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# On the State-Space Design of Optimal Controllers for Distributed Systems with Finite Communication Speed

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*Abstract*— **We consider the problem of designing optimal distributed controllers whose impulse response has limited propagation speed. We introduce a state-space framework in which such controllers can be described. We show that the optimal control problem is not convex with respect to certain state-space design parameters, and demonstrate a reasonable relaxation that renders the problem convex. This relaxation is associated with an iterative numerical scheme known as the Steiglitz-McBride (SM) algorithm. We improve the SM algorithm by using the algebraic Lyapunov equation to relieve time integration, thus significantly reducing computational costs.**

## I. INTRODUCTION

The synthesis problem of distributed control has received considerable attention in recent years [1]–[8]. In the control of distributed systems a desired scenario is to have each subsystem possess its own controller and each controller exchange information only within a prespecified "local" architecture. Standard optimal control design methods, when applied to distributed systems, yield "centralized" controllers [1]. In other words the controller of each subsystem demands information about the state of the entire system. Such solutions are undesirable from a practical point of view as they are expensive in hardware and computation requirements and demand excessive communication between different subsystems.

In the case of spatially invariant systems, [1] demonstrates that for optimal distributed controllers, the dependence of a controller on information coming from other parts of the system decays exponentially as one moves away from that controller. This motivates the search for inherently "localized" controllers. For example, one could search for optimal controllers that are subject to the condition that they communicate only to other controllers within a certain radius.

Optimal control problems are often reformulated in the "Youla parameter" domain, which allows for a closed-loop transfer function that is affine in the Youla parameter [9]. However, this generally comes at the expense of losing convexity of the constraint set to which the design parameter belongs. This is due to the nonlinearity of the mapping from the controller to the Youla parameter.

Recently, certain subspaces of localized systems which remain invariant under this nonlinear mapping have been characterized. References [2] and [3] introduce the subspaces of "cone causal" and "funnel causal" systems, respectively. These subspaces describe how information from every controller propagates through the distributed system. A similar but more general characterization, termed "quadratic invariance," is introduced in [4]. It is important to note that constructs such as cone and funnel causality lead to optimal control problems that are convex in the Markov (i.e., impulse response) parameters of the Youla variable and *not* its state-space parameters. Therefore, one is still faced with solving a realization problem for a distributed system.

In this paper we address the problem of designing structured optimal distributed controllers using a state-space framework. We show that not all controller design parameters appear quadratically in the objective function and we use a relaxation, associated with the the SM algorithm [10], to convexify the objective function. The SM–optimal coefficients are then obtained through an iterative numerical scheme. We improve upon existing SM algorithms [11] by using the algebraic Lyapunov equation to relieve time integration, thus significantly reducing the computational cost of the numerical scheme.

The paper is organized as follows. In Section II we describe the subspaces of distributed systems considered in this paper. In Section III we use the model-matching framework to find the optimal centralized controller, which we wish to approximate by a localized one. In Section IV we present a numerical algorithm for the design of structured decentralized controllers. We demonstrate our results by two illustrative examples in Section V and finish with conclusions in Section VI.

### *Preliminaries*

We consider discrete spatio-temporal systems, i.e., discrete time systems on a discrete one-dimensional spatial lattice. All systems are linear time invariant and spatially invariant. $\lambda$ denotes the temporal (one-sided) transform variable and $\zeta$ denotes the spatial (two-sided) transform variable. When evaluated on the unit circle, $\lambda$ and $\zeta$ are denoted by $e^{j\omega}$ and $e^{j\theta}$, respectively. $U^* = \overline{U}^T$ if $U$ is a constant matrix and $U(\zeta,\lambda)^* = \overline{U}(\zeta^{-1},\lambda^{-1})^T$ if $U$ is a spatio-temporal transfer function, where the bar over $U$ denotes complex conjugation and $^T$ denotes transposition. $U^\dagger$ denotes the pseudo-inverse of $U$.

## II. CONE CAUSAL AND $\mathscr{C}$–CAUSAL SYSTEMS

We begin by defining the class of cone causal systems introduced in [2].

*Definition 1:* A linear spatially invariant system is called *cone causal* if its spatio-temporal impulse response is of the form

$$G(\zeta,\lambda) \;=\; \sum_{k=0}^{\infty} g_k(\zeta)\,\lambda^k, \qquad (1)$$

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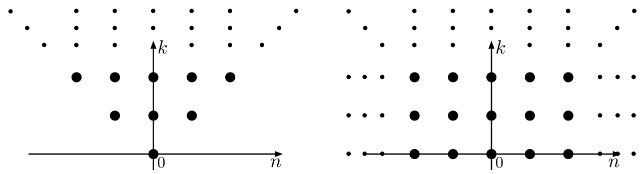


Fig. 1. The vertical axis denotes time and the horizontal axis denotes space. Left: The support of the spatio-temporal impulse response of a *cone causal* system. Right: The support of the spatio-temporal impulse response of a *centralized* system.

$$g_k(\zeta) \;=\; \sum_{n=-k}^{k} g_{nk}\,\zeta^n, \quad g_0(\zeta) \;=\; g_{00},$$

where $g_{nk}$ can be matrices in general.

Note that by the above definition, a spatio-temporal system can be cone causal *without* having to be stable. Cone causality is only a condition on the support of the impulse response in the spatio-temporal domain. The left picture in Figure 1 demonstrates the support of the spatio-temporal impulse response of a cone causal system.

