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$\mathcal{H}_2$ norm of linear time-periodic systems: a perturbation analysis

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***Abstract*— We consider a class of linear time-periodic systems in which dynamical generator $A(t)$ represents a sum of a stable time-invariant operator A_0 and a small amplitude zero-mean T-periodic operator $\epsilon A_p(t)$. We employ a perturbation analysis to develop a computationally efficient method for determination of the $\mathcal{H}_2$ norm. Up to a second order in perturbation parameter ϵ we show that: a) the $\mathcal{H}_2$ norm can be obtained from a conveniently coupled system of readily solvable Lyapunov and Sylvester equations; b) there is no coupling between different harmonics of $A_p(t)$ in the expression for the $\mathcal{H}_2$ norm. These two properties do not hold for arbitrary values of ϵ, and their derivation would not be possible if we tried to determine the $\mathcal{H}_2$ norm directly without resorting to perturbation analysis. Our method is well suited for identification of the values of period T that lead to the largest increase/reduction of the $\mathcal{H}_2$ norm. Two examples are provided to motivate the developments and illustrate the procedure.**

I. Introduction

Time-periodic systems arise in many important physical and engineering problems [1]. The examples of finite dimensional systems are given by the Hill and Mathieu equations, and the examples of infinite dimensional systems are given by the equations describing the periodic excitations of fluids, beams, plates, strings, and membranes. The *Floquet analysis* provides a theoretical framework for investigation of local stability properties of these systems [2]. On the other hand, the so-called *lifting technique* [3] and the *harmonic balance approach* [4] are most suitable for analysis of input-output properties of the linearized versions of these systems.

The utility of input-output analysis for linear time-invariant (LTI) systems is well documented [5]; we refer the reader to [6]–[8] for an example of how this analysis can be used to understand one of the oldest problems in fluid mechanics$-$*transition to turbulence in wall-bounded shear flows*. The $\mathcal{H}_2$ norm is an appealing measure of input-output amplification, as it quantifies the variance amplification of stochastically driven linear systems. For LTI systems, the $\mathcal{H}_2$ norm is determined by traces of controllability or observability Gramians which represent solutions to standard Lyapunov equations. On the other hand, the $\mathcal{H}_2$ norm of linear time-periodic (LTP) systems can be expressed in terms of a solution to the so-called *harmonic Lyapunov equation* [9]. Since the entries into this equation are bi-infinite matrices with, in general, operator valued elements, the computation of the $\mathcal{H}_2$ norm of LTP systems is a non-trivial exercise. Furthermore, the state-transition matrix of most LTP systems is difficult to obtain (either numerically or analytically) which additionally hinders analysis. The recent article [9] addressed these problems by: a) approximation of $A(t)$ in the state-equation by piecewise constant functions; b) truncation of bi-infinite matrices in harmonic Lyapunov equation. However, for systems described by partial integro-differential equations (PIDEs) even this approach would require solving a large-scale Lyapunov equation; for an accurate computation of the $\mathcal{H}_2$ norm of PIDEs with one spatial variable, the entries into this equation are typically matrices with at least several hundreds rows and columns.

In this paper, we study LTP systems in which $A(t)$ is given by a sum of a stable time-invariant operator A_0 and a small amplitude zero-mean T-periodic operator $\epsilon A_p(t)$. For example, these systems can be obtained by linearization of time-invariant nonlinear systems around small amplitude T-periodic trajectories. We employ a perturbation analysis to develop a computationally efficient method for determining the $\mathcal{H}_2$ norm. Up to a second order in perturbation parameter ϵ we show that: a) the $\mathcal{H}_2$ norm can be obtained from a conveniently coupled system of readily solvable Lyapunov and Sylvester equations; b) there is no coupling between different harmonics of $A_p(t)$ in the expression for the $\mathcal{H}_2$ norm. These two properties do not hold for arbitrary values of ϵ, and their derivation would not be possible if we tried to determine the $\mathcal{H}_2$ norm directly without resorting to perturbation analysis. Our method is well suited for identification of the values of period T that lead to the largest increase/reduction of the $\mathcal{H}_2$ norm. An immediate application domain is in fluid mechanics where temporally periodic excitations can be introduced either to suppress turbulence [10] or to enhance mixing.

We note that perturbation analysis used here has strong parallels with the approach of [11] for the $\mathcal{H}_2$ analysis of linear *spatially-periodic* systems. However, there are some important differences in the structure of frequency response operators for temporally and spatially periodic systems which necessitates separate treatments. For example, in spatially-periodic systems one often encounters cascades of spatially invariant differential and spatially periodic multiplication operators which somewhat complicates their analysis [12]. On the other hand, the state-space models of LTP systems do not contain cascades of differential and periodic operators, which imposes some additional structure that can be utilized in analysis (see, for example, Lemma 1).

Our presentation is organized as follows: in section II we formulate the problem and provide two examples that serve as a motivation for our analysis. In § III, we give a brief overview of a notion of the *frequency response* for exponentially stable LTP systems. In § IV, we define the $\mathcal{H}_2$ norm for LTP systems and employ a perturbation analysis to develop an efficient procedure for computing the $\mathcal{H}_2$ norm of these systems subject to small amplitude oscillations. In § V, we use the developed method to determine the second order corrections to the $\mathcal{H}_2$ norms of systems described in § II-A.1 and § II-A.2. In § VI, we end our presentation with some concluding remarks.

II. Problem formulation

Let a linear dynamical system be given by its state-space representation

$$\partial_t \psi \;=\; A(t)\psi \;+\; Bd, \tag{1a}$$
$$\phi \;=\; C\psi, \tag{1b}$$

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where ψ, ϕ, and d, respectively, denote the state, output, and input vector valued fields. We assume that $A(t)$ represents a time-periodic operator with a period $T = 2\pi/\omega_o$, $A(t) = A(t + T)$, that generates an exponentially stable strongly-continuous (C_o) semigroup on a Hilbert space $\mathbb{H}$. Input and output operators B and C are assumed to be time-invariant.

