


Lecture 18

03/29/23

LT) System: $\dot{x} = Ax$; $x(t) \in \mathbb{R}^n$

Quadratic forms: $V(x) = x^T \cdot P x$

$P = P^T > 0$: symmetric; positive definite
(e-values > 0)

$V(0) = 0$; $V(x) > 0$; $\forall x \in \mathbb{R}^n \setminus \{0\}$

$\lim_{\|x\| \rightarrow \infty} V(x) = +\infty$ ($V(x)$: positive definite, radially unbounded)

Ex: $n=2 \Rightarrow P = \begin{bmatrix} p_1 & p_0 \\ p_0 & p_2 \end{bmatrix}$

$$\Delta_1 = p_1 > 0$$

$$\Delta_2 = p_1 \cdot p_2 - p_0^2 > 0$$

conditions for $P = P^T > 0$
all principal minors
positive

①

Thm: LTI System $\dot{x} = Ax$ is stable
[i.e., $\bar{x} = 0$ is GAS; A is a Hurwitz matrix
(all e-values have negative real parts)]

$\forall Q = Q^T > 0, \exists P = P^T > 0$ s.t.

$$A^T P + PA = -Q. \quad \dots \text{(ALE)}$$

Moreover, such P is a unique (solution to ALE) and is given by:

$$P = \int_0^{\infty} e^{A^T t} Q e^{At} dt \quad \text{and}$$

$V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$. \square (2)

Proof of sufficiency:

Propose: $V(x) = x^T P x$

for $\dot{x} = A x$

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} =$$

$$= [A x]^T P x + x^T P [A x] =$$

$$= x^T A^T P x + x^T P A x =$$

$$= x^T \underbrace{[A^T P + P A]}_{-Q} x =$$

$$= -x^T Q x < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

Note: This for nonlinear systems gives sufficient conditions for stability but it DOES NOT tell us how to construct $V(x)$. (3)

In LTI case, we know how to construct $V(x)$; start with $Q = Q^T > 0$; solve (ALE) for $P \Rightarrow V(x) = x^T P x$.

Ex: $A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$; $\det(sI - A) = s^2 + \underline{b}s + \underline{a}$

$$P = \begin{bmatrix} p_1 & p_0 \\ p_0 & p_2 \end{bmatrix}$$

$$P \cdot A = \begin{bmatrix} p_1 & p_0 \\ p_0 & p_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} =$$

$$= \left[\begin{array}{c|c} -a p_0 & p_1 - b p_0 \\ \hline -a p_2 & p_0 - b p_2 \end{array} \right] \Rightarrow \textcircled{4}$$

\Rightarrow stability $a, b > 0$

$$A^T P + P A = \left[\begin{array}{c|c} -2ap_0 & p_1 - bp_0 - ap_2 \\ \hline * & 2(lp_2 - p_0) \end{array} \right] =$$

$$= \begin{bmatrix} -z_1 & -z_0 \\ -z_0 & -z_2 \end{bmatrix} = -Q$$

$$2ap_0 = z_1 \Rightarrow p_0 = \frac{z_1}{2a} > 0$$

$$p_1 - bp_0 - ap_2 = -z_0 \Rightarrow p_1 = bp_0 + ap_2 - z_0$$

$$2(lp_2 - p_0) = z_2 \quad \text{CAH let } z_0 = 0$$

$$\Rightarrow p_2 = \frac{1}{2l} \cdot \left[p_0 + \frac{z_2}{2} \right] = \frac{1}{2l} \cdot \left[\frac{z_1}{a} + z_2 \right] > 0 \quad (5)$$

$$x^T P x = \rho_1 x_1^2 + 2\rho_0 x_1 x_2 + \rho_2 x_2^2$$

$$\rho_1 > 0; \rho_1, \rho_2 > \rho_0^2$$

For $\varepsilon_0 = 0$

$$\rho_1 \cdot \rho_2 = \left[\rho_0 + \rho_2 \right] \cdot \frac{1}{\rho_0} \left[\rho_0 + \frac{\varepsilon_2}{2} \right] =$$

$$= \rho_0^2 + \frac{1}{2} \varepsilon_2 \rho_0 + \rho_2^2 > \rho_0^2$$

Thus, for $Q = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$ with
 $\varepsilon_1 > 0; \varepsilon_2 > 0$ and $a > 0, b > 0$

