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Exact computation of frequency responses for a class of infinite dimensional systems

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Abstract

We develop a method for the exact computation of the frequency responses for a class of infinite dimensional systems. In particular, we consider the distributed parameter systems in which a spatial independent variable belongs to a finite interval. We show that an explicit formula for the frequency responses can be derived whenever the underlying operators can be represented by a forced two point boundary value state-space realizations (TPBVSR). This formula involves finite dimensional computations with matrices whose dimension is at most four times larger than the order of the underlying differential operator. In this way an exact reduction of an infinite dimensional problem to a finite dimensional one is accomplished. We also provide several examples to illustrate the procedure.

1 Introduction

We study frequency responses of the distributed parameter systems in which a spatial independent variable belongs to a finite interval. Computation of frequency responses for this class of systems is usually done numerically by resorting to finite dimensional approximations of the underlying operators. We show that the spatial discretization can be circumvented and that frequency responses can be determined explicitly whenever the underlying operators can be represented by forced two point boundary value state-space realizations which are well posed.

Our results build on [1], where a formula for the trace of a class of differential operators defined by forced TPBVSR with constant coefficients has been derived. This formula has been used for computation of the $\mathcal{H}_2$ norm for a class of infinite dimensional systems in which the dynamical generators are normal (or self-adjoint). Here we study the *spatio-temporal frequency responses* of the distributed parameter systems with, in general, non-normal dynamical generators and non-constant coefficients in a spatially independent variable.

Our presentation is organized as follows: in section 2, we formulate the problem and briefly discuss the notion of frequency response for the distributed parameter systems. In § 3, we show how the frequency responses can be determined explicitly without resorting to the finite dimensional approximations. In § 4, we provide two examples to illustrate the application of the developed procedure. We conclude by summarizing major contributions in § 5.

2 Preliminaries

We consider distributed parameter systems of the form

$$\partial_t \psi(y,t) \;=\; [\mathcal{A}\psi(t)](y) \;+\; [\mathcal{B}d(t)](y), \qquad \text{(1a)}$$
$$\phi(y,t) \;=\; [\mathcal{C}\psi(t)](y), \qquad \text{(1b)}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are linear operators, d is a forcing term, and ψ and ϕ are fields of interest determined by the solution of the above equation. The spatial independent variable y is assumed to belong to a finite interval which has been normalized to $[-1,\,1]$. We remark that (1) can be also used to describe multi-dimensional distributed parameter systems that are spatially invariant in the remaining spatial directions [2]. In this case, the application of the spatial Fourier transform in these directions renders differential/integral operators into multiplication operators which results in a one-dimensional system parameterized by a vector of spatial frequencies κ

$$\partial_t \psi(y,t,\kappa) = [\mathcal{A}(\kappa)\psi(t,\kappa)](y) \;+\; [\mathcal{B}(\kappa)d(t,\kappa)](y), \quad \text{(2a)}$$
$$\phi(y,t,\kappa) = [\mathcal{C}(\kappa)\psi(t,\kappa)](y). \qquad \text{(2b)}$$

Our objective is to investigate dynamical properties of system (1) or system (2) by computing their frequency responses. The frequency response of (1) is determined by

$$\mathcal{H}(\omega) \;:=\; \mathcal{C}(i\omega I \;-\; \mathcal{A})^{-1}\mathcal{B},$$

where ω denotes the temporal frequency. Similarly, the *spatio-temporal frequency response* of system (2) is given by

$$\mathcal{H}(\omega,\kappa) \;:=\; \mathcal{C}(\kappa)(i\omega I \;-\; \mathcal{A}(\kappa))^{-1}\mathcal{B}(\kappa).$$

In the remainder of our paper we discuss the latter notion, because it is more general.

We remark that for any given pair $\{\omega,\kappa\}$, $\mathcal{H}(\omega,\kappa)$ represents an operator in y that maps $d(y,\omega,\kappa)$ into $\phi(y,\omega,\kappa)$ according to

$$\phi(y,\omega,\kappa) \;=\; \int_{-1}^{1} [\mathcal{H}(\omega,\kappa)]\,(y,\eta) d(\eta,\omega,\kappa)\; d\eta,$$

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where by an abuse of notation we use the symbol $[\mathcal{H}(\omega,\kappa)](y,\eta)$ to denote the kernel function representing the operator $\mathcal{H}(\omega,\kappa)$.