In this paper, we consider a class of LTP systems in which operator $A(t)$ can be represented as

$$A(t) \; = \; A_0 \; + \; \epsilon A_p(t),$$

where ϵ is a small parameter, A_0 is an exponentially stable time-invariant operator, and $A_p(t)$ is a zero-mean T-periodic operator. In other words, we assume that $A_p(t)$ can be expanded to its Fourier series, $A_p(t) = \sum_{n \in \mathbb{Z} \backslash 0} A_n \mathrm{e}^{\mathrm{j} n \omega_o t}$. Our objective is to derive a computationally efficient procedure for determination of the $\mathcal{H}_2$ norm of system (1) using a perturbation analysis.

We are particularly interested in distributed systems with one spatial variable $y \in [-1, 1]$. To highlight this, we rewrite (1) as

$$\begin{aligned} \partial_t \psi(y,t) \; &= \; A(t)\psi(y,t) \; + \; Bd(y,t), && \text{(2a)} \\ \phi(y,t) \; &= \; C\psi(y,t), && \text{(2b)} \end{aligned}$$

where for each t, $\psi(\cdot,t)$, $\phi(\cdot,t)$, and $d(\cdot,t)$ denote vector valued fields in $L^2[-1, 1]$. On the other hand, $A(t)$, B, and C are linear (integro-differential) operators in y, with $A(t) = A(t + T)$. The example presented in § II-A.2 illustrates structure of these operators for a system describing the evolution of velocity perturbations in a two-dimensional oscillating channel flow. We note that with a careful choice of notation all of our results hold for both finite dimensional LTP systems and LTP systems described by (2).

Let H denote the mapping from input d to output ϕ, $\phi = Hd$. We assume that H has a kernel representation given by

$$\phi(t) \; = \; \int_0^t H(t,\tau) d(\tau) \, \mathrm{d}\tau,$$

for finite dimensional systems of the form (1), and

$$\phi(y,t) \; = \; \int_0^t \int_{-1}^1 H(y,\eta;t,\tau) d(\eta,\tau) \, \mathrm{d}\eta \, \mathrm{d}\tau,$$

for infinite dimensional systems of the form (2). Here, with an abuse of notation, we use the same symbol for an operator and its kernel function. It is not difficult to show that the kernel function representing H is a doubly-periodic function in t and τ, i.e. $H(y,\eta;t,\tau) = H(y,\eta;t + nT, \tau + nT)$, $n \in \mathbb{N}_0$ [3]. We will consider a class of LTP systems whose kernel functions are bounded on bounded subsets of $\mathbb{R}^2$ (for finite dimensional systems) and $\mathbb{R}^2 \times [-1, 1] \times [-1, 1]$ (for system (2)), respectively. It is a standard fact that under these conditions the $\mathcal{H}_2$ norm of an L^2-stable system is well defined.

A. Motivating examples

We next provide two examples that serve as a motivation for our analysis. The first example represents a dissipative version of the well-known Mathieu equation, and the second example describes the dynamics of flow fluctuations in a two-dimensional channel flow subject to a streamwise pressure gradient and an oscillatory motion of the lower wall.

1) The dissipative Mathieu equation: The forced dissipative Mathieu equation is given by

$$\ddot{x} \, + \, 2b\dot{x} \, + \, (a \, - \, 2\epsilon \cos \omega_o t)x \; = \; d,$$

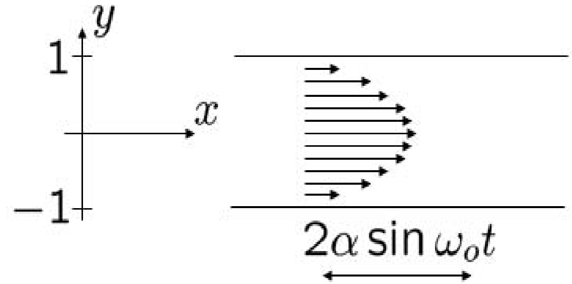

Fig. 1. A two-dimensional channel flow subject to a streamwise pressure gradient and an oscillatory motion of the lower wall.

where a and b denote positive parameters. By selecting

$$\psi(t) \; := \; [\; x(t) \quad \dot{x}(t) \;]^T, \quad \phi(t) \; := \; x(t),$$

this equation can be represented by (1) with

$$\begin{aligned} A(t) \quad &:= \quad \begin{bmatrix} 0 & 1 \\ -(a \, - \, 2\epsilon \cos \omega_o t) & -2b \end{bmatrix}, \\ B \quad &:= \quad [\; 0 \quad 1 \;]^T, \;\; C \; := \; [\; 1 \quad 0 \;]. \end{aligned}$$

Clearly, in this example $\mathbb{H} := \mathbb{R}^2$, and

$$\begin{aligned} A(t) \quad &:= \quad A_0 \, + \, \epsilon A_p(t) \\ &= \quad A_0 \, + \, \epsilon \left(A_{-1} \mathrm{e}^{-\mathrm{j}\omega_o t} \, + \, A_1 \mathrm{e}^{\mathrm{j}\omega_o t} \right), \end{aligned}$$

where

$$A_0 \; := \; \begin{bmatrix} 0 & 1 \\ -a & -2b \end{bmatrix}, \quad A_{\pm 1} \; := \; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

In § V-A, we consider the small amplitude oscillations and use perturbation analysis of § IV to determine how the $\mathcal{H}_2$ norm changes with forcing frequency ω_o.

2) An example from fluid mechanics: Consider a two-dimensional channel flow with geometry illustrated in Fig. 1. Let the flow be subject to a streamwise pressure gradient, $P_x = -2/R$, and an oscillatory motion of the lower wall, $U(y = -1, t) = 2\alpha \sin \omega_o t$. Here, t denotes the non-dimensional time, α and ω_o are, respectively, the non-dimensional amplitude and frequency of the wall oscillations, and R is the Reynolds number defined in terms of the centerline velocity and the channel half-width.