A spatio-temporal system described by (1) in which

$$g_k(\zeta) \;=\; \sum_{n=-\infty}^{\infty} g_{nk}\,\zeta^n, \qquad k \;=\; 0,1,2,\ldots,$$

is said to be *centralized*. In other words, a centralized system is one in which the impulse response has unbounded spatial spread at every time instant $k$; see the right picture in Figure 1.

*Subspace $\mathscr{C}$ and its State-Space Representation*

Consider a system $G$ with state-space representation

$$G \;=\; \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \;=\; D \,+\, \lambda\, C\,(I - \lambda\, A)^{-1} B. \quad (2)$$

*Definition 2:* We denote by $\mathscr{C}$ the set of systems that satisfy the following assumptions.

(i) $B$, $D$ are independent of $\zeta$.
(ii) $A$, $C$ have the form

$$\begin{aligned} A(\zeta) &= A_{-1}\,\zeta^{-1} \,+\, A_0 \,+\, A_1\,\zeta, \\ C(\zeta) &= C_{-1}\,\zeta^{-1} \,+\, C_0 \,+\, C_1\,\zeta, \end{aligned}$$

with $A_n$, $C_n$, $n = -1, 0, 1$ independent of $\zeta$.

The systems that belong to the set $\mathscr{C}$ are systems in which effects propagate at most one unit in space for every unit in time.

We refer to systems that belong to the set $\mathscr{C}$ as *$\mathscr{C}$–causal*. Furthermore, we denote by $\mathscr{C}_\mu$ the subset of $\mathscr{C}$–causal systems for which the matrix $A$ has Euclidean dimension equal to $\mu$. We refer to $\mu$ as the *temporal degree* or temporal order of $G$.

Of course, the above definition includes systems for which either or both of the matrices $A$ and $C$ are $\zeta$-independent.

*Proposition 1:* If $G \in \mathscr{C}$ then $G$ is cone causal.

*Proof:* Write the transfer function of $G$ in terms of its Markov parameters

$$G(\zeta,\lambda) \;=\; D \,+\, CB\,\lambda \,+\, CAB\,\lambda^2 \,+\, CA^2B\,\lambda^3 \,+\, \cdots.$$

It is clear that $G$ has the structure described in Definition 1. ∎

*Closure of $\mathscr{C}$ Under LFTs*

As we will show, the subspace $\mathscr{C}$ of cone causal systems is closed under addition, composition, and inversion of systems. Thus it is closed under feedback and linear fractional transformations (LFTs [12]).

Reference [2] demonstrates closure results for cone causal systems using Markov parameter descriptions. The following proposition proves closure results for $\mathscr{C}$–causal systems using state-space descriptions. Let $G^\dagger$ denote the right (left) inverse of $G$ and let $D^\dagger$ denote the right (left) inverse of $D$.

*Proposition 2:* Let $G$ be as in (2) and $\widetilde{G} = \left[\begin{array}{c|c} \widetilde{A} & \widetilde{B} \\ \hline \widetilde{C} & \widetilde{D} \end{array}\right]$, and assume that $D^\dagger$ exists. If $G$ and $\widetilde{G}$ belong to $\mathscr{C}$ then $G + \widetilde{G}$, $G\,\widetilde{G}$, and $G^\dagger$ belong to $\mathscr{C}$.

*Proof:* We have [12]

$$G + \widetilde{G} = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & \widetilde{A} & \widetilde{B} \\ \hline C & \widetilde{C} & D + \widetilde{D} \end{array}\right], \quad G\,\widetilde{G} = \left[\begin{array}{cc|c} A & B\widetilde{C} & B\widetilde{D} \\ 0 & \widetilde{A} & \widetilde{B} \\ \hline C & D\widetilde{C} & D\widetilde{D} \end{array}\right],$$

$$G^\dagger = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array}\right].$$

It is clear from Definition 2 and the state-space representations of $G + \widetilde{G}$, $G\,\widetilde{G}$, and $G^\dagger$ that they all belong to $\mathscr{C}$, and the proof is complete. ∎

## III. The Structured $\mathcal{H}^2$ Optimal Control Problem

Consider the system $G \in \mathscr{C}$,

$$G \;=\; \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \;=\; \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]. \quad (3)$$

Note that since $G \in \mathscr{C}$ then $B_w$, $B_u$, $D_{zu}$, $D_{yw}$ are independent of $\zeta$, and $A$, $C_z$, $C_y$ have $\zeta$-dependence of the form described in Definition 2. We also make the following simplifying assumptions.

*Assumption 1:* In system (3)

(i) $B_u$, $D_{zu}$ are column vectors.
(ii) $C_y$, $D_{yw}$ are row vectors.

Assumption 1 implies that the transfer functions $G_{22}$ from $u$ to $y$ is SISO (single input single output). Placing system $G$ in feedback with a SISO controller $K$ we obtain the closed-loop transfer function

$$G_{zw} \;=\; G_{11} \,+\, G_{12}\,K\,(I \,-\, G_{22}\,K)^{-1}\,G_{21}. \quad (4)$$

Before we discuss the optimal control problem of interest, we have to define the system norm we will be using.