Since $\mathcal{H}$ is an operator-valued function of two independent variables there is a variety of different ways to visualize system properties. For example, one can study the maximal singular values of the operator $\mathcal{H}$

$$\sigma^2_{\max}(\mathcal{H}(\omega,\kappa)) \;:=\; \lambda_{\max}\{\mathcal{H}(\omega,\kappa)\mathcal{H}^*(\omega,\kappa)\},$$

or compute the Hilbert–Schmidt norm of $\mathcal{H}(\omega,\kappa)$ provided that $\mathcal{H}(\omega,\kappa)\mathcal{H}^*(\omega,\kappa)$ represents a trace class operator on $L^2[-1,\,1]$

$$||\mathcal{H}(\omega,\kappa)||^2_{HS} \;:=\; \text{trace}(\mathcal{H}(\omega,\kappa)\mathcal{H}^*(\omega,\kappa)).$$

Computation of $||\mathcal{H}(\omega,\kappa)||_{HS}$ is usually done numerically for any given pair $\{\omega,\kappa\}$, after finite dimensional approximations of the underlying operators have been determined by the appropriate spatial discretization. In § 3, we show that the Hilbert–Schmidt norm of $\mathcal{H}(\omega,\kappa)$ can be determined explicitly without resorting to the finite dimensional approximations, whenever $\mathcal{H}(\omega,\kappa)$ can be represented by a TPBVSR which is well posed.

3 Computation of frequency responses from state-space realizations

We assume that the operator $\mathcal{H}(\omega,\kappa)$: $d(\omega,\kappa) \mapsto \phi(\omega,\kappa)$ can be represented by a well-posed TPBVSR of the following form

$$\begin{aligned}
\begin{bmatrix} x_1'(y) \\ x_2'(y) \end{bmatrix} &= A_0(y)\begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} \;+\; B_0(y)d(y),\\
\phi(y) &= C_0(y)\begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix},\\
x_1(\pm 1) &= 0, \qquad\qquad y \in [-1,\,1],
\end{aligned} \tag{3}$$

where the dependence on ω and κ is suppressed for notational convenience, and $x_k'(y) := dx_k(y)/dy$, $k = \{1,2\}$. Furthermore, A_0, B_0, and C_0 are matrices (with, in general, non-constant coefficients in y) of the appropriate dimensions for any given pair $\{\omega,\kappa\}$, and $x^T(y) := \begin{bmatrix} x_1^T(y) & x_2^T(y) \end{bmatrix}$ is an equal partitioning of the state variables. These realizations are chosen such that the first half of the state variables vanishes at the boundary points $y = \pm 1$, while the second half is free. We remark that a well-posed TPBVSR with a general linear constraint on the state at $y = \pm 1$ can be transformed into (3) by introducing an appropriate coordinate transformation, as illustrated in Appendix A.

It can be shown that the adjoint of the operator $\mathcal{H}(\omega,\kappa)$, $\mathcal{H}^*(\omega,\kappa)$: $f(\omega,\kappa) \mapsto d(\omega,\kappa)$, can be represented by a TPBVSR of the following form

$$\begin{aligned}
\begin{bmatrix} z_1'(y) \\ z_2'(y) \end{bmatrix} &= -A_0^*(y)\begin{bmatrix} z_1(y) \\ z_2(y) \end{bmatrix} \;-\; C_0^*(y)f(y),\\
d(y) &= B_0^*(y)\begin{bmatrix} z_1(y) \\ z_2(y) \end{bmatrix},\\
z_2(\pm 1) &= 0, \qquad\qquad y \in [-1,\,1],
\end{aligned} \tag{4}$$

where A_0^*, B_0^*, and C_0^* represent the adjoints (complex-conjugate-transpose matrices) of the matrices A_0, B_0, and C_0, respectively. Thus, we can obtain a state-space realization of $\mathcal{H}(\omega,\kappa)\mathcal{H}^*(\omega,\kappa)$: $f(\omega,\kappa) \mapsto \phi(\omega,\kappa)$, by combining (3) and (4)

$$\begin{aligned}
\begin{bmatrix} x_1'(y) \\ x_2'(y) \\ z_1'(y) \\ z_2'(y) \end{bmatrix} &= \begin{bmatrix} A_0(y) & B_0(y)B_0^*(y) \\ 0 & -A_0^*(y) \end{bmatrix}\begin{bmatrix} x_1(y) \\ x_2(y) \\ z_1(y) \\ z_2(y) \end{bmatrix}\\
&\quad + \begin{bmatrix} 0 \\ -C_0^*(y) \end{bmatrix} f(y),\\
\phi(y) &= \begin{bmatrix} C_0(y) & 0 \end{bmatrix}\begin{bmatrix} x(y) \\ z(y) \end{bmatrix},\\
x_1(\pm 1) &= z_2(\pm 1) \;=\; 0, \quad y \in [-1,\,1].
\end{aligned} \tag{5}$$