With the appropriate scaling of the Navier-Stokes (NS) equations, α and ω_o can be expressed as $\{\alpha = R_u/R, \, \omega_o = \Omega/R\}$, where R_u is the Reynolds number defined in terms of the wall oscillation amplitude (in physical units) and the channel half-width, and Ω is the Stokes number. Under these conditions, it is readily shown that the steady-state solution of the NS equations is given by

$$\begin{aligned} &U(y,t) \; = \; U_0(y) \, + \, 2(R_u/R)U_1(y,t), \, U_0(y) \; = \; 1 \, - \, y^2, \\ &U_1(y,t) \; = \; U_c(y) \cos{(\Omega/R)t} \; + \; U_s(y) \sin{(\Omega/R)t}, \end{aligned}$$

where $U_c(y)$ and $U_s(y)$ represent solutions to

$$\begin{aligned} U_s''(y) \; &= \; -\Omega U_c(y), \qquad U_c''(y) \; = \; \Omega U_s(y), \\ U_c(\pm 1) \; &= \; U_s(1) \; = \; 0, \quad U_s(-1) \; = \; 1. \end{aligned}$$

Here, $U_r''(y)$ denotes a second derivative of $U_r(y)$, that is $U_r''(y) := \mathrm{d}^2 U_r(y)/\mathrm{d}y^2$, $r = s$ or $r = c$.

The linearization of the NS equations around $U(y,t)$ in combination with the Fourier transform in x yields a one-dimensional PIDE (in y) parameterized by a wave-number

$k_x \in \mathbb{R}$. This PIDE has a state-space representation (2), with the state of the system determined by a scalar field $\psi(k_x, y, t)$ denoting the stream function. On the other hand, input and output fields d and ϕ are, respectively, defined as

$$\begin{aligned} d(k_x, y, t) &:= \begin{bmatrix} d_1(k_x, y, t) & d_2(k_x, y, t) \end{bmatrix}^T, \\ \phi(k_x, y, t) &:= \begin{bmatrix} u(k_x, y, t) & v(k_x, y, t) \end{bmatrix}^T, \end{aligned}$$

where d_1 and d_2 (u and v) denote body force (velocity) fluctuations in x and y. With this choice of the state, input, and output fields, operators A, B, and C are given by

$$\begin{aligned} A(t) &:= \Delta^{-1}\left(\tfrac{1}{R}\Delta^2 + \mathrm{j}k_x\big(U''(y,t) - U(y,t)\Delta\big)\right), \\ B &:= \Delta^{-1}\begin{bmatrix} \partial_y & -\mathrm{j}k_x \end{bmatrix}, \;\; C := \begin{bmatrix} \partial_y & -\mathrm{j}k_x \end{bmatrix}^T, \end{aligned}$$

where $\Delta := \partial_{yy} - k_x^2$, with homogenous Dirichlet boundary conditions, and $\Delta^2 := \partial_{yyyy} - 2k_x^2\partial_{yy} + k_x^4$, with homogenous Dirichlet *and* Neumann boundary conditions.

The underlying Hilbert space for A is given by [13]

$$\mathbb{H} := \left\{ g \in L^2[-1,1];\; g'' \in L^2[-1,1],\; g(\pm 1) = 0 \right\}.$$

This operator is unbounded, and it is defined on a domain

$$D(A) := \left\{ g \in \mathbb{H};\; g^{(4)} \in L^2[-1,1],\; g'(\pm 1) = 0 \right\}.$$

We endow $\mathbb{H}$ with the inner product

$$\langle \psi_1, \psi_2 \rangle_e := -\langle \psi_1, \Delta\psi_2 \rangle = -\int_{-1}^{1} \psi_1^* \Delta \psi_2 \,\mathrm{d}y,$$

and note that, for any k_x and t, $\langle \psi, \psi \rangle_e$ determines the kinetic energy of velocity perturbations. Here, $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard $L^2[-1,\,1]$ inner product. Based on this, the adjoints of operators A, B, and C can be determined from [8]

$$\begin{aligned} \langle \psi_1, A\psi_2 \rangle_e &= \langle A^*\psi_1, \psi_2 \rangle_e, \\ \langle \psi, Bd \rangle_e &= \langle B^*\psi, d \rangle, \\ \langle \phi, C\psi \rangle &= \langle C^*\phi, \psi \rangle_e, \end{aligned}$$

which yields $BB^* = C^*C = I$, and

$$A^*(t) = (1/R)\Delta^{-1}\Delta^2 + \mathrm{j}k_x(U(y,t) - \Delta^{-1}U''(y,t)).$$

Finally, we represent $A(t)$ in a form suitable for the $\mathcal{H}_2$ norm analysis

$$A(t) = A_0 + (R_u/R)(A_{-1}\mathrm{e}^{-\mathrm{j}(\Omega/R)t} + A_1\mathrm{e}^{\mathrm{j}(\Omega/R)t}),$$

where

$$\begin{aligned} A_0 &:= \Delta^{-1}\left(\tfrac{1}{R}\Delta^2 + \mathrm{j}k_x\big(U_0''(y) - U_0(y)\Delta\big)\right), \\ A_{\pm 1} &:= A_c \mp \mathrm{j}A_s, \\ A_r &:= \mathrm{j}k_x\Delta^{-1}\big(U_r''(y) - U_r(y)\Delta\big), \quad r = s, c. \end{aligned}$$

In § V-B, we consider wall oscillations of a small amplitude ($R_u \ll R$), and determine the $\mathcal{H}_2$ norm dependence on k_x and Ω at $R = 2000$ using a perturbation analysis of § IV.

III. Frequency response of LTP systems

We next provide a brief overview of a notion of the *frequency response* for exponentially stable LTP systems with period $T = 2\pi/\omega_o$. We refer the reader to [4], [9], [14]–[16] for additional information. In particular, the details about rigorous conditions for the existence of frequency response operators of the LTP systems can be found in [14].

It is a standard fact that a frequency response of a stable LTI system describes how a persistent harmonic input of a certain frequency propagates through the system in the steady-state. In other words, the steady-state response of a stable LTI system to an input signal of frequency ω, is a periodic signal of the same frequency, but with a modified amplitude and phase. The amplitude and phase of the output signal are precisely determined by the value of the frequency response at the input frequency ω.

