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Remarks on computing the $\mathcal{H}_2$ norm of incompressible fluids using descriptor state-space formulation

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Abstract— **This paper utilizes descriptor state-space formulation for computation of energy amplification in incompressible channel flows. The dynamics of velocity and pressure fluctuations in these flows are described by a system of partial differential-algebraic equations. Typically, the evolution model is obtained by projecting the velocity fluctuations on a divergence-free subspace which eliminates pressure from the equations. This procedure results into a standard state-space representation and the problem of quantifying receptivity of velocity fluctuations to stochastic exogenous disturbances is solved using well-known $\mathcal{H}_2$ formalism. In this paper, however, it is shown how energy amplification can be computed directly from the original system of the linearized Navier-Stokes and continuity equations. This approach avoids the need for finding the evolution model which is advantageous in many applications.**

Index Terms— **$\mathcal{H}_2$ norm; descriptor systems; distributed systems; incompressible fluids; Navier-Stokes equations.**

I. Introduction

Descriptor formulation of systems often arises from formulating the underlying equations in terms of their natural physical variables [1]. A particular class of physical problems that can be written in descriptor formulation are those with algebraic constraints. For example, systems of incompressible fluids require the velocity fluctuations to lie on a divergence-free subspace, hence, resulting in a differential-algebraic equation.

It is a standard task in fluid mechanics to eliminate the algebraic constraint from the equations and rewrite them in a standard state-space formulation [2]. However, working with the original descriptor formulation is advantageous in many applications where obtaining the evolution formulation is cumbersome and often results in a more complicated representation in terms of the order of differential operators involved.

In this paper, we consider channel flow of incompressible fluids and show how the problem at hand can be approached, alternatively, using the available theory of computing the $\mathcal{H}_2$ norm of finite dimensional descriptor systems [3], [4].

The paper is organized as follows: an introduction to finite dimensional descriptor systems is provided in § II. We briefly review the theory of computing the $\mathcal{H}_2$ norm of finite dimensional descriptor systems in § III. The system of incompressible channel flow is formulated in both descriptor and standard state-space formulations in § IV. In § V, we show how the underlying operators are approximated using collocation scheme. Comparison between the results obtained using the standard and descriptor formulations is given in § VI and concluding remarks are provided based on the above comparison.

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II. Preliminaries

Descriptor formulation is particularly suitable for defining systems of ordinary differential equations in time with algebraic constraints.

We consider the following linear time-invariant system

$$\begin{aligned} E\,\partial_t\psi(x,t) &= A\,\psi(x,t) + B\,u(x,t), \\ \phi(x,t) &= C\,\psi(x,t), \end{aligned} \tag{1}$$

where x and t denote spatial and temporal coordinates, ψ, u, and ϕ denote the spatio-temporal system state, input, and output, respectively, and E, A, B, and C are bounded operator valued matrices of appropriate dimensions. We denote by ∂_t the first derivative operator in time.

Remark 1: We note that if E is nonsingular, system (1) can be transformed into standard state-space representation by pre-multiplying the first equation by E^{-1} from the left. However, if E contains differential operators in x, one should be careful about implementation of boundary conditions when inverting E. This can be very difficult especially for problems involving complicated geometry or boundary conditions. Therefore, even when E is nonsingular, definition and analysis of certain systems in descriptor form is preferred.

The spatial differential operators involved in system (1) together with their boundary conditions can be approximated in descriptor form in a systematic way. However, this approximation is not always trivial if one decides to transform the system into a standard state-space representation by eliminating the constraints from the equations. Moreover, approximating the boundary conditions numerically may result in overspecification of the boundary conditions [5].

For computational purposes, we numerically approximate the underlying differential operators in (1). Once we obtain finite dimensional approximation of the infinite dimensional descriptor system (1), we can use the theory developed for the finite dimensional descriptor systems.