We now introduce a unitary coordinate transformation of the form

$$\begin{aligned}
q(y) &= \begin{bmatrix} q_1(y) \\ q_2(y) \end{bmatrix} := \begin{bmatrix} z_1(y) \\ x_2(y) \\ x_1(y) \\ z_2(y) \end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} x_1(y) \\ x_2(y) \\ z_1(y) \\ z_2(y) \end{bmatrix} =: T\begin{bmatrix} x_1(y) \\ x_2(y) \\ z_1(y) \\ z_2(y) \end{bmatrix},
\end{aligned}$$

and rewrite (5) as

$$\begin{aligned}
\begin{bmatrix} q_1'(y) \\ q_2'(y) \end{bmatrix} &= A(y)\begin{bmatrix} q_1(y) \\ q_2(y) \end{bmatrix} \;+\; B(y)f(y),\\
\phi(y) &= C(y)\begin{bmatrix} q_1(y) \\ q_2(y) \end{bmatrix},\\
q_2(\pm 1) &= 0, \qquad\qquad y \in [-1,\,1],
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
A(y) &:= T\begin{bmatrix} A_0(y) & B_0(y)B_0^*(y) \\ 0 & -A_0^*(y) \end{bmatrix}T,\\
B(y) &:= T\begin{bmatrix} 0 \\ -C_0^*(y) \end{bmatrix},\\
C(y) &:= \begin{bmatrix} C_0(y) & 0 \end{bmatrix}T.
\end{aligned}$$

We note that this TPBVSR of $\mathcal{H}\mathcal{H}^*$ (6) is such that the second half of the state variables vanishes at the boundary points $y = \pm 1$, while the first half is free. Using the argument similar to the one presented in [1] we are able to express trace$(\mathcal{H}(\omega,\kappa)\mathcal{H}^*(\omega,\kappa))$ as

$$\begin{aligned}
||\mathcal{H}(\omega,\kappa)||^2_{HS} &= \text{trace}\Big(\frac{1}{2}\int_{-1}^{1} C(y)B(y)\,dy \;-\\
&\int_{-1}^{1} C(y)\Phi(y,-1)\begin{bmatrix} 0 & \Phi_{21}^{-1}(1,-1) \\ 0 & 0 \end{bmatrix}\Phi(1,y)B(y)\,dy\Big)\\
&= -\,\text{trace}\Big(\int_{-1}^{1}\Phi(1,y)B(y)C(y)\Phi(y,-1)\,dy\;\times\\
&\qquad \begin{bmatrix} 0 & \Phi_{21}^{-1}(1,-1) \\ 0 & 0 \end{bmatrix}\Big),
\end{aligned} \tag{7}$$

where the state transition matrix $\Phi(y,\xi)$ of system (6) is partitioned conformably with the states q_1 and q_2, that is

$$\Phi(y,\xi) \;:=\; \begin{bmatrix} \Phi_{11}(y,\xi) & \Phi_{12}(y,\xi) \\ \Phi_{21}(y,\xi) & \Phi_{22}(y,\xi) \end{bmatrix}.$$

We have arrived at (7) using the commutativity property of the matrix trace and

$$\begin{aligned} C(y)B(y) &= \begin{bmatrix} C_0(y) & 0 \end{bmatrix} TT \begin{bmatrix} 0 \\ -C_0^*(y) \end{bmatrix} \\ &= \begin{bmatrix} C_0(y) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -C_0^*(y) \end{bmatrix} = 0. \end{aligned}$$

We now exploit the fact that the integral in (7) can be evaluated in terms of the response of an unforced dynamical system of the form

$$\begin{aligned} \begin{bmatrix} X_1'(y) \\ X_2'(y) \end{bmatrix} &= \begin{bmatrix} A(y) & 0 \\ B(y)C(y) & A(y) \end{bmatrix} \begin{bmatrix} X_1(y) \\ X_2(y) \end{bmatrix}, \\ Y(y) &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} X_1(y) \\ X_2(y) \end{bmatrix}, \\ \begin{bmatrix} X_1(-1) \\ X_2(-1) \end{bmatrix} &= \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad y \in [-1, 1], \end{aligned} \tag{8}$$

as

$$\begin{aligned} Y(1) &= \int_{-1}^{1} \Phi(1,y)B(y)C(y)\Phi(y,-1)\, dy \\ &= \begin{bmatrix} 0 & I \end{bmatrix} \Psi(1,-1) \begin{bmatrix} I \\ 0 \end{bmatrix}, \end{aligned} \tag{9}$$

where $\Psi(y,\xi)$ denotes the state transition matrix of system (8). Therefore, we obtain an expression for $||\mathcal{H}(\omega,\kappa)||_{HS}^2$ by combining (7) and (9)

$$||\mathcal{H}(\omega,\kappa)||_{HS}^2 = -\,\mathrm{trace}\left(\begin{bmatrix} 0 & I \end{bmatrix} \Psi(1,-1) \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \Phi_{21}^{-1}(1,-1) \\ 0 & 0 \end{bmatrix} \right). \tag{10}$$