*Definition 3:* Let $G_{zw}$ be a stable system. Then the spatio-temporal $\mathcal{H}^2$ norm of $G_{zw}$ is defined by [1]

$$\|G_{zw}\|^2_{\mathcal{H}^2} \;:=\; (\frac{1}{2\pi})^2 \int_0^{2\pi}\!\!\int_0^{2\pi} \mathrm{tr}\,[G_{zw}(\mathrm{e}^{\mathrm{j}\theta},\mathrm{e}^{\mathrm{j}\omega}) \; G_{zw}(\mathrm{e}^{\mathrm{j}\theta},\mathrm{e}^{\mathrm{j}\omega})^*]\,\mathrm{d}\theta\,\mathrm{d}\omega.$$

The problem we are interested in is the following. Given system $G \in \mathscr{C}$ find a stabilizing controller $K \in \mathscr{C}$ such that the closed-loop norm $\|G_{zw}\|^2_{\mathcal{H}^2}$ is minimized.

*Remark 1:* Structured optimal control problems such as the one posed above are hard to solve because of the nonlinear way in which the design parameter $K$ appears in the expression for $G_{zw}$; see (4). As we show below, a change of variables allows for a *new* design parameter $Q$ to appear affinely in $G_{zw}$, thus forming a convex objective function. However, the mapping from $K$ to $Q$ will be nonlinear, and therefore a convex constraint set for $K$ does not always get mapped to a convex constraint set for $Q$. This underlines the importance of subspaces such as cone causal [2], funnel causal [3], quadratically invariant [4], and $\mathscr{C}$–causal systems: they remain *invariant* under the map $K \mapsto Q$. Since every subspace is convex we thus end up with optimizing a convex objective over a convex set, which is a desired scenario. This remark is summarized in Theorem 3 below.

Using the "Youla parameterization", it is well-known [9, Chap. 3] that the transfer function of the closed-loop system (4) can be recast as

$$G_{zw} \;=\; T_1 \,-\, T_2\, Q\, T_3, \tag{5}$$

and thus the problem of minimizing $\|G_{zw}\|^2_{\mathcal{H}^2}$ can be rewritten as the so-called "model-matching problem"

$$\inf_{Q} \; \|T_1 \,-\, T_2\, Q\, T_3\|^2_{\mathcal{H}^2}. \tag{6}$$

The model-matching parameters $Q$ and $T_i$, $i = 1, 2, 3$ are all stable transfer functions. The $T_i$ have known state-space representations and can be found using only knowledge of the open-loop system $G$ (i.e., they are independent of $Q$). $Q$, often referred to as the Youla parameter, is unknown and depends on both the controller $K$ and the system $G$. Once problem (6) is solved and the optimal system $Q^{\rm opt}$ is found we obtain the optimal controller $K^{\rm opt}$ from $Q^{\rm opt}$, as discussed in [9].

By Assumption 1, $Q$ is a scalar and thus commutes with $T_3$. Defining $T = T_1$ and $U = T_2\, T_3$, problem (6) becomes

$$\inf_{Q} \; \|T \,-\, U\, Q\|^2_{\mathcal{H}^2}. \tag{7}$$

From [9, Chap. 4] it follows that

$$T \;=\; \left[\begin{array}{cc|c} A + B_u F & -B_u F & B_w \\ 0 & A + H C_y & B_w + H D_{yw} \\ \hline C_z + D_{zu} F & -D_{zu} F & 0 \end{array}\right], \tag{8}$$

$$U \;=\; \left[\begin{array}{cc|c} A + B_u F & B_u C_y & B_u D_{yw} \\ 0 & A + H C_y & B_w + H D_{yw} \\ \hline C_z + D_{zu} F & D_{zu} C_y & D_{zu} D_{yw} \end{array}\right], \tag{9}$$

where $F$ and $H$ are chosen such that $A + B_u F$ and $A + H C_z$ are stable, i.e., the matrices $[A + B_u F](\mathrm{e}^{\mathrm{j}\theta})$ and $[A + H C_z](\mathrm{e}^{\mathrm{j}\theta})$ have strictly negative eigenvalues for every $\theta \in [0, 2\pi]$. We make the following assumptions on $H$ and $F$.

*Assumption 2:* In system (3)

(i) A column vector $H$ independent of $\zeta$ can be found such that $A(\mathrm{e}^{\mathrm{j}\theta}) + H C_y(\mathrm{e}^{\mathrm{j}\theta})$ is a stable matrix for every $\theta \in [0, 2\pi]$.

(ii) A row vector $F(\zeta)$ of the form

$$F(\zeta) \;=\; F_{-1}\, \zeta^{-1} \,+\, F_0 \,+\, F_1\, \zeta,$$

with $F_n$, $n = -1, 0, 1$ independent of $\zeta$, can be found such that $A(\mathrm{e}^{\mathrm{j}\theta}) + B_u F(\mathrm{e}^{\mathrm{j}\theta})$ is a stable matrix for every $\theta \in [0, 2\pi]$.

We now state the main result of this section.

*Theorem 3:* Let the system $G \in \mathscr{C}$ with state-space representation (3) satisfy the conditions stated in Assumption 2. Then the mapping $Q \mapsto K$ is a bijection from $\mathscr{C}$ to itself. In particular, $K$ is stabilizing and belongs to $\mathscr{C}$ if and only if $Q$ is stable and belongs to $\mathscr{C}$.

