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On the ill-posedness of certain vehicular platoon control problems

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***Abstract*— We revisit the vehicular platoon control problems formulated by Levine & Athans [1] and Melzer & Kuo [2]. We show that in each case, these formulations are effectively ill-posed. Specifically, we demonstrate that in the first formulation, the system's stabilizability degrades as the size of the platoon increases, and that the system loses stabilizability in the limit of an infinite number of vehicles. We show that in the LQR formulation of Melzer & Kuo [2], the performance index is not detectable, leading to non-stabilizing optimal feedbacks. Effectively, these closed-loop systems do not have a uniform bound on the time constants of all vehicles. For the case of infinite platoons, these difficulties are easily exhibited using the theory of spatially invariant systems. We argue that the infinite case is a useful paradigm to understand large platoons. To this end, we illustrate numerically how stabilizability and detectability degrade as functions of a finite platoon size, implying that the infinite case is a reasonable approximation to the large, but finite case. Finally, we suggest a well-posed alternative formulation of the LQR problem based on penalizing absolute position errors in addition to relative ones in the performance objective.**

***Index Terms*— Vehicular Platoons; Optimal Control; Spatially Invariant Systems.**

I. INTRODUCTION

In this paper, we consider optimal control of vehicular platoons. This problem was originally studied by Levine & Athans [1], and for an infinite string of moving vehicles by Melzer & Kuo [2], both using LQR methods. We analyze the solutions to the LQR problem provided by these authors as a function of the size of the formation, and show that these control problem become effectively ill-posed as the size of the platoon increases. We investigate various ways of quantifying this ill-posedness. In section II, we show that essentially, the resulting closed-loop systems do not have a uniform bound on the rate of convergence of the regulated states to zero. In other words, as the size of the platoon increases, the closed-loop system has eigenvalues that limit to the imaginary axis.

In section II, we setup the problem formulations of [1], [2] and investigate the above mentioned phenomena for finite platoons numerically. In section III, we also consider the infinite platoon case as an insightful limit which can be treated analytically. We argue that the infinite platoons capture the essence of the large-but-finite platoons. The infinite problem is also more amenable to analysis using the recently developed theory for spatially invariant linear systems [3], which we use to show that the original solutions to this problem are not exponentially stabilizing in the case of an infinite number of vehicles. The reason for this is the lack of stabilizability or detectability of an underlying system. Thus, these control problems are inherently ill-posed even if methods other than LQR are used. We end in section IV with alternative problem formulations which are well-posed. The main feature of these alternative formulations is the addition of penalties on absolute position errors in the performance objective.

II. OPTIMAL CONTROL OF FINITE PLATOONS

In this section, we consider the LQR problem for finite vehicular platoons. This problem was originally studied by Levine & Athans [1] and subsequently by Melzer & Kuo [2], [4]. The main point of our study is to analyze the control strategies of [1], [2], and [4] as the number of vehicles in platoon increases. We show that the solutions provided by these authors yield the non-uniform rates of convergence towards the desired formation. In other words, we demonstrate that the time constant of the closed-loop system gets larger as the platoon size increases.

A system of M identical unit mass vehicles is shown in Fig. 1. The dynamics of this system can be obtained by

Fig. 1. Finite platoon of vehicles.

representing each vehicle as a moving mass with the second order dynamics

$$\ddot{x}_n + \kappa \dot{x}_n = u_n, \quad n \in \{1, \ldots, M\}, \tag{1}$$

where x_n represents the position of the n-th vehicle, u_n is the control applied on the n-th vehicle, and $\kappa \geq 0$ denotes the linearized drag coefficient per unit mass.

A control objective is to provide the desired cruising velocity v_d and to keep the distance between neighboring vehicles at a constant pre-specified level L. By introducing the absolute position and velocity error variables

$$\begin{aligned} \xi_n(t) &:= x_n(t) - v_d t + nL, & n \in \{1, \ldots, M\}, \\ \zeta_n(t) &:= \dot{x}_n(t) - v_d, & n \in \{1, \ldots, M\}, \end{aligned}$$

system (1) can be rewritten using a state-space realization of the form [2], [4]

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ 0 & -\kappa I \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \tilde{u} \\ &=: A\psi + B\tilde{u}, \end{aligned} \tag{2}$$

where $\xi := \begin{bmatrix} \xi_1 & \cdots & \xi_M \end{bmatrix}^*$, $\zeta := \begin{bmatrix} \zeta_1 & \cdots & \zeta_M \end{bmatrix}^*$, $\tilde{u} := \begin{bmatrix} \tilde{u}_1 & \cdots & \tilde{u}_M \end{bmatrix}^*$, and $\tilde{u}_n := u_n - \kappa v_d$. Alternatively, by introducing the relative position error variable

$$\eta_n(t) := x_n(t) - x_{n-1}(t) + L = \xi_n(t) - \xi_{n-1}(t),$$

for every $n \in \{2, \ldots, M\}$, and neglecting the position dynamics of the first vehicle, the system under study can be represented by a realization with $2M - 1$ state-space variables [1]

$$\begin{bmatrix} \dot{\eta} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & \bar{A}_{12} \\ 0 & -\kappa I \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \tilde{u} \quad (3)$$
$$=: \bar{A}\phi + \bar{B}\tilde{u},$$

with $\eta := \begin{bmatrix} \eta_2 & \cdots & \eta_M \end{bmatrix}^*$, and $\bar{A}_{12}$ being an $(M-1) \times M$ Toeplitz matrix with the elements on the main diagonal and the first upper diagonal equal to -1 and 1, respectively.