A. Finite dimensional descriptor form

Consider the following linear time-invariant system

$$E\,\frac{\mathrm{d}}{\mathrm{d}\,t}\psi(t) = A\,\psi(t) + B\,u(t), \quad \phi(t) = C\,\psi(t), \tag{2}$$

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where ψ, u, and ϕ denote system state vector, input, and output, respectively, and E, A, B, and C are matrices of appropriate dimensions. We assume that the pencil $(s\,E - A)$ is regular, i.e., there is a $s \in \mathbb{C}$ for which the resolvent operator $(s\,E - A)^{-1}$ exists. Then, $(s\,E - A)^{-1}$ can be uniquely written in terms of its Laurent parameters around $s = \infty$ [4]

$$(s\,E - A)^{-1} = s^{-1} \sum_{k=-\nu}^{\infty} \zeta_k\, s^{-k}, \tag{3}$$

where ζ_k denotes the Laurent parameters and ν is called the nilpotency index of the pencil $(s\,E - A)$. The Laurent parameters are very useful in analysis of descriptor systems because they separate the eigenspaces associated with finite (slow) and infinite (fast) generalized eigenvalues of the pencil $(s\,E - A)$ [3]. Another important property of the Laurent parameters is that $P_r := \zeta_0\,E$ and $Q_r := -\zeta_{-1}\,A$ are projections on the space spanned by the eigenvectors ξ_i corresponding with the finite and infinite eigenvalues λ_i of the eigenvalue problem $\lambda_i\,E\,\xi_i = A\,\xi_i$, respectively. Also $P_l := E\,\zeta_0$ and $Q_l := -A\,\zeta_{-1}$ are projections on the space spanned by the eigenvectors ξ_i corresponding with the finite and infinite eigenvalues λ_i of the eigenvalue problem $\lambda_i\,\xi_i\,E = \xi_i\,A$, respectively [4].

The Laurent parameters are determined from the elements of the Weierstrass canonical form of the pair (A, E). However, it is well-known that canonical forms involving Jordan structures are not numerically robust. Therefore, for computational purposes, it suffices to consider the *Weierstrass-like* canonical form of the pair (A, E) [6].

Weierstrass-like form: For a pair (A, E), one can find invertible matrices U and V such that

$$E = V \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} U, \quad A = V \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} U, \tag{4}$$

where E_f, A_f, and A_∞ are upper triangular invertible matrices and E_∞ is upper triangular and nilpotent. In the new coordinate system, we have

$$\psi = U^{-1} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},\ B = V \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} U,$$

where ψ_1 and ψ_2 denote slow (causal) and fast (impulsive or noncausal) parts of the state vector, respectively. The projections on the fast and slow subspaces in Weierstrass-like form are determined from

$$P_l = V \begin{bmatrix} E_f & 0 \\ 0 & 0 \end{bmatrix} V^{-1},\ P_r = U^{-1} \begin{bmatrix} E_f & 0 \\ 0 & 0 \end{bmatrix} U,$$
$$Q_l = V \begin{bmatrix} 0 & 0 \\ 0 & -A_\infty \end{bmatrix} V^{-1},\ Q_r = U^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -A_\infty \end{bmatrix} U.$$

Details regarding computation of the Weierstrass-like form is included in § III-A.

III. Computing the $\mathcal{H}_2$ norm

The $\mathcal{H}_2$ norm of (2) is determined by [4], [6], [7]

$$\|\mathcal{H}\|_2^2 = \operatorname{trace}(C\,(G_c + G_{nc})\,C^*), \tag{5}$$

where G_c and G_{nc}, the causal and non-causal reachability Graminas, satisfy the following generalized Lyapunov equations

$$\begin{aligned} E\,G_c\ A^* + A\,G_c\ E^* &= -P_l\,B\,B^*\,P_l^*, \\ E\,G_{nc}\,E^* - A\,G_{nc}\,A^* &= \quad Q_l\,B\,B^*\,Q_l^*. \end{aligned} \tag{6}$$

Under certain reachability conditions for the pairs (A_f, B_1) and (E_∞, B_2), unique projected solutions to (6) can be obtained by the following projections [4], [6]

$$G_c = P_r\,G_c\,P_r, \qquad G_{nc} = Q_r\,G_{nc}\,Q_r. \tag{7}$$

The approach towards solving the generalized Lyapunov equations relies heavily on the use of Weierstrass canonical form. Therefore, we first see how the Weierstrass transformation is obtained.