On the other hand, a steady-state response of a stable LTP system to a harmonic input of frequency ω contains an infinite number of harmonics separated by integer multiples of ω_o, that is $\omega + n\omega_o$, $n \in \mathbb{Z}$. Using this fact and the analogy with the LTI systems, the frequency response of an LTP system can be defined by introducing a notion of *exponentially modulated periodic* (EMP) signals. As shown in [4], the EMP signals are more suitable for the analysis of the LTP systems than the persistent complex exponentials. Namely, the steady-state response of (1) to an EMP signal

$$d(t) = \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{j}(n\omega_o + \theta)t}, \quad \theta \in [0, \omega_o),$$

is also an EMP signal

$$\phi(t) = \sum_{n=-\infty}^{\infty} \phi_n \mathrm{e}^{\mathrm{j}(n\omega_o + \theta)t}, \quad \theta \in [0, \omega_o).$$

The frequency response of (1) is an operator that maps a bi-infinite vector $\mathbf{d} := \mathrm{col}\,\{d_n\}_{n \in \mathbb{Z}}$ to a bi-infinite vector $\boldsymbol{\phi} := \mathrm{col}\,\{\phi_n\}_{n \in \mathbb{Z}}$, that is $\boldsymbol{\phi} = \mathcal{H}_\theta \mathbf{d}$.

For system (1), operator $\mathcal{H}_\theta$ can be expressed as

$$\mathcal{H}_\theta = \mathcal{C}(\mathcal{E}(\theta) - \mathcal{A})^{-1}\mathcal{B},$$

where $\mathcal{E}(\theta)$ is a block-diagonal operator given by $\mathcal{E}(\theta) := \mathrm{diag}\,\{\mathrm{j}(\theta + n\omega_o)I\}_{n \in \mathbb{Z}}$, and I is the identity operator. On the other hand, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ represent block-Toeplitz operators, e.g.

$$\mathcal{A} := \mathrm{toep}\,\{\cdots, A_2, A_1, \boxed{A_0}, A_{-1}, A_{-2}, \cdots\},$$

where the box denotes the element on the main diagonal of $\mathcal{A}$. This bi-infinite matrix representation is obtained by expanding operators A, B, and C in (1) into their Fourier series, e.g. $A(t) = \sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{\mathrm{j}n\omega_o t}$. Clearly, since B and C are time-invariant operators their block-Toeplitz representations simplify to block-diagonal representations, i.e. $\mathcal{B} = \mathrm{diag}\,\{B\}$ and $\mathcal{C} = \mathrm{diag}\,\{C\}$.

IV. $\mathcal{H}_2$ norm of LTP systems

The $\mathcal{H}_2$ norm of a T-periodic system (1) with $\phi = Hd$, is defined as [17]

$$\|\mathcal{H}\|_2^2 := \frac{1}{T}\int_0^T \int_0^\infty \left[\|H\|_{HS}^2\right](t,\tau)\,\mathrm{d}t\,\mathrm{d}\tau,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt (HS) norm. For finite dimensional systems with kernel functions $H(t,\tau)$, the HS norm simplifies to the Frobenius norm for matrices

$$\left[\|H\|_{HS}^2\right](t,\tau) := \mathrm{trace}\,\left(H^*(t,\tau)H(t,\tau)\right),$$

and for infinite dimensional systems with kernel function $H(y,\eta;t,\tau)$, the HS norm is given by

$$\begin{aligned} &\left[\|H\|_{HS}^2\right](t,\tau) := \\ &\int_{-1}^{1}\int_{-1}^{1} \mathrm{trace}\,\left(H^*(y,\eta;t,\tau)H(y,\eta;t,\tau)\right)\,\mathrm{d}\eta\,\mathrm{d}y. \end{aligned}$$

As shown in [17], the $\mathcal{H}_2$ norm of an LTP system can be interpreted as the square-average of the L^2 norms of the impulse responses to a set of input forcing functions applied over the entire interval $[0, T]$. This interpretation of the $\mathcal{H}_2$ norm of LTP systems represents the appropriate generalization of a well known deterministic interpretation of the $\mathcal{H}_2$ norm of LTI systems [5].

Equivalently, the $\mathcal{H}_2$ norm of an LTP system can be expressed in terms of its frequency response $\mathcal{H}_\theta$ as

$$\|\mathcal{H}\|_2^2 \;=\; \frac{1}{2\pi} \int_0^{\omega_o} \text{trace}\;(\mathcal{H}_\theta^* \mathcal{H}_\theta)\; \mathrm{d}\theta. \tag{4}$$

We note that integration over θ in (4) can be avoided in the computation of the $\mathcal{H}_2$ norm. Namely, the $\mathcal{H}_2$ norm of (1) can be expressed using a solution to either of the following *harmonic Lyapunov equations* [9]

$$\mathcal{F}\mathcal{V} \;+\; \mathcal{V}\mathcal{F}^* \;=\; -\,\mathcal{B}\mathcal{B}^*, \tag{5a}$$

$$\mathcal{F}^*\mathcal{W} \;+\; \mathcal{W}\mathcal{F} \;=\; -\,\mathcal{C}^*\mathcal{C}, \tag{5b}$$

as

$$\|\mathcal{H}\|_2^2 \;=\; \text{trace}\,(\mathcal{V}\mathbf{c}^*\mathbf{c}) \;=\; \text{trace}\,(\mathcal{W}\mathbf{b}\mathbf{b}^*),$$

where $\mathcal{F} \,:=\, \mathcal{A} \,-\, \mathcal{E}(0) \,=\, \mathcal{A} \,-\, \text{diag}\,\{\mathrm{j}n\omega_o I\}_{n\,\in\,\mathbb{Z}}$, and

$$\begin{aligned} \mathbf{c} \;&:=\; [\;\cdots\;\; C_1 \;\; C_0 \;\; C_{-1} \;\;\cdots\;] \\ &=\; [\;\cdots\;\; 0 \;\; C \;\; 0 \;\;\cdots\;], \\ \mathbf{b} \;&:=\; [\;\cdots\;\; B_{-1}^T \;\; B_0^T \;\; B_1^T \;\;\cdots\;]^T \\ &=\; [\;\cdots\;\; 0 \;\; B^T \;\; 0 \;\;\cdots\;]^T. \end{aligned}$$

For the LTI systems, the above formulae simplify to the well-known expressions that are commonly used for determination of the $\mathcal{H}_2$ norm [5].

The following Lemma, whose proof is omitted due to page constraints, shows that the solutions to harmonic Lyapunov equations (5a) and (5b) are block-Toeplitz operators.

Lemma 1: Let a T-periodic operator $A(t)$ with block-Toeplitz representation $\mathcal{A}$ be exponentially stable. Then, solutions $\mathcal{V}$ and $\mathcal{W}$ to harmonic Lyapunov equations (5a) and (5b) are self-adjoint block-Toeplitz operators.