Following [2], [4], fictitious lead and follow vehicles, respectively indexed by 0 and $M+1$, are added to the formation (see Fig. 2). These two vehicles are constrained to move at the desired velocity v_d and the relative distance between them is assumed to be equal to $(M+1)L$ for all times. In other words, it is assumed that

$$\{x_0(t) = v_d t, \ x_{M+1}(t) = v_d t - (M+1)L, \ \forall \, t \geq 0\},$$

or equivalently

$$\left. \begin{array}{lllll} \xi_0(t) & = & \xi_{M+1}(t) & = & 0 \\ \zeta_0(t) & = & \zeta_{M+1}(t) & = & 0 \end{array} \right\} \quad \forall \, t \geq 0. \quad (4)$$

A performance index of the form [2], [4]

$$J := \frac{1}{2}\int_0^\infty \Big(\sum_{n=1}^{M+1} q_1 \eta_n^2(t) + \sum_{n=1}^{M} (q_3 \zeta_n^2(t) + r\tilde{u}_n^2(t)) \Big) \, \mathrm{d}t \quad (5)$$

is associated with system (2). Using (4), J can be equivalently rewritten as

$$J = \frac{1}{2}\int_0^\infty (\psi^*(t) Q \psi(t) + \tilde{u}^*(t) R \tilde{u}(t)) \, \mathrm{d}t,$$

where matrices Q and R are determined by

$$Q := \begin{bmatrix} Q_1 & 0 \\ 0 & q_3 I \end{bmatrix}, \qquad R := rI,$$

with Q_1 being an $2M \times 2M$ tridiagonal symmetric Toeplitz matrix with the first row given by $\begin{bmatrix} 2q_1 & -q_1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{2M}$. The control problem is now in the standard LQR form.

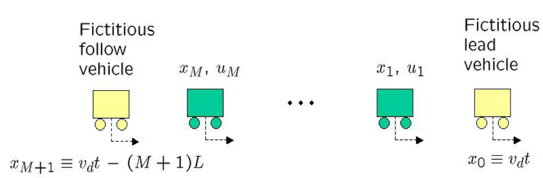

Fig. 2. Finite platoon with fictitious lead and follow vehicles.

The left and middle plots in Fig. 3 respectively show the dependence of the minimal and maximal eigenvalues of the solution to the LQR Algebraic Riccati Equation (ARE) P_M for system (2) with performance index (5), for $\{\kappa = 0, q_1 = q_3 = r = 1\}$. Clearly, $\lambda_{\min}\{P_M\}$ decays monotonically towards zero indicating that the pair (Q, A) gets closer to losing its detectability as the number of vehicles increases. On the other hand, $\lambda_{\max}\{P_M\}$ converges towards the constant value that determines the optimal value of objective (5) as M goes to infinity. The right plot in Fig. 3 illustrates the location of the dominant poles of system (2) connected in feedback with a controller that minimizes cost functional (5) for $\{\kappa = 0, q_1 = q_3 = r = 1\}$. The dotted line represents the function $-3.121/M$ which indicates that the least-stable eigenvalues of the closed-loop A matrix scale in an inversely proportional manner to the number of vehicles in platoon. Hence, the time constant of the closed-loop system gets larger as the size of platoon increases.

We remark that a formulation of the LQR problem for finite platoon (2) without the follow fictitious vehicle and appropriately modified cost functional of the form

$$J := \frac{1}{2}\int_0^\infty \Big(\sum_{n=1}^{M} q_1 \eta_n^2(t) + q_3 \zeta_n^2(t) + r\tilde{u}_n^2(t) \Big) \, \mathrm{d}t, \quad (6)$$

yields qualitatively similar results to the ones presented above. On the other hand, for system (2) without both lead and follow fictitious vehicles, the appropriately modified performance objective is given by (7). In this case, both the first and the last elements on the main diagonal of matrix Q_1 are equal to q_1. It can be shown that the pair (Q, A) for system (2,7) is practically not detectable [5], irrespective of the number of vehicles in formation.