*Proof:* See Appendix. ■

### *The Model-Matching Problem*

In this section we present the model-matching problem. We introduce an inner-outer factorization of $U$, $U = U_{\rm in} U_{\rm out}$, see [9]. In the following, we will use the isometry property of the inner function $U_{\rm in}(\mathrm{e}^{\mathrm{j}\theta}, \mathrm{e}^{\mathrm{j}\omega})$, $\theta, \omega \in [0, 2\pi]$, and the fact that

$$\|E\, G\|^2_{\mathcal{H}^2} \;=\; \|G\|^2_{\mathcal{H}^2}, \qquad E \,:=\, \left[\begin{array}{c} U^*_{\rm in} \\ I \,-\, U_{\rm in}\, U^*_{\rm in} \end{array}\right],$$

see [9, Lem. 1, Chap. 8]. We have

$$\begin{aligned}
&\|T \,-\, U\, Q\|^2_{\mathcal{H}^2} \;=\; \|E\,(T \,-\, U_{\rm in}\, U_{\rm out}\, Q)\|^2_{\mathcal{H}^2} \\
&= \left\| \left[\begin{array}{c} U^*_{\rm in}\, T \,-\, U_{\rm out}\, Q \\ (I \,-\, U_{\rm in}\, U^*_{\rm in})\, T \end{array}\right] \right\|^2_{\mathcal{H}^2} \\
&= \|(I \,-\, U_{\rm in}\, U^*_{\rm in})\, T\|^2_{\mathcal{H}^2} \,+\, \|U^*_{\rm in}\, T \,-\, U_{\rm out}\, Q\|^2_{\mathcal{H}^2} \\
&= \|(I \,-\, U_{\rm in}\, U^*_{\rm in})\, T\|^2_{\mathcal{H}^2} \,+\, \|[U^*_{\rm in}\, T]_{\rm un} \,+\, [U^*_{\rm in}\, T]_{\rm st} \,-\, U_{\rm out}\, Q\|^2_{\mathcal{H}^2} \\
&= \|(I - U_{\rm in}\, U^*_{\rm in})\, T\|^2_{\mathcal{H}^2} + \|[U^*_{\rm in}\, T]_{\rm un}\|^2_{\mathcal{H}^2} + \|[U^*_{\rm in}\, T]_{\rm st} - U_{\rm out}\, Q\|^2_{\mathcal{H}^2}
\end{aligned}$$

where $R := [U^*_{\rm in}\, T]_{\rm st}$ and $[U^*_{\rm in}\, T]_{\rm un}$ correspond to the stable and unstable parts of $U^*_{\rm in}\, T$, respectively; see [13, Chap. 6] for more details. The optimal solution (regardless of whether it does or does not belong to $\mathscr{C}$) is given by

$$Q^{\rm c} \;=\; U^{-1}_{\rm out}\, [U^*_{\rm in}\, T]_{\rm st}.$$

Note that $Q^{\rm c}$ is stable since $U^{-1}_{\rm out}$ is the inverse of a minimum phase system and thus stable.

The difficulty here is that once an inner-outer factorization of $U \in \mathscr{C}$ is performed, in general neither $U_{\rm in}$ nor $U_{\rm out}$ belongs to $\mathscr{C}$. In fact $Q^{\rm c}$ is a *centralized* system in general. This is due to $U_{\rm in}$ and $U_{\rm out}$ containing parameters that are found by solving an algebraic Riccati equation (ARE), and the solution $X$ of this ARE can *not* be expressed as a polynomial in $\zeta$. In particular, the state-space realizations of $U_{\rm in}$ and $U_{\rm out}$ do not satisfy conditions (i) and (ii) of Definition 2.

In this paper our aim is to find $Q \in \mathscr{C}$ that minimizes

$$\begin{aligned}
J \;&:=\; \|R \,-\, U_{\rm out}\, Q\|^2_{\mathcal{H}^2} && (10) \\
&=\; \|U_{\rm out}(U^{-1}_{\rm out} R \,-\, Q)\|^2_{\mathcal{H}^2} \\
&=\; \|U_{\rm out}(Q^{\rm c} \,-\, Q)\|^2_{\mathcal{H}^2}. && (11)
\end{aligned}$$

Thus we would like to find $Q \in \mathscr{C}$ that best approximates the centralized system $Q^{\rm c}$ in the sense of the weighted $\mathcal{H}^2$ norm. The norm in (11) is weighted by $U_{\rm out}$. Note that in (11) there is no restriction on the temporal order of $Q$. However one possibility is to choose the temporal degree of $Q$ to be equal to that of $Q^{\rm c}$, so that $Q$ imitates the temporal dynamics of $Q^{\rm c}$. We emphasize that there is no reason to expect that such a choice of temporal order is optimal.

*Literature Review*

To the best knowledge of the authors, no exact solution to the problem posed at the end of the previous section is known in general, and to find $Q \in \mathscr{C}$ one has to resort to some form of approximation.