In particular, for systems with constant coefficients $\Psi(1,-1)$ is determined by

$$\Psi(1,-1) = \exp\left\{ 2 \begin{bmatrix} A & 0 \\ BC & A \end{bmatrix} \right\}. \tag{11}$$

Hence, we have converted the problem of evaluating the trace of an essentially infinite dimensional object to computations with finite matrices. For systems with non-constant coefficients in y, at any given $\{\omega, \kappa\}$, we need to solve a differential equation whose order is four times larger than the order of the original differential operator to obtain the state transition matrix $\Psi(1,-1)$. On the other hand, for systems with constant coefficients in y this computation amounts to determination of the corresponding matrix exponential (11). Both computations can be easily performed using commercially available software such as MATLAB or MATHEMATICA.

In § 4 we illustrate the application of this procedure and exactly compute the frequency responses of two systems: a one-dimensional diffusion equation, and a system that describes the dynamics of velocity fluctuations in channel fluid flows.

4 Examples

In this section we illustrate with two examples how the frequency responses can be computed explicitly using the previously described procedure.

4.1 A one-dimensional diffusion equation

We consider a one-dimensional diffusion equation on $L^2[-1, 1]$ with Dirichlet boundary conditions

$$\begin{aligned} \partial_t \psi(y,t) &= [\partial_{yy}\psi(t)](y) + d(y,t), \\ \psi(\pm 1, t) &= 0. \end{aligned}$$

The application of the temporal Fourier transform yields

$$\psi(y,\omega) = [(i\omega I - \partial_{yy})^{-1} d(\omega)](y) =: [\mathcal{H}(\omega)d(\omega)](y).$$

Our objective is to compute the Hilbert–Schmidt norm of operator that maps d into ψ as a function of temporal frequency ω. A particular TPBVSR of $\mathcal{H}(\omega)$ is given by

$$\begin{aligned} \begin{bmatrix} x_1'(y,\omega) \\ x_2'(y,\omega) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ i\omega & 0 \end{bmatrix} \begin{bmatrix} x_1(y,\omega) \\ x_2(y,\omega) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(y,\omega), \\ \psi(y,\omega) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(y,\omega) \\ x_2(y,\omega) \end{bmatrix}, \\ x_1(\pm 1, \omega) &= 0, \quad y \in [-1, 1]. \end{aligned}$$

The application of procedure described in § 3 yields the realization of $\mathcal{H}(\omega)\mathcal{H}^*(\omega)$ in the form of (6) with

$$A = \begin{bmatrix} 0 & 0 & 0 & i\omega \\ 0 & 0 & i\omega & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T.$$

The application of (10,11), with the help of MATHEMATICA, yields

$$||\mathcal{H}(\omega)||_{HS}^2 = \frac{1}{2\omega^2}\left\{ -1 + \frac{\sqrt{2\omega}(\sinh(\sqrt{8\omega}) + \sin(\sqrt{8\omega}))}{\cosh(\sqrt{8\omega}) - \cos(\sqrt{8\omega})} \right\}. \tag{12}$$

It is noteworthy that $||\mathcal{H}(\omega)||_{HS}^2$ can be also obtained by doing a spectral decomposition of the operator ∂_{yy}. It is well known that this operator with Dirichlet boundary conditions has the following set of orthonormal eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$ with corresponding eigenvalues $\{\gamma_n\}_{n\in\mathbb{N}}$

$$\varphi_n(y) := \sin\left(\frac{n\pi}{2}(y+1)\right), \;\; \gamma_n := -\frac{n^2\pi^2}{4}, \;\; n \in \mathbb{N}.$$

It is easily shown that the eigenfunction expansion results into

$$||\mathcal{H}(\omega)||_{HS}^2 = \sum_{n\in\mathbb{N}} \frac{1}{\omega^2 + \left(\frac{n\pi}{2}\right)^4},$$

which can be summed to obtain (12). However, for operators that do not have explicit expressions for their eigenvalues, it is much easier to compute the trace explicitly as we illustrate in § 4.2.

4.2 A system from fluid dynamics

We consider the externally excited linearized Navier-Stokes (LNS) equations in channel flows in the presence of streamwise constant perturbations (that is, at $k_x = 0$). For background material on the use of system norms in transition to turbulence studies, we refer the reader to [3, 4, 5] and the references therein.

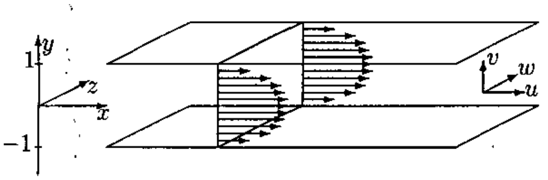

Figure 1: Three dimensional channel flow.