Levine & Athans [1] studied the finite string of M vehicles shown in Fig. 1, with state-space representation (3) expressed in terms of the relative position and absolute velocity error variables. In particular, the LQR problem with a quadratic performance objective of the form

$$J := \frac{1}{2}\int_0^\infty \Big(\sum_{n=2}^{M} q_1 \eta_n^2(t) + \sum_{n=1}^{M} (q_3 \zeta_n^2(t) + r\tilde{u}_n^2(t)) \Big) \, \mathrm{d}t, \quad (7)$$

was formulated. Furthermore, the solution was provided for a platoon with $M = 3$ vehicles and $\{\kappa = r = 1, q_1 = 10, q_3 = 0\}$. We take a slightly different approach and analyze the solution to this problem as a function of the number of vehicles in platoon.

The left and middle plots in Fig. 4 respectively show minimal and maximal eigenvalues of the solution to the ARE in LQR problem (3,7) with $\kappa = q_1 = q_3 = r = 1$. The right plot in the same figure shows the real parts of the least-stable poles of system (3) with a controller that minimizes (7). Clearly, $\lambda_{\max}\{P_M\}$ scales linearly with the number of vehicles and, thus, the optimal value of performance objective (7) gets larger as the size of a vehicular string grows. This is because the pair $(\bar{A}, \bar{B})$ gets closer to losing its stabilizability when the platoon size increases. Furthermore, in the right plot, the dotted line represents the function $-2.222/M$, which implies the inversely proportional relationship between the dominant eigenvalues of the closed-loop A matrix and the number of vehicles in platoon. This implies again that as the number of vehicles increases, there is no uniform bound on the decay rates of regulated states to zero.

The results of this section clearly indicate that control strategies of [1], [2], [4] lead to closed-loop systems with arbitrarily slow decay rates as the number of vehicles increases. In § III, we show analytically that the absence of a uniform rate of convergence in finite platoons manifests itself as the absence of exponential stability in the limit of an infinite vehicular strings.

III. Optimal control of infinite platoons

In this section, we consider the LQR problem for infinite vehicular platoons. This problem was originally studied by Melzer & Kuo [2]. Using recently developed theory for spatially invariant linear systems [3], we show that the controller obtained by these authors does not provide exponential stability of the closed-loop system due to the lack of detectability of the pair (Q, A) in their LQR problem. We further demonstrate that the infinite platoon size limit of the problem formulation

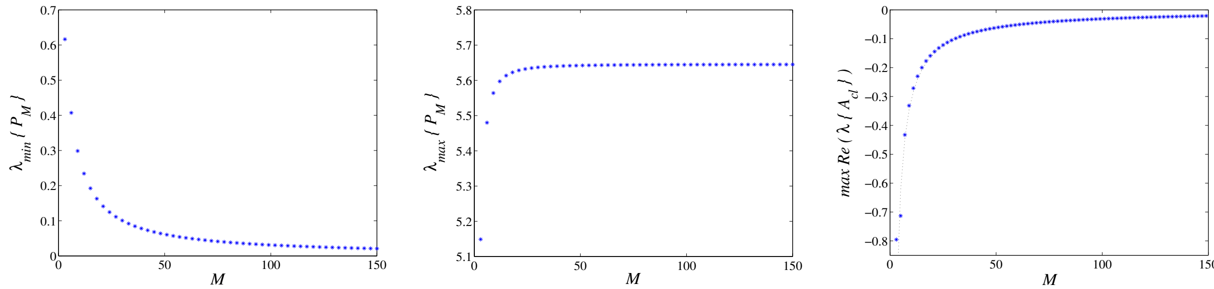

Fig. 3. The minimal (left plot) and maximal eigenvalues (middle plot) of the ARE solution P_M for system (2) with performance objective (5), and the dominant poles of LQR controlled platoon (2,5) (right plot) as functions of the number of vehicles for $\{\kappa = 0,\, q_1 = q_3 = r = 1\}$.

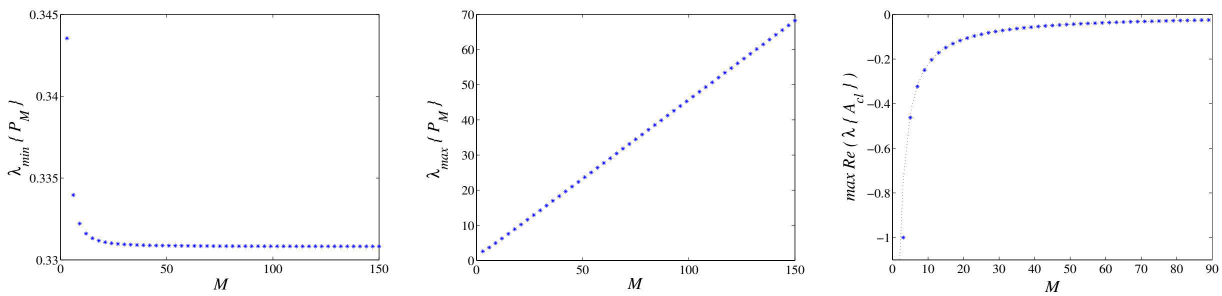

Fig. 4. The minimal (left plot) and maximal eigenvalues (middle plot) of the ARE solution P_M for system (3) with performance objective (7), and the dominant poles of LQR controlled platoon (3,7) (right plot) as functions of the number of vehicles for $\kappa = q_1 = q_3 = r = 1$.

of Levine & Athans [1] yields an infinite-dimensional system which is not stabilizable.