Voulgaris *et al.* [2] consider this problem in the Markov-parameter setting using the projection theorem for Hilbert spaces. More specifically, they obtain the spatio-temporal Markov parameters $q_{nk}$, up to time $k = \kappa$, of an FIR (finite impulse response) cone causal system $Q^\kappa$

$$Q^\kappa(\zeta,\lambda) \;=\; \sum_{k=0}^{\kappa} q_k(\zeta)\,\lambda^k, \qquad q_k(\zeta) \;=\; \sum_{n=-k}^{k} q_{nk}\,\zeta^n,$$

such that $Q^\kappa$ minimizes

$$\|R \,-\, U_{\text{out}}\,Q\|^2_{\mathcal{H}^2}, \qquad Q \text{ cone causal},$$

if the $\mathcal{H}^2$ norm is computed up to time $k = \kappa$. Furthermore, they show weak convergence of $Q^\kappa$ to the unique optimal cone causal system $Q^{\text{opt}}$ as $\kappa \to \infty$.

In a mathematical sense [2] solves the optimal control problem. But the difficulty with the approach of [2] is with regards to the implementation of the resulting controller. The state-space realization of $Q^\kappa$ is a dead-beat system of order $\kappa$. If $\kappa$ is taken to be large to achieve a small closed-loop norm, $Q^\kappa$ and thus the controller $K^\kappa$ will have large temporal degrees, in general.

This motivates the problem of solving the structured optimal control problem not with respect to the Markov parameters of $Q$, but with respect to its state-space or transfer function representation. This will be our aim in the following section, where given a temporal order $\mu$, we present a numerical algorithm for computing $Q \in \mathscr{C}_\mu$ that minimize a relaxation of the objective function $J$ in (10).

## IV. A Numerical Algorithm for Computing $Q$

The SM (Steiglitz-McBride) algorithm is an iterative numerical optimization scheme originally used for the identification of linear systems [10]. Recently it has been further developed and coupled with other numerical methods for the purpose of designing IIR (infinite impulse response) digital filters [11]. In this section we use this algorithm to find $Q \in \mathscr{C}$ that minimizes $J$ in (10). We improve upon existing SM algorithms by using the algebraic Lyapunov equation to relieve time integration, thus significantly reducing the computational cost of the numerical scheme.

For the sake of clarity we first describe the basic idea of the SM algorithm in the transfer function setting. We then derive the computational procedure in state-space. We assume all transfer functions are SISO.

Let $Q(\zeta,\lambda) = N(\zeta,\lambda)/M(\zeta,\lambda)$, where $N(\zeta,\lambda)$ and $M(\zeta,\lambda)$ are scalar polynomial functions in $\zeta^k\lambda^l$, and consider

$$\begin{aligned} J \;&=\; \|R \,-\, U_{\text{out}}\,Q\|^2_{\mathcal{H}^2} \;=\; \|R \,-\, U_{\text{out}}\,\frac{N}{M}\|^2_{\mathcal{H}^2} \\ &=\; \|\frac{1}{M}\,(R\,M \,-\, U_{\text{out}}\,N)\|^2_{\mathcal{H}^2}. \end{aligned} \tag{12}$$

It is desired to find the coefficients of $N$ and $M$ so that $Q = \frac{N}{M}$ belongs to $\mathscr{C}_\mu$ and minimizes $J$.

The difficulty here is that $J$ is *not convex* in the coefficients of $M$. The SM algorithm circumvents this issue by relaxing the objective function (12) to

$$J_{\text{SM}} \;=\; \|\frac{1}{\widetilde{M}}\,(R\,M \,-\, U_{\text{out}}\,N)\|^2_{\mathcal{H}^2},$$

where $\widetilde{M}$ corresponds to $M$ obtained from the *previous* iteration. At each step $J_{\text{SM}}$ is convex in the unknown coefficients, since $N$ and $M$ both appear affinely inside the norm and the norm is a convex function of its argument.

We next describe a state-space method of implementing the SM algorithm.

Consider the problem of minimizing (10) with

$$Q \;=\; e \,+\, \frac{\lambda\,p_1(\zeta) + \lambda^2\,p_2(\zeta) + \cdots + \lambda^\eta\,p_\eta(\zeta)}{1 + \lambda\,q_1(\zeta) + \lambda^2\,q_2(\zeta) + \cdots + \lambda^\mu\,q_\mu(\zeta)},$$

where $\mu > \eta$. $Q$ belongs to $\mathscr{C}_\mu$ with

$$q_k(\zeta) \;=\; \sum_{n=-k}^{k} q_{nk}\,\zeta^n, \qquad p_k(\zeta) \;=\; \sum_{n=-k}^{k} p_{nk}\,\zeta^n, \tag{13}$$

and $e$ is independent of $\zeta$.

Let us introduce a controller canonical form realization of $R - U_{\text{out}}\,Q$,

$$R \,-\, U_{\text{out}}\,Q \;=\; \left[\begin{array}{c|c} \Lambda & \Phi \\ \hline \Psi & \Delta \end{array}\right].$$

Our goal is to minimize

$$\begin{aligned} J \;&=\; \|\left[\begin{array}{c|c} \Lambda(\zeta) & \Phi \\ \hline \Psi(\zeta) & \Delta(\zeta) \end{array}\right]\|^2_{\mathcal{H}^2} \\ &=\; \|\left[\begin{array}{c|c} \Lambda(\zeta) & \Phi \\ \hline \Psi(\zeta) & 0 \end{array}\right]\|^2_{\mathcal{H}^2} \,+\, \|\,\Delta(\zeta)\,\|^2_{\mathcal{H}^2} \\ &=: \; J_{\text{SM}} \,+\, J_\Delta. \end{aligned}$$

We relax the problem of minimizing $J$ to one in which we first minimize $J_\Delta$ and then minimize $J_{\text{SM}}$.