The forced LNS equations for streamwise constant flow perturbations are described by [6]

$$\partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{R}\mathcal{L} & 0 \\ \mathcal{C}_p & \frac{1}{R}\mathcal{S} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 & \mathcal{B}_y & \mathcal{B}_z \\ \mathcal{B}_x & 0 & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}, \quad (13a)$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{C}_u \\ \mathcal{C}_v & 0 \\ \mathcal{C}_w & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad (13b)$$

where u, v, and w (d_x, d_y, and d_z) denote velocity (forcing) field components in the x, y, and z directions respectively (see Figure 1 for geometry in Poiseuille flow). These equations are parameterized by two important parameters: the spanwise wave number k_z, and the Reynolds number R. The wall-normal velocity and vorticity fields are denoted by $\psi_1(y,t,k_z)$ and $\psi_2(y,t,k_z)$, respectively, with the boundary conditions

$$\psi_1(\pm 1,t,k_z) = \partial_y \psi_1(\pm 1,t,k_z) = \psi_2(\pm 1,t,k_z) = 0.$$

The underlying operators in (13) are defined by

$$\begin{aligned} &\mathcal{L} := \Delta^{-1}\Delta^2, \quad \mathcal{S} := \Delta^{-1}, \quad \mathcal{C}_p := -ik_z U'(y), \\ &\mathcal{B}_x := ik_z, \quad \mathcal{B}_y := -k_z^2\Delta^{-1}, \quad \mathcal{B}_z := -ik_z\Delta^{-1}\partial_y, \\ &\mathcal{C}_u := -\frac{i}{k_z}, \quad \mathcal{C}_v := I, \quad \mathcal{C}_w := \frac{i}{k_z}\partial_y, \end{aligned}$$

where $U(y)$ denotes the nominal velocity profile, $U' := dU/dy$, and Δ represents the Laplacian, $\Delta := \partial_{yy} - k_z^2$. It is well known that system (13) is stable for any pair $\{k_z, R\}$ [3].

Application of the temporal Fourier transform and elimination of $\begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix}^T$ from (13) gives operator $\mathcal{H}(\omega,k_z,R)$ that maps $\begin{bmatrix} d_x & d_y & d_z \end{bmatrix}^T$ into $\begin{bmatrix} u & v & w \end{bmatrix}^T$ (see Appendix B)

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{ux} & \mathcal{H}_{uy} & \mathcal{H}_{uz} \\ \mathcal{H}_{vx} & \mathcal{H}_{vy} & \mathcal{H}_{vz} \\ \mathcal{H}_{wx} & \mathcal{H}_{wy} & \mathcal{H}_{wz} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} := \begin{bmatrix} R\bar{\mathcal{H}}_{ux} & R^2\bar{\mathcal{H}}_{uy} & R^2\bar{\mathcal{H}}_{uz} \\ 0 & R\bar{\mathcal{H}}_{vy} & R\bar{\mathcal{H}}_{vz} \\ 0 & R\bar{\mathcal{H}}_{wy} & R\bar{\mathcal{H}}_{wz} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}, \quad (14)$$

where

$$\begin{aligned} \bar{\mathcal{H}}_{ux}(\Omega,k_z) &:= \mathcal{C}_u(i\Omega I - \mathcal{S})^{-1}\mathcal{B}_x, \\ \bar{\mathcal{H}}_{uy}(\Omega,k_z) &:= \mathcal{C}_u(i\Omega I - \mathcal{S})^{-1}\mathcal{C}_p(i\Omega I - \mathcal{L})^{-1}\mathcal{B}_y, \\ \bar{\mathcal{H}}_{uz}(\Omega,k_z) &:= \mathcal{C}_u(i\Omega I - \mathcal{S})^{-1}\mathcal{C}_p(i\Omega I - \mathcal{L})^{-1}\mathcal{B}_z, \\ \bar{\mathcal{H}}_{rs}(\Omega,k_z) &:= \mathcal{C}_r(i\Omega I - \mathcal{L})^{-1}\mathcal{B}_s, \; \{r = v,w; \; s = y,z\}, \end{aligned}$$

and $\Omega := \omega R$.

Therefore, we are able to state the following theorem for streamwise constant perturbations in any channel flow with nominal velocity profile $U(y)$.