A system of identical unit mass vehicles in an infinite string is shown in Fig. 5. The infinite dimensional equivalents of (2)

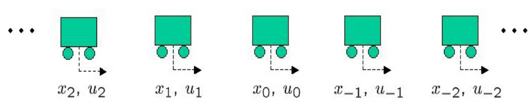

Fig. 5. Infinite platoon of vehicles.

and (3) are respectively given by

$$\begin{bmatrix} \dot{\xi}_n \\ \dot{\zeta}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} \xi_n \\ \zeta_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}_n \qquad (8)$$
$$=: A_n \psi_n + B_n \tilde{u}_n, \quad n \in \mathbb{Z},$$

and

$$\begin{bmatrix} \dot{\eta}_n \\ \dot{\zeta}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 - T_{-1} \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} \eta_n \\ \zeta_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}_n$$
$$=: \bar{A}_n \phi_n + \bar{B}_n \tilde{u}_n, \quad n \in \mathbb{Z}, \qquad (9)$$

where T_{-1} is the operator of translation by -1 (in the vehicle's index). As in [2], [6], we consider a quadratic cost functional of the form

$$J := \frac{1}{2}\int_0^\infty \sum_{n\in\mathbb{Z}} (q_1\eta_n^2(t) + q_3\zeta_n^2(t) + r\tilde{u}_n^2(t))\, \mathrm{d}t, \qquad (10)$$

with q_1, q_3, and r being positive design parameters.

We utilize the fact that systems (8) and (9) have spatially invariant dynamics over a discrete spatial lattice $\mathbb{Z}$ [3]. This implies that the appropriate Fourier transform (in this case the bilateral Z-transform evaluated on the unit circle) can be used to convert analysis and quadratic design problems into those for a parameterized family of finite-dimensional systems. This transform, which we refer to here as the Z_θ-transform, is defined by

$$\hat{x}_\theta := \sum_{n\in\mathbb{Z}} x_n \mathrm{e}^{-jn\theta}.$$

Using this, system (8) and cost functional (10) transform to

$$\begin{bmatrix} \dot{\hat{\xi}}_\theta \\ \dot{\hat{\zeta}}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} \hat{\xi}_\theta \\ \hat{\zeta}_\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\tilde{u}}_\theta \qquad (11)$$
$$=: \hat{A}_\theta \hat{\psi}_\theta + \hat{B}_\theta \hat{\tilde{u}}_\theta, \quad 0 \le \theta < 2\pi,$$

and

$$J = \frac{1}{4\pi}\int_0^\infty \int_0^{2\pi} \left(\hat{\psi}_\theta^*(t)\hat{Q}_\theta\hat{\psi}_\theta(t) + \hat{\tilde{u}}_\theta^*(t)\hat{R}_\theta\hat{\tilde{u}}_\theta(t)\right) \mathrm{d}\theta\, \mathrm{d}t,$$

where $\hat{R}_\theta := r$, and

$$\hat{Q}_\theta := \begin{bmatrix} 2q_1(1 - \cos\theta) & 0 \\ 0 & q_3 \end{bmatrix}, \quad 0 \le \theta < 2\pi.$$

Similarly, system (9) and cost functional (10) transform to

$$\begin{bmatrix} \dot{\hat{\eta}}_\theta \\ \dot{\hat{\zeta}}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 - \mathrm{e}^{-j\theta} \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} \hat{\eta}_\theta \\ \hat{\zeta}_\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\tilde{u}}_\theta$$
$$=: \hat{\bar{A}}_\theta \hat{\phi}_\theta + \hat{\bar{B}}_\theta \hat{\tilde{u}}_\theta, \quad 0 \le \theta < 2\pi, \qquad (12)$$

with

$$J = \frac{1}{4\pi}\int_0^\infty \int_0^{2\pi} \left(\hat{\phi}_\theta^*(t)\hat{\bar{Q}}_\theta\hat{\phi}_\theta(t) + \hat{\tilde{u}}_\theta^*(t)\hat{R}_\theta\hat{\tilde{u}}_\theta(t)\right) \mathrm{d}\theta\, \mathrm{d}t,$$

where

$$\hat{Q}_\theta \ := \ \begin{bmatrix} q_1 & 0 \\ 0 & q_3 \end{bmatrix}, \quad \hat{R}_\theta \ := \ r, \quad 0 \ \leq \ \theta \ < \ 2\pi.$$