*Minimizing $J_\Delta$*

We find the value of $e$ that minimizes

$$J_\Delta \;=\; \|\,\Delta(\zeta)\,\|^2_{\mathcal{H}^2} \;=\; \frac{1}{2\pi}\int_0^{2\pi} \Delta(\mathrm{e}^{\mathrm{j}\theta})\,\Delta(\mathrm{e}^{\mathrm{j}\theta})^*\,\mathrm{d}\theta.$$

Substituting $\Delta = d_R - d_U\,e$ and setting

$$\frac{\partial}{\partial e} J_\Delta \;=\; 0$$

we obtain

$$e^{\text{SM}} \;=\; \frac{\operatorname{Re}\{\int_0^{2\pi} d_R(\mathrm{e}^{\mathrm{j}\theta})\,d_U(\mathrm{e}^{\mathrm{j}\theta})^*\,d\theta\}}{\int_0^{2\pi} d_U(\mathrm{e}^{\mathrm{j}\theta})\,d_U(\mathrm{e}^{\mathrm{j}\theta})^*\,d\theta}. \tag{14}$$

Note that there is no iteration involved in finding $e^{\text{SM}}$.

*Minimizing $J_{SM}$*

We now minimize $J_{\text{SM}}$ while assuming $e = e^{\text{SM}}$. We consider again the state-space realization

$$\left[\begin{array}{c|c} \Lambda & \Phi \\ \hline \Psi & 0 \end{array}\right],$$

and make the following observations.

(a) Since $q_k(\zeta)$, $k = 1, \ldots, \mu$ appear in the denominator of $R - U_{\text{out}}\,Q$, they also show up inside the matrix $\Lambda$. However, the SM algorithm is based on replacing every $q_k(\zeta)$ with it previous estimate $\widetilde{q}_k(\zeta)$, so that only $\widetilde{q}_k(\zeta)$, $k = 1, \ldots, \mu$ appear in $\Lambda$. This is the key attribute of the SM algorithm and is responsible for rendering the optimization scheme convex.
(b) From (12) it is clear that $q_k(\zeta)$, $k = 1, \ldots, \mu$ and $p_k(\zeta)$, $k = 1, \ldots, \eta$ also appear in the numerator of $R - U_{\text{out}}\,Q$, and thus they show up affinely in the output matrix $\Psi$. We can extract the coefficients $q_{nk}$ and $p_{nk}$ of $q_k(\zeta)$ and $p_k(\zeta)$ from $\Psi$ and form a quadratic problem in these coefficients (since $\Psi$ appears quadratically in the expression of the $\mathcal{H}^2$ norm).

We now describe items (a) and (b) above in more detail. It is known [1] that

$$\begin{aligned} J_{\text{SM}} \;&=\; \left\| \left[\begin{array}{c|c} \Lambda(\zeta) & \Phi \\ \hline \Psi(\zeta) & 0 \end{array}\right] \right\|_{\mathcal{H}^2}^2 \\ &=\; \frac{1}{2\pi}\int_0^{2\pi} \Psi(\mathrm{e}^{\mathrm{j}\theta})\,\Pi(\mathrm{e}^{\mathrm{j}\theta})\,\Psi(\mathrm{e}^{\mathrm{j}\theta})^*\,\mathrm{d}\theta \end{aligned} \tag{15}$$

where $\Pi$ is the solution of the algebraic Lyapunov equation

$$\Lambda(\zeta)\,\Pi(\zeta)\,\Lambda(\zeta)^* \,-\, \Pi(\zeta) \;=\; -\,\Phi\,\Phi^*. \tag{16}$$

Thus the optimization problem has simplified to choosing the coefficients $q_{nk}$ and $p_{nk}$ of $q_k(\zeta)$ and $p_k(\zeta)$ that appear in $\Psi$ so as to minimize $\int_0^{2\pi}\Psi\,\Pi\,\Psi^*\,\mathrm{d}\theta$.[1] Note that since the realization is a controller canonical form, $\Phi$ is a constant matrix independent of $\zeta$ and the unknown parameters.

The output matrix $\Psi(\zeta)$ depends affinely on $q_k(\zeta)$, $p_k(\zeta)$, and $q_k(\zeta)$, $p_k(\zeta)$ depend linearly on their coefficients $q_{nk}$, $p_{nk}$. Therefore it is possible to reorganize $\Psi$ so that it can be written as

$$\Psi(\zeta) \;=\; [\,q_{\text{par}}\;\; p_{\text{par}}\,]\,\Sigma(\zeta) \,+\, \sigma(\zeta), \tag{17}$$

where $q_{\text{par}}$ and $p_{\text{par}}$ denote row vectors stacked with the unknown coefficients $q_{nk}$ and $p_{nk}$ of the denominator and numerator of $Q$, respectively,

$$\begin{aligned} q_{\text{par}} \;&=\; [\,q_{-11},\, q_{01},\, q_{11}\,|\,\cdots\,|\,q_{-\mu\mu},\,\cdots,\,q_{0\mu},\,\cdots,\,q_{\mu\mu}\,], \\ p_{\text{par}} \;&=\; [\,p_{-11},\, p_{01},\, p_{11}\,|\,\cdots\,|\,p_{-\eta\eta},\,\cdots,\,p_{0\eta},\,\cdots,\,p_{\eta\eta}\,]. \end{aligned}$$