A. Obtaining the Weierstrass-like form

We use MATLAB's QZ algorithm to compute the generalized Schur form of (A, E) such that

$$E = W \begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix} T, \quad A = W \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix} T, \tag{8}$$

where E_f, A_f, and A_∞ are nonsingular upper-triangular matrices and E_∞ is nilpotent upper-triangular. To this end, one needs to use ordqz.m function in order to put all the finite generalized eigenvalues of (A, E) in E_f.

The invertible matrices U and V in the Weierstrass-like form (4) are determined by

$$V = W \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}, \quad U = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} T,$$

where Z and Y are obtained from the following system of generalized Sylvester equation [6]

$$\begin{aligned} E_f\,Y - Z\,E_\infty &= -E_u, \\ A_f\,Y - Z\,A_\infty &= -A_u. \end{aligned} \tag{9}$$

A fast algorithm for solving the system of Sylvester equations is given in [8].

B. Solving the generalized Lyapunov equation

We use the Weierstrass-like form discussed in § II-A to reduce the generalized projected Lyapunov equations (6,7) to standard Lyapunov equations. This is not a trivial task, without using the Weierstrass transformation.

The causal reachability Gramian is determined by

$$E\,P_r\,G_c\,P_r^*\,A^* + A\,P_r\,G_c\,P_r^*\,E^* = -P_l\,B\,B^*\,P_l^*.$$

We substitute the underlying matrices with their Weierstrass-like form. After initial manipulation, we get

$$\begin{bmatrix} E_f\,E_f & 0 \\ 0 & 0 \end{bmatrix} \tilde{G}_c \begin{bmatrix} E_f^*\,A_f^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_f\,E_f & 0 \\ 0 & 0 \end{bmatrix} \tilde{G}_c \begin{bmatrix} E_f^*\,E_f^* & 0 \\ 0 & 0 \end{bmatrix}$$
$$= -\begin{bmatrix} E_f & -E_f\,Z \\ 0 & 0 \end{bmatrix} \tilde{F} \begin{bmatrix} E_f^* & 0 \\ -Z^*\,E_f^* & 0 \end{bmatrix}, \tag{10}$$

where

$$\tilde{G}_c \;:=\; \begin{bmatrix} \tilde{G}_{c1} & \tilde{G}_{c2} \\ \tilde{G}_{c3} & \tilde{G}_{c4} \end{bmatrix} \;:=\; U\,G_c\,U^*,$$
$$\tilde{F} \;:=\; \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ \tilde{F}_3 & \tilde{F}_4 \end{bmatrix} \;:=\; W^*\,B\,B^*\,W.$$

Note that because of the projections (7), only $\tilde{G}_{c1}$ survives in (10) and we obtain the following standard Lyapunov equation

$$\begin{aligned}(E_f^{-1}\,A_f)\,\hat{G}_{c1} \,+\, \hat{G}_{c1}\,(E_f^{-1}\,A_f)^* \;=\; \\ -(\tilde{F}_1 \,-\, \tilde{F}_2\,Z^* \,-\, Z\,\tilde{F}_3 \,+\, Z\,\tilde{F}_4\,Z^*),\end{aligned} \tag{11}$$

where $\hat{G}_{c1} := E_f\,\tilde{G}_{c1}\,E_f^*$.

Therefore, $\tilde{G}_{c1} \;=\; E_f^{-1}\,\hat{G}_{c1}\,E_f^{-*}$. The projection defined in (7) requires all the other blocks of $\tilde{G}_c$ to be zero. Finally,

$$G_c \;=\; U^{-1}\,\tilde{G}_c\,U^{-*} \;=\; T^*\begin{bmatrix} \tilde{G}_{c1} & 0 \\ 0 & 0 \end{bmatrix} T. \tag{12}$$

Note that uniqueness of $\tilde{G}_c$ is a direct consequence of the uniqueness of the solution of the standard Lyapunov equation (11).