If $(\hat{A}_\theta, \hat{B}_\theta)$ is stabilizable and $(\hat{Q}_\theta, \hat{A}_\theta)$ is detectable for all $\theta \in [0,\, 2\pi)$, then the θ-parameterized ARE

$$\hat{A}_\theta^* \hat{P}_\theta \ + \ \hat{P}_\theta \hat{A}_\theta \ + \ \hat{Q}_\theta \ - \ \hat{P}_\theta \hat{B}_\theta \hat{R}_\theta^{-1} \hat{B}_\theta^* \hat{P}_\theta \ = \ 0,$$

has a unique positive definite solution for every $\theta \in [0,\, 2\pi)$. This positive definite matrix determines the optimal stabilizing feedback for system (11) for every $\theta \in [0,\, 2\pi)$

$$\hat{\tilde{u}}_\theta \ := \ \hat{K}_\theta \hat{\psi}_\theta \ = \ - \ \hat{R}_\theta^{-1} \hat{B}_\theta^* \hat{P}_\theta \hat{\psi}_\theta, \quad 0 \ \leq \ \theta \ < \ 2\pi.$$

If this is the case, then there exist an exponentially stabilizing feedback for system (8) that minimizes (10) [3]. This optimal stabilizing feedback for (8) is given by

$$\tilde{u}_n \ = \ \sum_{k \in \mathbb{Z}} K_{n-k} \psi_k, \quad n \in \mathbb{Z},$$

where

$$K_n \ = \ \frac{1}{2\pi} \int_0^{2\pi} \hat{K}_\theta \mathrm{e}^{jn\theta} \, \mathrm{d}\theta, \quad n \in \mathbb{Z}.$$

It is easily shown that the pair $(\hat{A}_\theta, \hat{B}_\theta)$ is controllable for every $\theta \in [0,\, 2\pi)$. On the other hand, the pair $(\hat{Q}_\theta, \hat{A}_\theta)$ is not detectable at $\theta = 0$. In particular, the solution to the ARE at $\theta = 0$ is given by

$$\hat{P}_0 \ := \ \begin{bmatrix} 0 & 0 \\ 0 & r(-\kappa \ + \ \gamma) \end{bmatrix},$$

which yields a closed-loop A matrix at $\theta = 0$ of the form

$$\hat{A}_{\mathrm{cl}0} \ = \ \begin{bmatrix} 0 & 1 \\ 0 & -\gamma \end{bmatrix}, \tag{13}$$

with $\gamma := \frac{1}{r}\sqrt{(\kappa r)^2 \ + \ r q_3}$. Therefore, matrix $\hat{A}_{\mathrm{cl}0}$ is not Hurwitz, which implies that the solution to the LQR problem does not provide an exponentially stabilizing feedback for the original system [3]. We remark that this fact has not been realized in [2] and [6]. The spectrum of the closed-loop generator for $\{\kappa = 0,\, q_1 = q_3 = r = 1\}$ is shown in Fig. 6 to illustrate the absence of exponential stability.

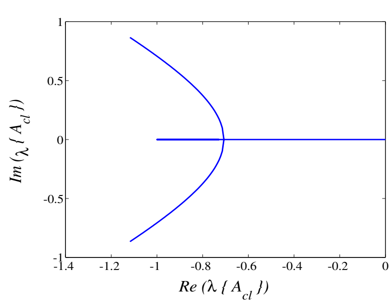

Fig. 6. The spectrum of the closed-loop generator in an LQR controlled spatially invariant string of vehicles (8) with performance objective (10) and $\{\kappa = 0,\, q_1 = q_3 = r = 1\}$.

It is instructive to consider the initial states that are not stabilized by this LQR feedback. Based on (13), it follows that the solution of system (11) at $\theta = 0$ with the controller of [2] is determined by

$$\begin{aligned} \hat{\zeta}_0(t) \ &= \ \mathrm{e}^{-\gamma t} \hat{\zeta}_0(0), \\ \hat{\xi}_0(t) \ &= \ \hat{\xi}_0(0) \ - \ \frac{1}{\gamma}(1 \ - \ \mathrm{e}^{-\gamma t}) \hat{\zeta}_0(0). \end{aligned}$$

We have assumed that $\gamma \neq 0$, which can be accomplished for any $\kappa \geq 0$ by choosing $q_3 > 0$. Thus,

$$\sum_{n \in \mathbb{Z}} \xi_n(t) \ = \ \sum_{n \in \mathbb{Z}} \xi_n(0) \ - \ \frac{1}{\gamma}(1 \ - \ \mathrm{e}^{-\gamma t}) \sum_{n \in \mathbb{Z}} \zeta_n(0),$$

which implies that $\lim_{t \to \infty} \sum_{n \in \mathbb{Z}} \xi_n(t) \neq 0$ unless

$$\sum_{n \in \mathbb{Z}} \xi_n(0) \ - \ \frac{1}{\gamma} \sum_{n \in \mathbb{Z}} \zeta_n(0) \ \equiv \ 0. \tag{14}$$