Since the entries into the harmonic Lyapunov equation are bi-infinite matrices with, in general, operator valued elements, determination of the $\mathcal{H}_2$ norm of the LTP systems is arguably a computationally intensive undertaking. In view of this, we will consider a problem where operator $A(t)$ can be represented as a sum of a time-invariant operator A_0 and a zero-mean time-periodic operator $\epsilon A_p(t)$, where ϵ denotes a small parameter. For this special case, we will employ a perturbation analysis to develop a computationally efficient method for determination of the $\mathcal{H}_2$ norm. We will show that the $\mathcal{H}_2$ norm can be obtained by solving a conveniently coupled system of Lyapunov and Sylvester equations. The entries into these equations are determined by the elements of bi-infinite matrices in (5). For the oscillating channel flow example, the underlying Lyapunov and Sylvester equations are operator valued equations in the wall-normal direction (y). A discretization in y can be used to obtain a set of matrix valued equations that can be easily solved in e.g. MATLAB. The order of these equations is determined by the size of discretization in the wall-normal direction (typically around 50).

A. Perturbation analysis

Using the structure of $A(t)$, we represent operator $\mathcal{F}$ in (5) as $\mathcal{F} = \mathcal{F}_0 + \epsilon \sum_{m\,\in\,\mathbb{N}} \mathcal{F}_m$, where $\mathcal{F}_0 := \text{diag}\,\{F(n)\}_{n\,\in\,\mathbb{Z}} = \text{diag}\,\{A_0 - \mathrm{j}n\omega_o I\}_{n\,\in\,\mathbb{Z}}$. On the other hand, for each $m \in \mathbb{N}$, $\mathcal{F}_m$ represents a block-Toeplitz operator with A_{-m} and A_m on the mth upper and lower block sub-diagonals, respectively. For example,

$$\begin{aligned} \mathcal{F}_1 \;&:=\; \text{toep}\,\{\cdots, 0, A_1, \boxed{0}, A_{-1}, 0, \cdots\}, \\ \mathcal{F}_2 \;&:=\; \text{toep}\,\{\cdots, 0, A_2, 0, \boxed{0}, 0, A_{-2}, 0, \cdots\}, \end{aligned}$$

and similarly for the other $\mathcal{F}_m$'s. In view of the above decomposition of operator $\mathcal{F}$, we rewrite harmonic Lyapunov equation (5a)

$$(\mathcal{F}_0 \,+\, \epsilon \sum_{m\,\in\,\mathbb{N}} \mathcal{F}_m)\mathcal{V} \;+\; \mathcal{V}(\mathcal{F}_0^* \,+\, \epsilon \sum_{m\,\in\,\mathbb{N}} \mathcal{F}_m^*) \;=\; -\,\mathcal{B}\mathcal{B}^*,$$

and represent $\mathcal{V}$ as

$$\mathcal{V} \;:=\; \sum_{n\,\in\,\mathbb{N}_0} \epsilon^n \mathcal{V}_n \;=\; \mathcal{V}_0 \,+\, \epsilon\mathcal{V}_1 \,+\, \epsilon^2\mathcal{V}_2 \,+\, \cdots. \tag{6}$$

The self-adjoint block-Toeplitz operators $\{\mathcal{V}_n\}_{n\,\in\,\mathbb{N}_0}$ satisfy the following sequence of operator Lyapunov equations

$$\mathcal{F}_0\mathcal{V}_0 + \mathcal{V}_0\mathcal{F}_0^* \;=\; -\,\mathcal{B}\mathcal{B}^*, \tag{7a}$$

$$\mathcal{F}_0\mathcal{V}_i + \mathcal{V}_i\mathcal{F}_0^* \;=\; -\sum_{m\,\in\,\mathbb{N}} (\mathcal{F}_m\mathcal{V}_{i-1} + \mathcal{V}_{i-1}\mathcal{F}_m^*), \tag{7b}$$

for each $i \in \mathbb{N}$. Since $\mathcal{F}_0$ and $\mathcal{B}$ are block-diagonal operators, it follows from (7a) that $\mathcal{V}_0$ is also a block-diagonal operator, $\mathcal{V}_0 := \text{diag}\,\{X\}$, with

$$A_0 X \;+\; X A_0^* \;=\; -BB^*.$$

Using linearity of (7b) we express $\mathcal{V}_1$ as

$$\mathcal{V}_1 \;=\; \sum_{m\,\in\,\mathbb{N}} \mathcal{V}_1^{(m)},$$

where

$$\mathcal{F}_0\mathcal{V}_1^{(m)} \;+\; \mathcal{V}_1^{(m)}\mathcal{F}_0^* \;=\; -\,(\mathcal{F}_m\mathcal{V}_0 \;+\; \mathcal{V}_0\mathcal{F}_m^*). \tag{8}$$

Here, for each $m \in \mathbb{N}$, $\mathcal{V}_1^{(m)}$ denotes a self-adjoint block-Toeplitz operator with non-zero elements on the mth block sub-diagonals; this structure of $\mathcal{V}_1^{(m)}$ follows directly from (8), Lemma 1, and a simple observation that a product between a block-diagonal and a block-Toeplitz operator with non-zero elements on the mth block sub-diagonals yields an operator with non-zero elements on the mth block sub-diagonals. Thus, $\mathcal{V}_1$ is a trace-less operator, and each $\mathcal{V}_1^{(m)}$ is a self-adjoint block-Toeplitz operator with Y_m on the mth upper block sub-diagonal. Furthermore, operator Y_m represents a solution to the following Sylvester equation

$$(A_0 \,+\, \mathrm{j}m\omega_o I)Y_m \;+\; Y_m A_0^* \;=\; -\,(A_{-m}X + XA_m^*)\,.$$

Based on linearity of (7b) and the above representation of $\mathcal{V}_1$ it follows that $\mathcal{V}_2$ can be expressed as

$$\mathcal{V}_2 \;=\; \sum_{m\,\in\,\mathbb{N}} \sum_{k\,\in\,\mathbb{N}} \mathcal{V}_2^{(m,k)},$$

where

$$\mathcal{F}_0\mathcal{V}_2^{(m,k)} \;+\; \mathcal{V}_2^{(m,k)}\mathcal{F}_0^* \;=\; -\,(\mathcal{F}_m\mathcal{V}_1^{(k)} \;+\; \mathcal{V}_1^{(k)}\mathcal{F}_m^*). \tag{9}$$