Similar result is obtained for the noncausal reachability Gramian determined by

$$E\,Q_r\,G_{nc}\,Q_r^*\,E^* \,+\, A\,Q_r\,G_{nc}\,Q_r^*\,A^* \;=\; Q_l\,B\,B^*\,Q_l^*.$$

After similar manipulations to those carried to obtain G_c, we have

$$G_{nc} \;=\; U^{-1}\,\tilde{G}_{nc}\,U^{-*} \;=\; T^*\begin{bmatrix} Y\,\tilde{G}_{nc4}\,Y^* & Y\,\tilde{G}_{nc4} \\ \tilde{G}_{nc4}\,Y^* & \tilde{G}_{nc4} \end{bmatrix} T, \tag{13}$$

where $\tilde{G}_{nc4} \;=\; A_\infty^{-1}\,\hat{G}_{nc4}\,A_\infty^{-*}$ and $\tilde{G}_{nc4}$ is obtained by solving the following standard Lyapunov equation

$$(A_\infty^{-1}\,E_\infty)\,\hat{G}_{nc4}\,(A_\infty^{-1}\,E_\infty)^* \,-\, \hat{G}_{nc4} \;=\; \tilde{F}_4. \tag{14}$$

Below, we summarize the procedure of computing the $\mathcal{H}_2$ norm for the descriptor system (2)

1) Compute the generalized Schur form of the pair (A, E) given in (8).
2) Solve the system of generalized Sylvester equation (9) for Y and Z.
3) Compute G_c and G_{nc} from (11)-(14).
4) Finally, the $\mathcal{H}_2$ norm is determined by (5).

IV. Incompressible plane channel flow

We consider motion of incompressible Newtonian fluids between two infinite planes shown in Fig. 1. The linearized Navier-Stokes equations govern evolution of velocity and pressure fluctuations $(\mathbf{v}, p)$ around nominal velocity and pressure $(\bar{\mathbf{u}}, P)$ in the presence of forcing fluctuations $\mathbf{d}$, where $\mathbf{v} := \begin{bmatrix} u & v & w \end{bmatrix}^T$ and $\mathbf{d} := \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}^T$. We assume the following nominal velocity profile

$$\bar{\mathbf{u}} \;=\; \begin{bmatrix} U(y) & 0 & 0 \end{bmatrix}^T, \quad U(y) \;=\; 1 \,-\, y^2.$$

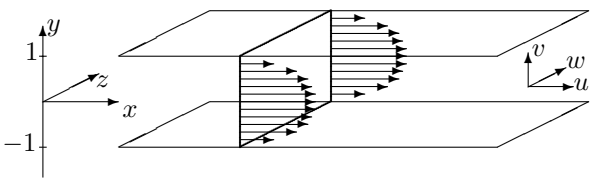

Fig. 1. Three dimensional channel flow.

Navier-Stokes equations for velocity fluctuations of incompressible fluids linearized around above profile are given by

$$\begin{aligned}\partial_t\,u \;&=\; (\tfrac{1}{R}\,\Delta \,-\, U\,\partial_x)\,u \,-\, U'\,v \,-\, \partial_x\,p \,+\, d_1,\\ \partial_t\,v \;&=\; (\tfrac{1}{R}\,\Delta \,-\, U\,\partial_x)\,v \,-\, \partial_y\,p \,+\, d_2,\\ \partial_t\,w \;&=\; (\tfrac{1}{R}\,\Delta \,-\, U\,\partial_x)\,w \,-\, \partial_z\,p \,+\, d_3,\end{aligned} \tag{15}$$

$$0 \;=\; \partial_x\,u \,+\, \partial_y\,v \,+\, \partial_z\,w, \tag{16}$$

where $\Delta := \partial_{xx} \,+\, \partial_{yy} \,+\, \partial_{zz}$ is the Laplacian operator with homogenous Dirichlet boundary conditions and R is the Reynolds number. Equation (16) is an algebraic constraint in time, i.e., it does not involve partial derivatives in time.

The set of equations (15) and (16) can be summarized in the following descriptor formulation

$$E\,\partial_t\,\psi \;=\; A\,\psi \,+\, B\,\mathbf{d}, \qquad \mathbf{v} \;=\; C\,\psi, \tag{17}$$

where $\psi \;:=\; \begin{bmatrix} \mathbf{v} & p \end{bmatrix}^T$ denotes vector of system states. Moreover,

$$E \;:=\; \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\; A \;:=\; \begin{bmatrix} \bar{A} & -D^T \\ D & 0 \end{bmatrix},\; B \,=\, C^T \;:=\; \begin{bmatrix} I \\ 0 \end{bmatrix},$$

where

$$\bar{A} \;:=\; \begin{bmatrix} \frac{1}{R}\,\Delta \,-\, U\,\partial_x & U' & 0 \\ 0 & \frac{1}{R}\,\Delta \,-\, U\,\partial_x & 0 \\ 0 & 0 & \frac{1}{R}\,\Delta \,-\, U\,\partial_x \end{bmatrix},$$

and $D := \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix}$.