Therefore, if the initial condition of system (8) does not satisfy (14) than $\sum_{n \in \mathbb{Z}} \xi_n(t)$ cannot be asymptotically driven towards zero. It is not difficult to construct a physically relevant initial condition that violates (14). For example, this situation will be encountered if the string of vehicles at $t = 0$ cruises at the desired velocity v_d with all the vehicles being at their desired spatial locations except for a single vehicle. In other words, even for a seemingly benign initial condition of the form $\{\zeta_n(0) \equiv 0;\ \xi_n(0) = 0,\ \forall\, n \in \mathbb{Z} \setminus 0;\ \xi_0(0) = S \neq 0\}$ there exist at least one vehicle whose absolute position error does not converge to zero as time goes to infinity when the control strategy of Melzer & Kuo [2] is employed. This non-zero mean position initial condition is graphically illustrated in Fig. 7.

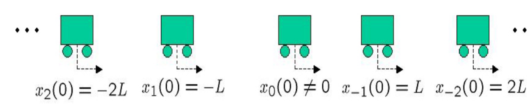

Fig. 7. An example of a position initial condition for which there is at least one vehicle whose absolute position error does not asymptotically converge to zero when the control strategy of [2] is used.

We note that system (12) is not stabilizable at $\theta = 0$, which prevents system (9) from being stabilizable [3]. Hence, when infinite vehicular platoons are considered the formulation of design problem of Levine & Athans [1] is ill-posed (that is, unstabilizable). In particular, the solution of the '$\hat{\eta}$-subsystem' of (12) at $\theta = 0$ does not change with time, that is $\hat{\eta}_0(t) \equiv \hat{\eta}_0(0)$, which indicates that $\sum_{n \in \mathbb{Z}} \eta_n(t) \ \equiv \ \sum_{n \in \mathbb{Z}} \eta_n(0)$. Therefore, for a non-zero mean initial condition $\{\eta_n(0)\}_{n \in \mathbb{Z}}$, the sum of all relative position errors is identically equal to a non-zero constant determined by $\sum_{n \in \mathbb{Z}} \eta_n(0)$. An example of such initial condition is given by $\{\eta_n(0) \ = \ 0,\ \forall\, n \in \mathbb{Z} \setminus 0;\ \eta_0(0) = S \neq 0\}$, and it is illustrated in Fig. 8. It is quite remarkable that the control strategy of Levine & Athans [1] is not able to asymptotically steer all relative position errors towards zero in an infinite platoon with this, at first glance, innocuously looking initial condition.

Therefore, we have shown that exponential stability of an LQR controlled infinite platoon cannot be achieved due to the lack of detectability (in the case of [2], [4]) and stabilizability (in the case of [1]). These facts have very important practical implications for optimal control of large vehicular platoons. Namely, our analysis clarifies results of § II, where we have observed that decay rates of a finite platoon with controllers of [1], [2], [4] become smaller as the platoon size increases.

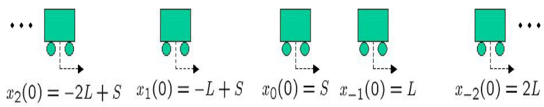

Fig. 8. An example of a position initial condition for which there is at least a pair of vehicles whose relative distance does not asymptotically converge to the desired inter-vehicular spacing L when a control strategy of [1] is used.

In § IV, we demonstrate that exponential stability of an infinite string of vehicles can be guaranteed by accounting for position errors with respect to absolute desired trajectories in both the state-space representation and the performance criterion.

IV. Alternative problem formulations

In this section, we propose an alternative formulation of the LQR problem for vehicular platoons. We show that the problems discussed in § II and § III can be overcome by accounting for the absolute position errors in both the state-space realization and the performance criterion. Since consideration of infinite platoons is better suited for analysis, we first study the LQR problem for a spatially invariant string of vehicles, and then discuss practical implications for optimal control of finite vehicular platoons. We also briefly remark on the choice of the appropriate state-space.

We represent system shown in Fig. 5 by its state-space representation (8) expressed in terms of absolute position and velocity error variables, and propose the quadratic performance objective of the form

$$J := \frac{1}{2}\int_0^\infty \sum_{n\in\mathbb{Z}} \left(q_1\eta_n^2(t) + q_2\xi_n^2(t) + q_3\zeta_n^2(t) + r\tilde{u}_n^2(t)\right)\mathrm{d}t \tag{15}$$

with q_1, q_2, q_3, and r being positive design parameters. It should be noted that in (15) we account for both absolute position errors ξ_n and relative position errors η_n. This is in contrast to performance index (10) considered by Melzer & Kuo [2] and Chu [6], where only relative position errors are penalized in J. The main point of this section is to show that if one accounts for absolute position errors (in addition to the relative ones) in the cost functional, then LQR feedback will be exponentially stabilizing.