Theorem 1 *For any channel flow with nominal velocity profile $U(y)$, the componentwise Hilbert-Schmidt norms of streamwise constant perturbations are given by*

$$\begin{bmatrix} \|\mathcal{H}_{ux}\|_{HS}^2 & \|\mathcal{H}_{uy}\|_{HS}^2 & \|\mathcal{H}_{uz}\|_{HS}^2 \\ \|\mathcal{H}_{vx}\|_{HS}^2 & \|\mathcal{H}_{vy}\|_{HS}^2 & \|\mathcal{H}_{vz}\|_{HS}^2 \\ \|\mathcal{H}_{wx}\|_{HS}^2 & \|\mathcal{H}_{wy}\|_{HS}^2 & \|\mathcal{H}_{wz}\|_{HS}^2 \end{bmatrix} = \begin{bmatrix} R^2 f_{ux}(\Omega,k_z) & R^4 f_{uy}(\Omega,k_z) & R^4 f_{uz}(\Omega,k_z) \\ 0 & R^2 f_{vy}(\Omega,k_z) & R^2 f_{vz}(\Omega,k_z) \\ 0 & R^2 f_{wy}(\Omega,k_z) & R^2 f_{wz}(\Omega,k_z) \end{bmatrix},$$

where

$$\begin{aligned} f_{rs}(\Omega,k_z) &:= \|\bar{\mathcal{H}}_{rs}(\Omega,k_z)\|_{HS}^2, \quad \Omega := \omega R, \\ &\{r = u,v,w; \; s = x,y,z\}. \end{aligned}$$

The overall Hilbert-Schmidt norm for streamwise constant perturbations is given by

$$\|\mathcal{H}(\omega,k_z,R)\|_{HS}^2 = g(\Omega,k_z)R^2 + h(\Omega,k_z)R^4, \quad (15)$$

where

$$\begin{aligned} g &:= f_{ux} + f_{vy} + f_{vz} + f_{wy} + f_{wz}, \\ h &:= f_{uy} + f_{uz}. \end{aligned}$$

Remark 1 *Functions f_{rs}, for every $\{r = u,v,w; \; s = x,y,z\}$, depend only on two parameters: the spanwise wave number, and the product between temporal frequency ω and the Reynolds number R. Thus, for high Reynolds number channel flows the influence of small temporal frequencies dominates the evolution of the velocity perturbations. Therefore, one should expect the dominance of the effects with large time constants in channel flows.*

We remark that $\bar{\mathcal{H}}_{ux}(\Omega,k_z)$ has a second order state-space realization, the operators $\bar{\mathcal{H}}_{rs}(\Omega,k_z)$, $\forall\{r = v,w; \; s = y,z\}$, can be represented by the fourth order TPBVSR, while $\bar{\mathcal{H}}_{uy}(\Omega,k_z)$ and $\bar{\mathcal{H}}_{uz}(\Omega,k_z)$ can be described by minimal realizations with six states. These realizations can be chosen in a number of different ways. The particular realizations that we use are given in Appendix B. It is noteworthy that the Hilbert-Schmidt norms of these operators for any given pair of $\{\Omega,k_z\}$ can be determined explicitly using the procedure of § 3 which involves finite dimensional computations with matrices whose dimension is at most four times larger than the order of the underlying operator. Therefore, in the worst case we need to determine the state transition matrices that belong to $\mathbb{C}^{24\times 24}$, which can be easily done, for example in MATLAB. This illustrates the power of the developed procedure and circumvents the need for numerical approximation of the underlying operators.

In the particular case of Couette flow, which represents the exact steady-state solution of the Navier-Stokes equations of the form $U(y) := y$, the coupling operator simplifies to $\mathcal{C}_p = -ik_z$. Therefore, for this flow, all operators in (13) have the constant coefficients in the wall-normal direction. This implies that the nominal-flow-dependent quantities f_{uy} and f_{uz} can be determined explicitly using (10,11).

Figure 2 illustrates the (Ω, k_z)-parameterized plots of f_{rs}, for every $\{r = u, v, w;\ s = x, y, z\}$. These computations are performed in MATLAB for any given pair $\{\Omega, k_z\}$ using the procedure of § 3. The nominal-flow-dependent quantities f_{uy} and f_{uz} are determined for the Couette flow.

5 Concluding remarks

We have developed a procedure for the explicit determination of the frequency responses for a class of infinite dimensional systems. This procedure avoids the need for spatial discretization and provides an exact reduction of the infinite dimensional problem to the problem in which only matrices of finite dimensions are involved. The order of these matrices is at most four times larger than the order of the differential operator at hand.

We have also illustrated application of this technique by providing two examples: a one-dimensional diffusion equation, and a system obtained by linearization of the Navier-Stokes equations in channel flows around a given nominal velocity profile. In the latter case, we have derived important relationships that illustrate the parametric dependence of the frequency responses on the temporal frequency ω, the spanwise wave-number k_z, and the Reynolds number R.

