 $\dot{x}_2 = -g(x_1) - b \cdot x_2$

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow 0 = -g(x_1) - b \cdot x_2$$
$$\Rightarrow g(x_1) = 0 \Rightarrow x_1 = 0 \text{ is the only point in } (-a, a) \text{ for which } g(x_1) = 0$$

From here $\Rightarrow M = \{(0, 0)\} \Rightarrow$

$\lim_{t \rightarrow \infty} x(t) = 0 \Rightarrow \text{LAS of } \bar{x} = 0.$

In (+) case; $g(x_1) = a \cdot x_1 ; a > 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a > 0; b > 0 \\ \text{SAS of } \bar{x} = 0 \\ \text{via} \\ \text{Routh-Hurwitz} \end{bmatrix}$$
$$\Rightarrow \det(S^T I - A) = S^2 + bS + a$$

$$V(x_1, x_2) = \frac{a}{2} x_1^2 + \frac{1}{2} x_2^2 \Rightarrow \dot{V} = -bx_2^2 \leq 0$$

Note:

$$\dot{V} = -[x_1 \quad x_2] \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & b \end{bmatrix}}_Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$= -[x_1 \quad x_2] \cdot \underbrace{\begin{bmatrix} 0 \\ \sqrt{\epsilon_0} \end{bmatrix}}_{C^T} \cdot \underbrace{\begin{bmatrix} 0 & \sqrt{\epsilon_0} \end{bmatrix}}_C \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} =$$

$$= -C\mathbf{x}^T \cdot (C\mathbf{x}) \Rightarrow$$

where

$\gamma = -y^T \cdot y$

$\gamma = C\mathbf{x} =$
 $= [0 \quad \sqrt{\epsilon_0}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

We'll come back to this example soon to discuss details about L₂-S_Me's Invariance Principle for LTI systems.

But First: (brief summary)

- Lyapunov functions for LT systems
details in EE 535
- $\dot{x} = Ax ; f(x) = Ax ; A \in \mathbb{R}^{n \times n}$
fixed matrix

For LT systems, we can restrict our attention to the class of quadratic Lyapunov functions:

$$V(x) = x^T P x ; \text{ quadratic form on } \mathbb{R}^n$$

$P = P^T$ is a symmetric positive definite matrix.

Fact: Any matrix can be written as

$$P = P_S + P_A = \frac{1}{2}(P + P^T) + \frac{1}{2}(P - P^T)$$

Show that

$$\alpha^T P_A \alpha = 0$$

$$\begin{aligned}\frac{1}{2} \alpha^T (P - P^T) \alpha &= \frac{1}{2} \alpha^T P \alpha - \frac{1}{2} \alpha^T P^T \alpha = \\ &= \frac{1}{2} \alpha^T P \alpha - \frac{1}{2} (\alpha^T P \alpha)^T \alpha = \\ &= \frac{1}{2} \alpha^T P \alpha - \frac{1}{2} \alpha^T P \alpha = 0 \Rightarrow \\ \alpha^T P \alpha &= \alpha^T P_S \alpha \quad [\text{Enough to work w/ symmetric matrices}]\end{aligned}$$

Fact: Any symmetric matrix has real eigenvalues and is uniformly diagonalizable

$$P = V \Lambda V^T ; V V^T = V^T V = I$$

$$P \cdot \begin{bmatrix} | \\ v_i \end{bmatrix} = \lambda_i \begin{bmatrix} | \\ v_i \end{bmatrix} ; V = \begin{bmatrix} | \\ v_1, \dots, v_n \end{bmatrix}$$

Value:

$$\langle P x \rangle = x^T P x = x^T V \cdot \Lambda \cdot V^T \cdot x =$$

$$= z^T \cdot \Lambda \cdot z = z$$

$$= \sum_{i=1}^n \lambda_i \cdot z_i^2$$

$$\sqrt{\lambda_i} = 0$$

$$\langle P x \rangle > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

i.e. $P = P^T$ positive definite $\Rightarrow \lambda_i > 0 ; \forall i = 1, \dots, n$

Fact:

$$\lambda_{\min}(P) \cdot \|x\|_2^2 \leq x^T P x \leq \lambda_{\max}(P) \|x\|_2^2$$

If $P = P^T > 0$ positive-definite \Rightarrow

$\lambda_{\min}(P) > 0 \Rightarrow \lim_{\|x\| \rightarrow \infty} V(x) = +\infty$
V is always unbounded

Summary: For any $P = P^T > 0$
(symmetric matrix w/ positive eigenvalues)

$V(x) = x^T P x$ is a legitimate Lyapunov function candidate
 for GLOBAL stability analysis.