Now, since $\mathcal{F}_m$ and $\mathcal{V}_1^{(k)}$ are, respectively, block-Toeplitz operators with non-zero elements on the mth and kth block sub-diagonals, their product will have non-zero elements on the main-diagonal if and only if $m = k$. Thus, we see from (9) that only operators $\mathcal{V}_2^{(m,m)}$ have a non-zero trace; for $m \neq k$, operators $\mathcal{V}_2^{(m,k)}$ are trace-less. In view of this, we disjoin the block-diagonal part of $\mathcal{V}_2^{(m,m)}$ from the rest of it

$$\mathcal{V}_2^{(m,m)} \;=\; \mathrm{diag}\left\{Z_m\right\} \;+\; \bar{\mathcal{V}}_2^{(m,m)},$$

and derive the following Lyapunov equation for operator Z_m

$$\begin{aligned} A_0 Z_m \,+\, Z_m A_0^* \;=\; -\,(A_m Y_m \,+\, Y_m^* A_m^* \,+ \\ A_{-m} Y_m^* \,+\, Y_m A_{-m}^*). \end{aligned}$$

Hence, up to a second order in perturbation parameter ϵ, the $\mathcal{H}_2$ norm can be expressed as

$$\begin{aligned} \|\mathcal{H}\|_2^2 \;&=\; \sum_{n \,\in\, \mathbb{N}_0} \epsilon^n \mathrm{trace}\left(\mathcal{V}_n \mathbf{c}^* \mathbf{c}\right) \\ &=\; \mathrm{trace}\left(\big(X + \epsilon^2 \sum_{m \,\in\, \mathbb{N}} Z_m\big) C^* C\right) + O(\epsilon^3). \end{aligned}$$

Based on the above, we state the following result.

Theorem 2: Up to a second order in perturbation parameter ϵ, the $\mathcal{H}_2$ norm of system (1) with

$$A(t) \;=\; A_0 \,+\, \epsilon \sum_{n \,\in\, \mathbb{Z}\backslash 0} A_n \mathrm{e}^{\mathrm{j} n \omega_o t},$$

is given by

$$\|\mathcal{H}\|_2^2 \;=\; \mathrm{trace}\left(\big(X + \epsilon^2 \sum_{m \,\in\, \mathbb{N}} Z_m\big) C^* C\right) + O(\epsilon^3),$$

where

$$\begin{aligned} A_0 X + X A_0^* \;&=\; -\,BB^*, \\ (A_0 + \mathrm{j} m \omega_o I) Y_m + Y_m A_0^* \;&=\; -\left(A_{-m} X + X A_m^*\right), \\ A_0 Z_m + Z_m A_0^* \;&=\; -\,(A_m Y_m + Y_m^* A_m^* + \\ &\qquad A_{-m} Y_m^* + Y_m A_{-m}^*). \end{aligned}$$

Thus, up to a second order in perturbation parameter ϵ, there is no coupling between different harmonics of $A_p(t)$ in the expression for the $\mathcal{H}_2$ norm. This decoupling between different harmonics does not hold for arbitrary values of ϵ, and its derivation would not be possible if we tried to solve the harmonic Lyapunov equation directly without resorting to perturbation analysis.

When $A(t)$ contains only the first harmonic ω_o, i.e.

$$A(t) \;=\; A_0 \,+\, \epsilon\left(A_{-1} \mathrm{e}^{-\mathrm{j}\omega_o t} \,+\, A_1 \mathrm{e}^{\mathrm{j}\omega_o t}\right),$$

operator $\mathcal{F}$ can be represented as $\mathcal{F} = \mathcal{F}_0 + \epsilon \mathcal{F}_1$, where $\mathcal{F}_0$ and $\mathcal{F}_1$ are defined in the beginning of this section. Now, using the structure of operators $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{V}_{i-1}$ in (7b) we can establish that:

- For any $n \in \mathbb{N}_0$, $\mathcal{V}_{2n}$ in (6) is a self-adjoint block-Toeplitz operator with non-zero elements on block sub-diagonals $2k$, $k = 0, \ldots, n$, that is

  $$\begin{aligned} \mathcal{V}_{2n} \;&=\; \mathrm{diag}\left\{V_{2n,0}\right\} + \textstyle\sum_{k\,=\,1}^{n} \mathcal{S}_{2k}\, \mathrm{diag}\left\{V_{2n,2k}\right\} + \\ &\quad \textstyle\sum_{k\,=\,1}^{n} \mathrm{diag}\left\{V_{2n,2k}^*\right\} \mathcal{S}_{2k}^*, \end{aligned}$$

  where $\mathcal{S}_{2k}$ denotes a bi-infinite block-Toeplitz operator with identity operators on the upper block sub-diagonal $2k$. Notation $V_{n,k}$ indicates that $V_{n,k}$ belongs to the kth upper block sub-diagonal of $\mathcal{V}_n$, and, for any $\{n \in \mathbb{N}_0,\; k \;=\; 0, \ldots, n\}$, operators $V_{2n,2k}$ represent the solutions to Lyapunov and Sylvester equations given in Theorem 3.
- For any $n \in \mathbb{N}$, $\mathcal{V}_{2n-1}$ in (6) is a self-adjoint block-Toeplitz operator with non-zero elements on block sub-diagonals $2k-1$, $k \,=\, 1, \ldots, n$, that is

  $$\begin{aligned} \mathcal{V}_{2n-1} \;&=\; \textstyle\sum_{k\,=\,1}^{n} \mathcal{S}_{2k-1}\, \mathrm{diag}\left\{V_{2n-1,2k-1}\right\} + \\ &\quad \textstyle\sum_{k\,=\,1}^{n} \mathrm{diag}\left\{V_{2n-1,2k-1}^*\right\} \mathcal{S}_{2k-1}^*, \end{aligned}$$

  where $\mathcal{S}_{2k-1}$ denotes a bi-infinite block-Toeplitz operator with identity operators on the upper block sub-diagonal $2k-1$. Notation $V_{n,k}$ indicates that $V_{n,k}$ belongs to the kth upper block sub-diagonal of $\mathcal{V}_n$. Thus, $\mathrm{trace}\left(\mathcal{V}_{2n-1}\right) \;\equiv\; 0$, and, for any $\{n \;\in\; \mathbb{N},\; k \;=\; 1, \ldots, n\}$, operators $V_{2n-1,2k-1}$ represent the solutions to Lyapunov and Sylvester equations given in Theorem 3.