Application of Z_θ-transform renders (8) into (11), whereas (15) simplifies to

$$J = \frac{1}{4\pi}\int_0^\infty\int_0^{2\pi}\left(\hat{\psi}_\theta^*(t)\hat{Q}_\theta^\star\hat{\psi}_\theta(t) + \hat{\tilde{u}}_\theta^*(t)\hat{R}_\theta\hat{\tilde{u}}_\theta(t)\right)\mathrm{d}\theta\,\mathrm{d}t,$$

where $\hat{R}_\theta := r$, and

$$\hat{Q}_\theta^\star := \begin{bmatrix} q_2 + 2q_1(1-\cos\theta) & 0 \\ 0 & q_3 \end{bmatrix},\ 0 \le \theta < 2\pi.$$

As shown in § III, the pair $(\hat{A}_\theta, \hat{B}_\theta)$ is controllable for every $\theta \in [0, 2\pi)$. Furthermore, it is easily established that the pair $(\hat{Q}_\theta^\star, \hat{A}_\theta)$ is detectable if and only if

$$q_2 + 2q_1(1-\cos\theta) \neq 0, \quad \forall\,\theta\in[0, 2\pi).$$

Even if q_1 is set to zero, this condition is satisfied as long as $q_2 > 0$. However, in this situation the inter-vehicular spacing is not penalized in the cost functional which may result into an unsafe control strategy. Because of that, as in [2], [6], we assign a positive value to q_1. In this case, if $q_2 = 0$, the pair $(\hat{Q}_\theta^\star, \hat{A}_\theta)$ is not detectable at $\theta = 0$, which implies that accounting for the absolute position errors in the performance criterion is essential for obtaining a stabilizing solution to the LQR problem. The spectrum of the closed-loop generator shown in Fig. 9 illustrates exponential stability of infinite string of vehicles (8) combined in feedback with a controller that minimizes performance objective (15) for $\{\kappa = 0,\, q_1 = q_2 = q_3 = r = 1\}$.

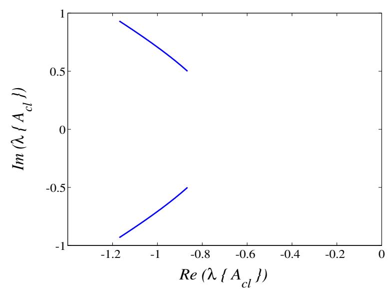

Fig. 9. The spectrum of the closed-loop generator in an LQR controlled spatially invariant string of vehicles (8) with performance objective (15) and $\{\kappa = 0,\, q_1 = q_2 = q_3 = r = 1\}$.

For a finite platoon with M vehicles the appropriately modified version of (15) is obtained by adding an additional term that accounts for absolute position errors to the right-hand side of (5)

$$\begin{aligned} J \;:=\; & \frac{1}{2}\int_0^\infty \Big(\sum_{n=1}^{M+1} q_1\eta_n^2(t) \;+ \\ & \sum_{n=1}^{M}\left(q_2\xi_n^2(t) + q_3\zeta_n^2(t) + r\tilde{u}_n^2(t)\right)\Big)\,\mathrm{d}t. \end{aligned} \tag{16}$$

Equivalently, (16) can be rewritten as

$$J = \frac{1}{2}\int_0^\infty \left(\psi^*(t)Q^\star\psi(t) + \tilde{u}^*(t)R\tilde{u}(t)\right)\mathrm{d}t,$$

where $Q^\star$ and R are $2M\times 2M$ and $M\times M$ matrices given by

$$Q^\star := \begin{bmatrix} Q_1 + q_2 I & 0 \\ 0 & q_3 I \end{bmatrix}, \qquad R := rI,$$

respectively. Toeplitz matrix Q_1 has the same meaning as in § II.

The left and middle plots in Fig. 10 respectively show the minimal and maximal eigenvalues of the ARE solution for system (2) with performance objective (16), and the right plot in the same figure shows the real parts of the least-stable poles in an LQR controlled string of vehicles (2,16) for $\{\kappa = 0,\, q_1 = q_2 = q_3 = r = 1\}$. Clearly, when the absolute position errors are accounted for in both the state-space realization and J, the problems addressed in § II are easily overcome. In particular, the least-stable closed-loop eigenvalues converge towards a non-zero value determined by the dominant pole of the spatially invariant system.

We note that qualitatively similar results are obtained if LQR problem is formulated for system (2) with either functional (6) or functional (7) augmented by a term penalizing absolute position errors.

In § IV-A, we briefly comment on the initial conditions that cannot be dealt with the quadratically optimal controllers.