Substituting (17) into (15) and assuming that the coefficients $p_{nk}$ and $q_{nk}$ are all real, we arrive at the quadratic problem

$$J_{\text{SM}} \;=\; \frac{1}{2}\,[\,q_{\text{par}}\;\; p_{\text{par}}\,]\,\Gamma\,[\,q_{\text{par}}\;\; p_{\text{par}}\,]^T \,+\, [\,q_{\text{par}}\;\; p_{\text{par}}\,]\,\rho \,+\, \tau,$$

where

$$\Gamma \;=\; \frac{1}{\pi}\int_0^{2\pi} \Sigma\,\Pi\,\Sigma^*\,\mathrm{d}\theta, \tag{18}$$

$$\rho \;=\; \frac{1}{\pi}\,\mathrm{Re}\left\{\int_0^{2\pi} \Sigma\,\Pi\,\sigma^*\,\mathrm{d}\theta\right\}, \tag{19}$$

$$\tau \;=\; \frac{1}{2\pi}\int_0^{2\pi} \sigma\,\Pi\,\sigma^*\,\mathrm{d}\theta. \tag{20}$$

Finally, the SM–optimal values of the parameters are given by setting

$$\frac{\partial}{\partial q_{nk}} J_{\text{SM}} \;=\; 0, \qquad \frac{\partial}{\partial p_{nk}} J_{\text{SM}} \;=\; 0, \qquad \text{for all } \; q_{nk},\, p_{nk}$$

which gives

$$[\,q_{\text{par}}\;\; p_{\text{par}}\,]^{\text{SM}} \;=\; \frac{1}{2}\,\rho\,\Gamma^{-1}. \tag{21}$$

Note that these parameter values are the result of just one iteration and can now be used to initialize the next iteration, and so on.

Let us summarize the state-space SM algorithm.

(1) Compute $e^{\text{SM}}$ from (14). Choose initial values for the coefficients $q_{nk}$, $p_{nk}$.
(2) Set $\widetilde{q}_k(\zeta) = \sum_{n=-k}^{k} q_{nk}\,\zeta^n$, $k = 1, \ldots, \mu$ using the current estimate of the coefficients $q_{nk}$.
(3) Form the matrix $\Lambda(\zeta)$ and solve the algebraic Lyapunov equation (16) to find $\Pi(\zeta)$.
(4) Compute $\Gamma$ and $\rho$ from equations (18)–(19).
(5) Find the next estimate of the coefficients $q_{nk}$, $p_{nk}$ from (21). If $q_k(\zeta) - \widetilde{q}_k(\zeta)$, $k = 1, \ldots, \mu$ and $p_k(\zeta) - \widetilde{p}_k(\zeta)$, $k = 1, \ldots, \eta$ are sufficiently small in norm, stop. Otherwise go to step 2.

## V. Examples

*Example 1*

Let

$$G \;=\; \left[\begin{array}{c|cc} a & 1 & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & 0 & 0 \end{array}\right], \qquad a(\zeta) \;=\; \zeta^{-1}/4 \,+\, 1/4 \,+\, \zeta/4.$$

The system is open-loop stable and we have

$$T \;=\; \begin{bmatrix} \dfrac{\lambda}{1-\lambda\,a} \\ 0 \end{bmatrix}, \qquad U \;=\; -\begin{bmatrix} \dfrac{\lambda}{1-\lambda\,a} \\ 1 \end{bmatrix} \frac{\lambda}{1-\lambda\,a}.$$

Performing an inner-outer factorization on $U$ and carrying out the steps described in Section III, we arrive at

$$R \;=\; d_R \,+\, \frac{\lambda\,c_R}{1-\lambda\,a_R}, \tag{22}$$

$$U_{\text{out}} \;=\; d_U \,+\, \frac{\lambda\,c_{1U} + \lambda^2\,c_{2U}}{(1-\lambda\,a_{1U})(1-\lambda\,a_{2U})}, \tag{23}$$

where

$$\begin{aligned} a_R \;&=\; a, \quad c_R \;=\; 1/(\gamma^* - \kappa^*/a), \quad d_R \;=\; 1/\gamma^*, \\ a_{1U} \;&=\; a, \quad a_{2U} \;=\; a, \quad d_U \;=\; \kappa, \\ c_{1U} \;&=\; 2\,a\,\kappa - \gamma, \quad c_{2U} \;=\; -\,a^2\,\kappa, \end{aligned}$$

and

$$\kappa \;=\; \sqrt{1 + a^*a/2 + \sqrt{1 + (a^*a)^2/4}}, \quad \gamma \;=\; a/\kappa^*.$$

The optimal values of the parameters of $Q \in \mathscr{C}_1$, as given by the SM algorithm, are

$$\begin{aligned} &q_{-1} \;=\; -0.1417, \quad q_0 \;=\; -0.1133, \quad q_1 \;=\; -0.1417, \\ &p_{-1} \;=\; 0.0249, \quad p_0 \;=\; -0.9455, \quad p_1 \;=\; 0.0249, \\ &e \;=\; 0.2667, \end{aligned}$$

[1] We henceforth drop the "$(\mathrm{e}^{\mathrm{j}\theta})$" notation from inside all integrals, with the understanding that any function of $\zeta$ inside an integral is evaluated on the unit circle.

which result in

$$J_{\text{SM}} = \|R - U_{\text{out}}\, Q^{\text{SM}}\|^2_{\mathcal{H}^2} = 0.3943.$$

For this example, the SM algorithm was iterated $40$ times. But the parameters converged to values very close to those given above in less than $5$ iterations.