The above observations for time-periodic operators $A(t)$ with a single harmonic ω_o are summarized in Theorem 3.

Theorem 3: The $\mathcal{H}_2$ norm of system (1) with

$$A(t) \;=\; A_0 \,+\, \epsilon\left(A_{-1} \mathrm{e}^{-\mathrm{j}\omega_o t} \,+\, A_1 \mathrm{e}^{\mathrm{j}\omega_o t}\right),\; 0 \,<\, \epsilon \,\ll\, 1,$$

is given by

$$\|\mathcal{H}\|_2^2 \;=\; \sum_{n\,=\,0}^{\infty} \epsilon^{2n}\, \mathrm{trace}\,\left(V_{2n,0}\, C^* C\right),$$

where

$$\begin{aligned} &A_0 V_{0,0} \,+\, V_{0,0} A_0^* \;=\; -\,BB^*, \\ &A_0 V_{2n,0} \,+\, V_{2n,0} A_0^* \;=\; -\,(A_1 V_{2n-1,1} \,+\, V_{2n-1,1}^* A_1^* \,+ \\ &A_{-1} V_{2n-1,1}^* \,+\, V_{2n-1,1} A_{-1}^*), \\ &(A_0 \,+\, \mathrm{j} l \omega_o I) V_{l,l} \,+\, V_{l,l} A_0^* \;=\; -\,(A_{-1} V_{l-1,l-1} \,+ \\ &V_{l-1,l-1} A_1^*), \quad l \,\in\, \mathbb{N}, \\ &(A_0 \,+\, \mathrm{j} m \omega_o I) V_{l,m} \,+\, V_{l,m} A_0^* \;=\; -\,(A_{-1} V_{l-1,m-1} \,+ \\ &V_{l-1,m-1} A_1^* \,+\, A_1 V_{l-1,m+1} \,+\, V_{l-1,m+1} A_{-1}^*), \\ &m \;=\; \begin{cases} 2, 4, \ldots, l-2 & l \;-\; \text{even}, \\ 1, 3, \ldots, l-2 & l \;-\; \text{odd}. \end{cases} \end{aligned}$$

Application of Theorem 3 is illustrated in § V on two examples: the dissipative Mathieu equation of § II-A.1 and the two-dimensional oscillating channel flow of § II-A.2.

V. Examples

In this section, we employ Theorem 3 to determine the second order corrections to the $\mathcal{H}_2$ norms of systems described in § II-A.1 and § II-A.2.

A. The dissipative Mathieu equation

The $\mathcal{H}_2$ norm of dissipative Mathieu equation subject to small amplitude oscillations (see § II-A.1) is given by

$$\|\mathcal{H}\|_2^2 \;=\; f_0 \,+\, \epsilon^2 f_2(\omega_o) \,+\, O(\epsilon^4),$$

where $f_0 \,=\, 1/(4ab)$, and

$$f_2(\omega_o) \;=\; \frac{64ab^2 \,+\, 4\left(3a \,-\, 4b^2\right)\omega_o^2 \,-\, \omega_o^4}{2a^2 b\left(4b^2 + \omega_o^2\right)\left(16a^2 - 8\left(a - 2b^2\right)\omega_o^2 + \omega_o^4\right)}.$$

The formula for $f_2(\omega_o)$ is obtained from Theorem 3 with the help of MATHEMATICA.

Plots of $f_2(\omega_o)$ and $\log_{10}|f_2(\omega_o)|$ in the expression for the $\mathcal{H}_2$ norm of dissipative Mathieu equation with $a \,=\, 1$ and $b \,=\, 0.2$ are shown in Fig. 2. We observe two resonant peaks: the positive at $\omega_o \;\approx\; 1.88$, and the negative at $\omega_o \;\approx\; 4.23$. As can be seen from the plot of $\log_{10}|f_2(\omega_o)|$, the latter resonant peak has a very small magnitude compared to the peak at $\omega_o \approx 1.88$ and its contribution to the $\mathcal{H}_2$ norm is not likely to be significant.

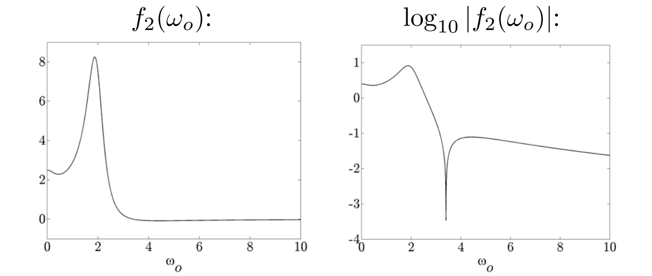

Fig. 2. Plots of $f_2(\omega_o)$ and $\log_{10}|f_2(\omega_o)|$ in the expression for the $\mathcal{H}_2$ norm of dissipative Mathieu equation with $a = 1$ and $b = 0.2$.

B. Two-dimensional oscillating channel flow

The $\mathcal{H}_2$ norm of a two-dimensional oscillating channel flow is parameterized by the wave-number k_x, the Stokes number Ω, and the Reynolds numbers R and R_u (see § II-A.2). For small amplitude oscillations of the lower wall ($R_u \ll R$), we use Theorem 3 to obtain

$$\left[\|\mathcal{H}\|_2^2\right](k_x) = f_0(k_x) + R_u^2 f_2(k_x, \Omega) + O(R_u^4),$$

where functions $f_0(k_x)$ and $f_2(k_x, \Omega)$ also depend on the Reynolds number R.