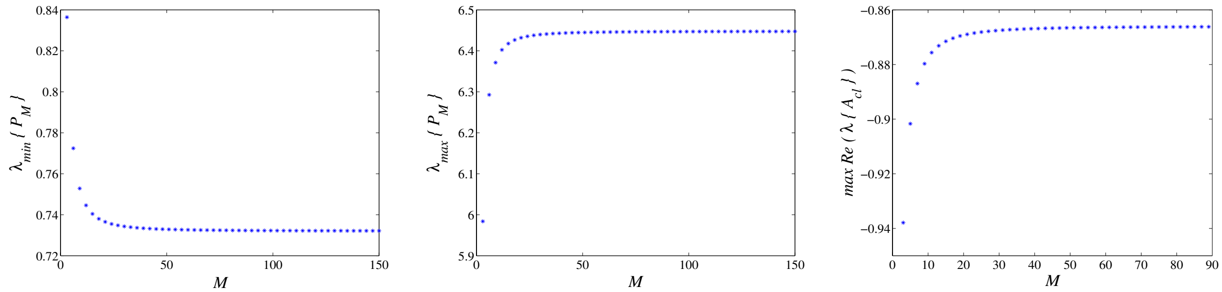

Fig. 10. The minimal (left plot) and maximal eigenvalues (middle plot) of the ARE solution P_M for system (2) with performance objective (16), and the dominant poles of LQR controlled platoon (2,16) (right plot) as functions of the number of vehicles, for $\{\kappa = 0, q_1 = q_2 = q_3 = r = 1\}$.

A. *On the choice of the appropriate state-space*

In this subsection, we remark on the initial conditions that are not square summable (in a space of the absolute errors) and as such are problematic for optimal controllers involving quadratic criteria.

Motivated by example shown in Fig. 8 we note that the initial relative position errors $\{\eta_n(0)\}_{n\in\mathbb{Z}}$ cannot be non-zero mean unless the absolute position errors at $t = 0$ sum to infinity. Namely, if $\sum_{n\in\mathbb{Z}} \xi_n(0)$ is bounded, then

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \eta_n(0) \;&:=\; \sum_{n\in\mathbb{Z}} (\xi_n(0) \;-\; \xi_{n-1}(0)) \\ &=\; \sum_{n\in\mathbb{Z}} \xi_n(0) \;-\; \sum_{n\in\mathbb{Z}} \xi_{n-1}(0) \;=\; 0. \end{aligned}$$

Clearly, for the initial condition shown in Fig. 8, that is,

$$\begin{aligned} x_n(0) \;&=\; -\,nL \;+\; S, \quad \forall\, n \in \mathbb{N}_0 \\ x_n(0) \;&=\; -\,nL, \qquad\quad \forall\, n \in \mathbb{Z} \setminus \mathbb{N}_0, \end{aligned}$$

sequence $\{\xi_n(0)\}_{n\in\mathbb{Z}}$ sums to infinity if $S \neq 0$. In addition to that, $\{\xi_n(0)\}_{n\in\mathbb{Z}} \notin l_2$, whereas $\{\eta_n(0)\}_{n\in\mathbb{Z}} \in l_2$. Therefore, despite the fact that the inter-vehicular spacing for all but a single vehicle is kept at the desired level L, a relevant non-square summable initial condition[1] is easily constructed. It is worth noting that the controller of § IV is derived under the assumption of the square summable initial conditions and as such cannot be used for guarding against an entire class of physically relevant initial states. This illustrates that a Hilbert space l_2 may represent a rather restrictive choice for the underlying state-space of system (8). Perhaps the more appropriate state-space for this system is a Banach space l_∞. The control design on this state-space is outside the scope of this work. We refer the reader to [5], [7] for additional details.

V. Concluding remarks

We have illustrated potential difficulties in the control of large or infinite vehicular platoons. In particular, shortcomings of previously reported solutions to the LQR problem have been exhibited. By considering the case of infinite platoons as the limit of the large-but-finite case, we have shown analytically how the aforementioned formulations lack stabilizability or detectability. We argued that the infinite case is a useful abstraction of the large-but-finite case, in that it explains the almost loss of stabilizability or detectability in the large-but-finite case, and the arbitrarily slowing rate of convergence towards desired formation observed in numerical studies of finite platoons of increasing sizes. Finally, using the infinite platoon formulation, we showed how incorporating absolute position errors in the cost functional alleviates these difficulties and provides uniformly bounded rates of convergence. We refer the reader to [5] where well-posed formulations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems for infinite vehicular strings are proposed.

The literature on control of platoons is quite extensive, and we have not attempted a thorough review of all the proposed control schemes here. However, it is noteworthy that the early work of [1], [2], [4], [6] is very widely cited, but to our knowledge these serious difficulties with their methods have not been previously pointed out in the literature.

As a further note, we point out that in a recent article [7] it was shown that imposing a uniform rate of convergence for all vehicles towards their desired trajectories may generate large control magnitudes for certain physically realistic initial conditions. Therefore, even though the formulation of an optimal control problem suggested in § IV circumvents the weak points of [1], [2], [4], [6], additional care should be exercised in control of vehicular platoons.

Acknowledgment

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[1] In a space of the absolute position errors.