$\Rightarrow P = P^T > 0 \Rightarrow$ LTI system stable. (6)

Recall: with $V(x) = \frac{a}{2} x_1^2 + \frac{1}{2} x_2^2$
 $p_1 = \frac{a}{2}; p_0 = 0; p_2 = \frac{1}{2}$

we have:

$$\dot{V} = -y^T y = -(Cx)^T (Cx) \leq 0$$

$$C = \begin{bmatrix} 0 & \sqrt{b} \end{bmatrix} \Rightarrow Q = C^T C = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$Q = Q^T \geq 0$ but not positive
positive semi-definite (for $b > 0$) definite

CANNOT use previous Thm

but can invoke LaSalle's
invariance principle.

(7)

$$\dot{V} = 0 \Rightarrow y = Cx = 0$$

$$\left. \begin{array}{l} \dot{x} = Ax \\ y = Cx \end{array} \right) \Rightarrow \boxed{y(t) = Ce^{At}x_0}$$

$$x_0 = 0 \Rightarrow y(t) \equiv 0$$

Q: Is there any $x_0 \neq 0$ s.t.

$$y(t) = Ce^{At}x_0 \equiv 0?$$

If Yes, (LTI) system is NOT observable.

Observability of $(A, C) \Rightarrow$ Lyapunov

stable of $\dot{x} = 0$ of $\dot{x} = Ax$

Summary: For $\dot{x} = Ax$ with

$$\dot{V}(x) = -x^T Q x = -[c x]^T c x \leq 0$$

LaSalle $\Rightarrow \dot{x} = Ax$ is stable if (A, c) observable

IF EESSS

Kalman rank test $\text{rank} \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} = n$

PBH test
for all $s \in \text{eigenvalues of } A$,
 $\text{rank} \begin{bmatrix} (sI - A) \\ C^T \end{bmatrix} = n$

$$A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}; \quad c = [0 \quad \sqrt{b}]$$

$$cA = [0 \quad \sqrt{b}] \cdot \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} =$$

$$= [-a\sqrt{b} \quad -b\sqrt{b}]$$

$$\begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{b} \\ -a\sqrt{b} & -b\sqrt{b} \end{bmatrix}$$

$$\det \begin{bmatrix} c \\ cA \end{bmatrix} = a \cdot b \neq 0$$

$a, b \neq 0$

$a > 0; b > 0$
from

$p = p^T > 0; \quad a = a^T \geq 0$

(10)

Comments:

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases} \Rightarrow$$

Observability
gramian

$$P = \int_0^{\infty} e^{At} C^T C e^{At} dt$$

$$A^T P + P A = -C^T C \quad \textcircled{Q}$$

$$2^{\circ} \dot{x} = Ax + Bu ; \quad Ax + x A^T = -BB^T$$

Controllability
gramian

$$X = \int_0^{\infty} e^{At} B B^T e^{A^T t} dt$$

1) u : white noise, with zero mean

and identity covariance matrix

$$E\{u(t)u^T(\tau)\} = I \cdot \delta(t-\tau)$$

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1) $\dot{x} = Ax$ is stable

$$X = \lim_{t \rightarrow \infty} \{E\} x(t) \cdot x^T(t)$$

3^o For DT LTI systems

$$x(t+1) = Ax(t) \quad ; \quad t = 0, 1, 2, 3, \dots$$

$$A^T P A - P = -Q \quad \text{v.s.} \quad \underbrace{A^T P + P A = -Q}_{\text{C.T.}}$$

$$V(x) = x^T P x$$
$$V(x(t+1)) - V(x(t)) = \underbrace{[Ax(t)]^T}_{x^T(t+1)} P \underbrace{Ax(t)}_{x(t+1)} - x^T(t) \cdot P x(t)$$

$$P = \sum_{t=0}^{\infty} (A^T)^t \cdot Q \cdot A^t$$

v.s.

$$P = \int_0^{\infty} e^{A^T t} Q e^{A t} dt$$

Q.T.