Note that we do not claim global optimality for the above solution. However, we formed 1000 systems $Q^{\text{pert}}$ by perturbing the parameters of $Q^{\text{SM}}$ around their SM–optimal values, and we observed that $\|R - U_{\text{out}}\, Q^{\text{pert}}\|^2_{\mathcal{H}^2}$ was always larger that $J_{\text{SM}}$.

Finally note that for $Q = 0$ (open-loop system) we have

$$\|R - U_{\text{out}}\, Q\|^2_{\mathcal{H}^2} = \|R\|^2_{\mathcal{H}^2} = 2.6627.$$

*Example 2*

We consider the example given in Voulgaris *et al.* [2]

$$T = \frac{\lambda}{1 - \lambda\, r}, \qquad U = \frac{\lambda^2}{(1-\lambda\,\rho)(1-\lambda\, r)},$$

with

$$\begin{aligned} \rho(\zeta) &= \zeta^{-1}/6 + 1/3 + \zeta/6, \\ r(\zeta) &= \zeta^{-1}/8 + 1/4 + \zeta/8. \end{aligned}$$

The transfer functions $R$ and $U_{\text{out}}$ for this problem have the same form as in (22) and (23) with

$$\begin{aligned} a_R &= r, \quad c_R = r^2, \quad d_R = r, \\ a_{1U} &= \rho, \quad a_{2U} = r, \quad d_U = 1, \\ c_{1U} &= \rho + r, \quad c_{2U} = -\rho\, r. \end{aligned}$$

The optimal values of the parameters of $Q$, as given by the SM algorithm, are

$$\begin{aligned} &q_{-1} = 0.0873, \quad q_0 = 0.1985, \quad q_1 = 0.0873, \\ &p_{-1} = -0.0137, \quad p_0 = -0.0564, \quad p_1 = -0.0137, \\ &e = 0.25, \end{aligned}$$

which result in

$$\|T - UQ^{\text{SM}}\|^2_{\mathcal{H}^2} = 1.0318.$$

This is an improvement on the “truncated 2–relaxed” solution $Q^{\text{Vou}}$ presented in [2], for which

$$\|T - UQ^{\text{Vou}}\|^2_{\mathcal{H}^2} = 1.0659.$$

Let $Q^{\text{opt}}$ denote the globally optimal cone causal $Q$ as discussed at the end of Section III, i.e.,

$$Q^{\text{opt}} = \arg \inf_{\text{cone causal } Q} \|T - UQ\|^2_{\mathcal{H}^2}.$$

Voulgaris *et al.* show that

$$\|T - UQ^{\text{opt}}\|^2_{\mathcal{H}^2} = 1.0157.$$

It can be seen that $Q^{\text{SM}} \in \mathscr{C}_1$ gives a value of the closed-loop $\mathcal{H}^2$ norm that is within 2% of the optimal value.

## VI. Conclusions

We consider the design of optimal distributed controllers with finite communication speed. These are controllers whose impulse response has support inside a cone in the spatio-temporal domain. This problem has been previously considered by [2] in the context of cone causal systems. We part from [2] by searching for the optimal controller parameters in state-space. We achieve convexity by relaxing the optimal control objective, and use an iterative numerical scheme to compute the state-space parameters.

## VII. Appendix

*Proof of Theorem 3:*

The basic idea of the proof can be found in [3]. By Assumption 2 we can find $H$ and $F$ such that $A + HC_y$ and $A + B_uF$ are stable. From [12, Thm. 12.8], [13, Thm. 5.4.1] all stabilizing controllers ($\mathscr{C}$ causal or not) can be parameterized by

$$\begin{aligned} K &= J_{11} + J_{12}\, Q\, (I - J_{22}\, Q)^{-1}\, J_{21}, \\ J &= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A + B_uF + HC_y & -H & B_u \\ \hline F & 0 & I \\ -C_y & I & 0 \end{array}\right], \\ Q & \ \text{stable}, \end{aligned}$$

and any $K$ found from the above relation is stabilizing if and only if its corresponding $Q$ is stable.

Next we bring into consideration the spatial structure of $K$ and $Q$, and show that the mapping $Q \mapsto K$ is a bijection on $\mathscr{C}$.

From $G \in \mathscr{C}$, Assumption 2 on the matrices $H$ and $F$, and the state-space representation of $J$, it follows that $J \in \mathscr{C}$. Now, assume $Q \in \mathscr{C}$. Since $K$ is given by a linear fractional transformation of $Q$ with coefficients $J_{ij} \in \mathscr{C}$, $i, j = 1, 2$ then $K \in \mathscr{C}$. Conversely, assume $K \in \mathscr{C}$. From [13, Thm. 5.4.1] we have

$$Q = J_{12}^{-1}\,(K - J_{11})\, J_{21}^{-1}\,[I + J_{12}^{-1}\,(K - J_{11})\, J_{21}^{-1}\, J_{22}]^{-1}.$$

Since $J_{ij} \in \mathscr{C}$, $i, j = 1, 2$ then $Q \in \mathscr{C}$. The proof is thus complete. ■

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