A TPBVSR with a linear boundary constraint on the state

Consider the operator $\mathcal{H}(\omega, \kappa) : d(\omega, \kappa) \mapsto \phi(\omega, \kappa)$ with a well-posed TPBVSR of the form

$$\begin{aligned}
\begin{bmatrix} p_1'(y) \\ p_2'(y) \end{bmatrix} &= A_1(y) \begin{bmatrix} p_1(y) \\ p_2(y) \end{bmatrix} + B_1(y) d(y), \\
\phi(y) &= C_1(y) \begin{bmatrix} p_1(y) \\ p_2(y) \end{bmatrix}, \qquad (16) \\
N p(\pm 1) &= 0, \quad y \in [-1, 1],
\end{aligned}$$

where N is a constant matrix. It is readily shown that a coordinate transformation of the form

$$\begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} := T_1 \begin{bmatrix} p_1(y) \\ p_2(y) \end{bmatrix} = \begin{bmatrix} N \\ M \end{bmatrix} \begin{bmatrix} p_1(y) \\ p_2(y) \end{bmatrix},$$

renders (16) into (3) where matrix M is chosen such that T_1 is invertible. Similarly, for the following class of systems

$$\begin{aligned}
\begin{bmatrix} p_1'(y) \\ p_2'(y) \end{bmatrix} &= A_1(y) \begin{bmatrix} p_1(y) \\ p_2(y) \end{bmatrix} + B_1(y) d(y), \\
\phi(y) &= C_1(y) \begin{bmatrix} p_1(y) \\ p_2(y) \end{bmatrix}, \\
N_- p(-1) &= 0, \\
N_+ p(+1) &= 0, \qquad y \in [-1, 1],
\end{aligned}$$

a y-dependent change of coordinates can be used to rewrite them in the form suitable for the application of the procedure described in § 3.

B A TPBVSR of the LNS equations at $k_x = 0$

In this section we describe the TPBVSR that are used for the frequency response computations of the streamwise constant LNS equations. Once the temporal Fourier transform has been applied, (13a) can be rewritten as

$$\psi_1 = R(i\Omega I - \mathcal{L})^{-1}(\mathcal{B}_y d_y + \mathcal{B}_z d_z), \qquad (17a)$$

$$\psi_2 = R(i\Omega I - \mathcal{S})^{-1}(\mathcal{C}_p \psi_1 + \mathcal{B}_x d_x), \qquad (17b)$$

with $\Omega := \omega R$, where ω denotes the temporal frequency. By combining (17a) and (17b) with (13b) we arrive at (14). Since the Reynolds number enters nicely into the equations, we can study (17) at $R = 1$. In the particular case of Couette flow system (17) can be rewritten as

$$\begin{aligned}
&(\partial_{yyyy} - (2k_z^2 + i\Omega)\partial_{yy} + k_z^2(k_z^2 + i\Omega))\psi_1 = \\
&\begin{bmatrix} k_z^2 & 0 \end{bmatrix} \begin{bmatrix} d_y \\ d_z \end{bmatrix} + \begin{bmatrix} 0 & ik_z \end{bmatrix} \begin{bmatrix} \partial_y d_y \\ \partial_y d_z \end{bmatrix}, \qquad (18a) \\
&(\partial_{yy} - (k_z^2 + i\Omega))\psi_2 = ik_z \psi_1 - ik_z d_x, \qquad (18b)
\end{aligned}$$

with the boundary conditions $\psi_1(\pm 1) = \partial_y \psi_1(\pm 1) = \psi_2(\pm 1) = 0$.

In particular, system (18,13b) can be represented by a TPBVSR (3), which is a suitable form for the application of the procedure presented in § 3, with matrices A_0, B_0, and C_0 determined by

$$A_0 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2k_z^2 + i\Omega & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_z^2(k_z^2 + i\Omega) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ ik_z & 0 & k_z^2 + i\Omega & 0 & 0 & 0 \end{bmatrix},$$

$$B_0 := \begin{bmatrix} B_{x0} & B_{y0} & B_{z0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_z^2 & 0 \\ 0 & 0 & ik_z \\ -ik_z & 0 & 0 \end{bmatrix},$$

$$C_0 := \begin{bmatrix} C_{u0} \\ C_{v0} \\ C_{w0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{i}{k_z} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{k_z} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The realizations of $\bar{\mathcal{H}}_{rs}(\Omega, k_z)$ can be now easily determined for every $\{r = u, v, w;\ s = x, y, z\}$. For example, TPBVSR of $\bar{\mathcal{H}}_{uy}(\Omega, k_z)$ is given by

$$\begin{aligned}
\begin{bmatrix} x_1'(y) \\ x_2'(y) \end{bmatrix} &= A_0 \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} + B_{y0} d_y(y), \\
u(y) &= C_{u0} \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix}, \\
x_1(\pm 1) &= 0, \qquad y \in [-1, 1].
\end{aligned}$$