Since the underlying differential operators are spatially invariant in x and z directions, one can apply spatial Fourier transform in x and z and algebrize the differential operators in these directions. Differential operators in y direction will be approximated numerically. Details of the numerical issues are discussed in § V.

With a slight abuse of notation, we denote by (17) both physical and frequency representations of our system. The choice of either is clear from the context. Therefore, the underlaying operators in frequency domain are determined from

$$\bar{A} \;:=\; \begin{bmatrix} \frac{1}{R}\,\Delta \,-\, \mathrm{i}\,k_x\,U & U' & 0 \\ 0 & \frac{1}{R}\,\Delta \,-\, \mathrm{i}\,k_x\,U & 0 \\ 0 & 0 & \frac{1}{R}\,\Delta \,-\, \mathrm{i}\,k_x\,U \end{bmatrix},$$
$$D \;:=\; \begin{bmatrix} \mathrm{i}\,k_x\,I & \partial_y & \mathrm{i}\,k_z\,I \end{bmatrix},$$

where $\Delta := -k^2 + \partial_{yy}$ and k_x and k_z denote wave numbers in x and z directions, respectively, $k^2 \;:=\; k_x^2 \,+\, k_z^2$ and $\mathrm{i} = \sqrt{-1}$.

The $\mathcal{H}_2$ norm of system (17) is interpreted as energy

amplification of stochastic disturbances which are white zero-mean in y and t and are harmonic in x and z coordinates

$$E(k_x, k_z) := \operatorname{trace}\left(\lim_{t\to\infty} \mathcal{E}\left\{\mathbf{v}(k_x,\cdot,k_z,t) \otimes \mathbf{v}(k_x,\cdot,k_z,t)\right\}\right),$$

where $\mathcal{E}$ is the expectation operator, and $\otimes$ denotes the tensor product operator. Another interpretation of $E(k_x, k_z)$ is that it determines energy of the impulse response of (17)

$$E(k_x, k_z) := \int_0^\infty \operatorname{trace}\left(\mathbf{v}(k_x,\cdot,k_z,t) \otimes \mathbf{v}(k_x,\cdot,k_z,t)\right) \mathrm{d}t.$$

In § VI, we utilize the method discussed in § III to compute the $\mathcal{H}_2$ norm of linearized plane channel flow given in descriptor form (17). We will test our results by comparing the $\mathcal{H}_2$ norm obtained from the descriptor formulation of the system with the $\mathcal{H}_2$ norm obtained from the system formulated in standard state-space formulation.

It is a standard task in fluid mechanics to write (15) and (16) in standard state-space formulation [2]. This is done by eliminating pressure p from (15) by writing p in terms of velocity fluctuations and then using the algebraic constraint (16) to reduce the number of unknown fields to two. Therefore, only two state fields suffice to fully represent the system in standard formulation. A common choice of state variables is $\Psi := \begin{bmatrix} v & \eta \end{bmatrix}^T$, where $\eta := u_z - w_x$ is the wall-normal vorticity. The standard state-space formulation of (17) is determined by

$$\partial_t \Psi = \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d}, \qquad \mathbf{v} = \mathcal{C}\,\Psi, \tag{18}$$

where

$$\mathcal{A} := \begin{bmatrix} -\Delta^{-1}(\mathrm{i}\,k_x\,(U\,\Delta + U'') + \frac{1}{R}\,\Delta^2) & 0 \\ -\mathrm{i}\,k_z\,U' & -\mathrm{i}\,k_x\,U + \frac{1}{R}\,\Delta \end{bmatrix},$$

$$\mathcal{B} := \begin{bmatrix} -\mathrm{i}\,k_x\,\Delta^{-1}\,\partial_y & -k^2\,\Delta^{-1} & -\mathrm{i}\,k_z\,\Delta^{-1}\,\partial_y \\ \mathrm{i}\,k_z & 0 & -\mathrm{i}\,k_x \end{bmatrix},$$

$$\mathcal{C} := \frac{1}{k^2}\begin{bmatrix} \mathrm{i}\,k_x\,\partial_y & k^2 & \mathrm{i}\,k_z \\ -\mathrm{i}\,k_z & 0 & \mathrm{i}\,k_x \end{bmatrix}^T,$$

where $\Delta^2 := \partial_{yyyy} + 2\,k^2\,\partial_{yy} + (k^2)^2$ with Dirichlet and Neumann boundary conditions.