Fig. 3 illustrates plots of $f_0(k_x)$ and $f_2(k_x, \Omega)$ in the two-dimensional oscillating channel flow with $R = 2000$. These two functions are determined numerically using a Matlab Differentiation Matrix Suite [18] with 50 collocation points in the wall-normal direction. We observe a peak in the plot of $f_0(k_x)$ which is caused by 'poorly damped modes' of parallel channel flow $U_0(y)$. Clearly, depending on the value of Stokes number Ω this peak can be attenuated or amplified in the presence of wall oscillations. For small values of Ω (approximately $\Omega < 20$) 'the periodic feedback' leads to a reduction in the $\mathcal{H}_2$ norm, whereas for large values of Ω (approximately $20 < \Omega < 250$) it increases the $\mathcal{H}_2$ norm. Thus, the perturbation analysis facilitates identification of the Stokes numbers (i.e. the wall oscillation frequencies) that lead to amplification or attenuation (relative to $U_0(y)$) of background disturbances. Once the wall oscillation frequency is selected using perturbation analysis, the influence of the wall oscillation amplitude on the $\mathcal{H}_2$ norm can be studied using, for example, the truncation of bi-infinite operators in the harmonic Lyapunov equation or so-called 'approximate modeling approach' [9]. We note that our analysis provides a computationally efficient method for determination of the $\mathcal{H}_2$ norm of periodic systems subject to small amplitude oscillations without resorting to either of these two approaches. The only approximation in our analysis arises due to discretization of channel flow system in the wall-normal direction. As far as temporal dynamics is concerned, our analysis is *exact*.

VI. Concluding remarks

We use a perturbation analysis to develop an efficient method for computation of the $\mathcal{H}_2$ norm of LTP systems with small amplitude oscillations. We show that, up to a second order in perturbation parameter, the $\mathcal{H}_2$ norm can be determined by solving a conveniently coupled system of Lyapunov and Sylvester equations whose structure is determined by the structure of unperturbed LTI system.

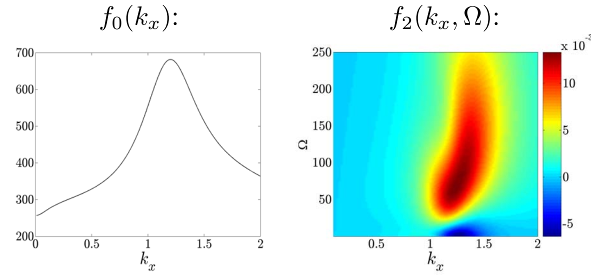

Fig. 3. Plots of $f_0(k_x)$ and $f_2(k_x, \Omega)$ in the expression for the $\mathcal{H}_2$ norm of two-dimensional oscillating channel flow with $R = 2000$.

For finite dimensional systems, the size of these equations corresponds to the size of matrices in the original LTI system. For infinite dimensional systems, these equations are operator valued and they are typically solved by resorting to finite dimensional approximation of the underlying operators. In the channel flow example, this amounts to solving Lyapunov and Sylvester equations of the size determined by discretization in the wall-normal direction (typically less than 50). The developed procedure is suitable for identification of forcing frequency $\omega_o := 2\pi/T$ that leads to the largest $\mathcal{H}_2$ norm reduction/increase.

References

[1] A. H. Nayfeh and D. T. Mook, *Nonlinear oscillations*. New York: John Wiley & Sons, 1979.

[2] M. Farkas, *Periodic motions*. New York: Springer-Verlag, 1994.

[3] B. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with applications to H^∞ sampled-data control," *IEEE Transactions on Automatic Control*, vol. 37, no. 4, pp. 418–435, April 1992.

[4] N. M. Wereley and S. R. Hall, "Frequency response of linear time periodic systems," in *Proceedings of the 29th IEEE Conference on Decision and Control*, Honolulu, HI, 1990, pp. 3650–3655.

[5] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Prentice Hall, 1996.

[6] B. Bamieh and M. Dahleh, "Energy amplification in channel flows with stochastic excitation," *Phys. Fluids*, vol. 13, no. 11, pp. 3258–3269, 2001.

[7] M. R. Jovanović, "Modeling, analysis, and control of spatially distributed systems," Ph.D. dissertation, University of California, Santa Barbara, 2004.

[8] M. R. Jovanović and B. Bamieh, "Componentwise energy amplification in channel flows," *J. Fluid Mech.*, vol. 534, pp. 145–183, July 2005.

[9] J. Zhou, T. Hagiwara, and M. Araki, "Trace formula of linear continuous-time periodic systems via the harmonic Lyapunov equation," *Int. J. Control*, vol. 76, no. 5, pp. 488–500, 2003.

[10] M. R. Jovanović, "Turbulence suppression in channel flows by small amplitude transverse wall oscillations," in *Proceedings of the 2006 American Control Conference*, Minneapolis, MN, 2006.

[11] M. Fardad and B. Bamieh, "A perturbation approach to the H_2 analysis of spatially periodic systems," in *Proceedings of the 2005 American Control Conference*, Portland, OR, 2005, pp. 4838–4843.

[12] M. Fardad, M. R. Jovanović, and B. Bamieh, "Frequency analysis and norms of distributed spatially periodic systems," *IEEE Transactions on Automatic Control*, 2005, submitted.

[13] S. C. Reddy and D. S. Henningson, "Energy growth in viscous channel flows," *J. Fluid Mech.*, vol. 252, pp. 209–238, 1993.

[14] J. Zhou and T. Hagiwara, "Existence conditions and properties of the frequency response operators of continuous-time periodic systems," *SIAM J. Control Optim.*, vol. 40, no. 6, pp. 1867–1887, 2002.

[15] H. Sandberg, E. Möllerstedt, and B. Bernhardsson, "Frequency-domain analysis of linear time-periodic systems," in *Proceedings of the 2004 American Control Conference*, Boston, MA, 2004, pp. 3357–3362.

[16] H. Sandberg, "Model reduction for linear time-varying systems," Ph.D. dissertation, Department of Automatic Control, Lund Institute of Technology, Sweden, December 2004.

[17] B. Bamieh and J. B. Pearson, "The H^2 problem for sampled-data systems," *Systems and Control Letters*, vol. 19, no. 1, pp. 1–12, July 1992.

[18] J. A. C. Weideman and S. C. Reddy, "A MATLAB differentiation matrix suite," *ACM Transactions on Mathematical Software*, vol. 26, no. 4, pp. 465–519, December 2000.