As previously mentioned, $\bar{\mathcal{H}}_{ux}(\Omega, k_z)$ and $\bar{\mathcal{H}}_{rs}(\Omega, k_z)$, for every $\{r = v, w;\ s = y, z\}$, are the second and

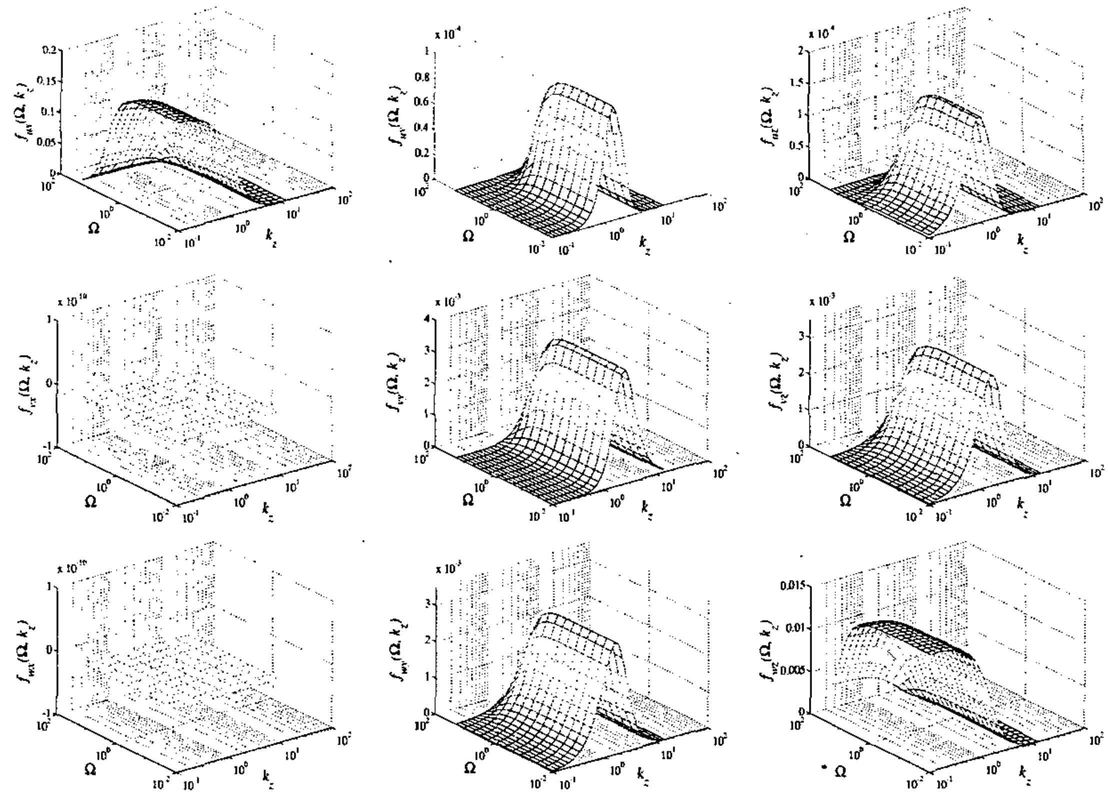

Figure 2: Plots of $f_{rs}(\Omega, k_z)$. $f_{uy}(\Omega, k_z)$ and $f_{uz}(\Omega, k_z)$ are determined for the Couette flow.

the fourth order operators, respectively. We note that their minimal realizations can be obtained by combining the respective realizations of (18b) and (18a) with the corresponding rows of (13b). For example, a minimal realization of $\bar{\mathcal{H}}_{ux}(\Omega, k_z)$ is given by

$$\begin{bmatrix} \eta_1'(y) \\ \eta_2'(y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k_z^2 + i\Omega & 0 \end{bmatrix} \begin{bmatrix} \eta_1(y) \\ \eta_2(y) \end{bmatrix} + \begin{bmatrix} 0 \\ -ik_z \end{bmatrix} d_x(y),$$

$$u(y) = \begin{bmatrix} -\frac{i}{k_z} & 0 \end{bmatrix} \begin{bmatrix} \eta_1(y) \\ \eta_2(y) \end{bmatrix},$$

$$\eta_1(\pm 1) = 0, \quad y \in [-1, 1].$$

Alternatively, $||\bar{\mathcal{H}}_{ux}(\Omega, k_z)||_{HS}^2$ can be determined from the following non-minimal realization of $\bar{\mathcal{H}}_{ux}(\Omega, k_z)$

$$\begin{aligned} \begin{bmatrix} x_1'(y) \\ x_2'(y) \end{bmatrix} &= A_0 \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} + B_{x0} d_x(y), \\ u(y) &= C_{u0} \begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix}, \\ x_1(\pm 1) &= 0, \quad y \in [-1, 1]. \end{aligned}$$

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