Remark 2: We note that while the number of elements of the state vector in standard formulation is two, the descriptor formulation requires four elements in its state vector. Therefore, assuming that each of these infinite dimensional elements are approximated with vectors of the same length, the problem size in descriptor formulation will have twice the size of the problem in standard formulation. In spite of the above mentioned advantage of the standard formulation, one sees that the entries in matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are substantially more complicated operators than those in E, A, B, or C. We note the following

- The operator-valued matrices in the standard formulation involve differential operators with two degrees higher than those in the descriptor formulation. Moreover, they contain integral operators like Δ^{-1} which add to complexity of the standard representation. We note that these issues are treated very well for problems with certain boundary conditions such as Dirichlet and Neumann boundary conditions.
- We also note that we have arrived at the standard formulation after certain analytical and algebraic manipulations on the descriptor formulation of the system [9]. For problems with more complicated nominal velocity profiles that involve components which also vary along other spatial coordinates such as x and z, this is an arduous undertaking [10], [11].

Therefore, simplicity of defining systems with algebraic constraints in the descriptor formulation together with reduction in the order of numerically approximated differential operators serve as important motivations for development of efficient analysis tools for these systems. In § VI, we show that the $\mathcal{H}_2$ norm obtained from the descriptor formulation of plane channel flow compares very well to that obtained from the standard formulation.

V. Numerical approximation of spatial operators

Over the past decades, many different schemes are developed for numerical approximation of differential operators, in particular those with non-periodic boundary conditions [12]. These boundary conditions arise in applications with finite domains where the differential operators lack invariance under spatial shifting transformations. Many fluid systems with bounded geometry like the channel flow system are of this kind.

We use the matrix differentiation suite developed by Weideman and Reddy [13] to numerically approximate differential equations in wall-normal direction, y. This tools is based on computation of differentiation matrices using collocation method. Chebyshev polynomials are selected as basis functions, since these polynomials are most appropriate for bounded non-periodic domains.

In collocation methods, one-to-one mapping is established between the selected basis function and a non-uniform set of grid points chosen such that the approximation error decays exponentially as the number of grid points increases. Therefore, convergence rate of spectral methods $(\mathcal{O}(\mathrm{e}^{-c\,N}))$ are by far superior to that of finite difference methods $(\mathcal{O}(N^{-c}))$, where N is the number of degrees of freedom in the expansion series for spectral methods or the number of grid points in the case of finite difference and spectral collocation methods.

Boundary conditions are implemented either with Galerkin schemes based on choice of Chebyshev polynomials or with boundary bordering method that eliminates first and last rows and columns of the differentiation matrix in the case that boundary points satisfy the boundary conditions.

A. Operators and their adjoints

We consider $(N+2)$ grid points to approximate differential operators in wall-normal direction using the Galerkin spectral collocation scheme with Chebyshev polynomials. Effectively, functions and operators are approximated by finite vectors and matrices, respectively. We note the following

- Elements of the velocity and vorticity fields (u, v, w, η) satisfy Dirichlet boundary conditions and the first and last entries in their corresponding vector approximation are always zero and they can be removed. Therefore, the velocity and vorticity fields are approximated with N-dimensional vectors. On the other hand, there is no boundary condition on pressure p, nor there is one on input $\mathbf{d}$. So, the pressure and input fields are approximated with $(N+2)$-dimensional vectors.
- The first and last columns of the differentiation matrices acting on these functions can be removed due to the Dirichlet boundary conditions. Also the first and last rows of the operators resulting in velocity and vorticity fields can be removed.

Therefore, for example, E and A are $(4N+2)\times(4N+2)$, $\bar{A}$ is $3N\times 3N$, D is $(N+2)\times 3N$, and B is $(4N+2)\times(3N)$, etc. Note that we have used the same notation for the functions and operators and their numerical approximations.

Operator adjoints are needed for the purpose of norm computations. To obtain the matrices approximating operator adjoints, one can analytically find the adjoint operators and then approximate them numerically. An alternative way is computing the adjoint operators directly from approximated operators.

Adjoints of A and E are determined from

$$\langle \psi_1, A\,\psi_2\rangle \;=\; \left\langle A^+\,\psi_1, \psi_2\right\rangle, \quad \langle \psi_1, E\,\psi_1\rangle \;=\; \left\langle E^+\,\psi_1, \psi_2\right\rangle, \tag{19}$$

where $^+$ denotes the adjoint operator and $\langle\cdot,\cdot\rangle$ denotes the $L^2[-1,1]$ inner product

$$\langle f, g\rangle \;:=\; \int_{-1}^{1} f^*\, g\,\mathrm{d}y. \tag{20}$$

To numerically approximate the integral in (20), we use the integration matrix that contains appropriate integrating weights to account for the non-uniform distribution of the grid points in y [14]. Therefore, numerically

$$\langle f, g\rangle \;\approx\; f^*\,S\,g \;=\; \sum_{i\,=\,1}^{N} f(y_i)\,s_i\,g(y_i), \tag{21}$$

where y_i's denote selected grid points, s_i's denote the appropriate integrating weights, and S is the positive definite diagonal matrix containing the integrating weights, s_i. We note that size of the integration weights are determined by size of the corresponding numerically approximated functions, i.e., N for (u, v, w) and $(N+2)$ for p. Let $S(N)$ be the appropriate integrating weights for functions approximated by N grid points. To compute the necessary integrals needed for computing the adjoints numerically, we need the following integrating weight matrices

$$S_1 \;:=\; \begin{bmatrix} S(N) & 0 & 0 \\ 0 & S(N) & 0 \\ 0 & 0 & S(N) \end{bmatrix},$$

$$S_2 \;:=\; \begin{bmatrix} S(N) & 0 & 0 & 0 \\ 0 & S(N) & 0 & 0 \\ 0 & 0 & S(N) & 0 \\ 0 & 0 & 0 & S(N+2) \end{bmatrix}.$$

From (19) and (21), we have

$$\psi_1^*\,S_2\,A\,\psi_2 \;=\; (A^+\,\psi_1)^*\,S_2\,\psi_2.$$

Finally, adjoints of A and E are determined by

$$A^+ \;=\; S_2^{-1}\,A^*\,S_2, \qquad E^+ \;=\; S_2^{-1}\,E^*\,S_2.$$

Adjoint of ζ_k can be obtained from A^+ and E^+ from (3). Adjoint of the projection operators onto subspaces associated with the finite and infinite eigenvalues of (A, E) are obtained similarly. For example, ζ_k^+ and P_r^+ are numerically approximated by

$$\begin{aligned} \zeta_k^+ \;&=\; S_2^{-1}\,\zeta_k^*\,S_2, \\ P_r^+ \;&=\; (\zeta_0\,E)^+ \;=\; E^+\,\zeta_0^+ \;=\; S_2^{-1}\,P_r^*\,S_2. \end{aligned}$$

Adjoints of B and C are determined form

$$\langle \psi, B\,\mathbf{d}\rangle \;=\; \left\langle B^+\,\psi, \mathbf{d}\right\rangle, \quad \langle \mathbf{v}, C\,\psi\rangle \;=\; \left\langle C^+\,\mathbf{v}, \psi\right\rangle.$$

and are numerically approximated by

$$B^+ \;=\; S_1^{-1}\,B^*\,S_2, \qquad C^+ \;=\; S_2^{-1}\,C^*\,S_1.$$

Remark 3: Note that the appropriate inner products for definition of $\mathcal{A}^+$, $\mathcal{B}^+$, and $\mathcal{C}^+$ in the standard state-space formulation (18) are different from that in (20). This is because the state vector, Ψ, is not in $L^2[-1,1]$. Therefore, one should use a weighted inner product that yields the definition of energy of velocity fluctuations. See [9] for more details.

B. Change of variables

The method of computing the $\mathcal{H}_2$ norm discussed in § III is based on finite dimensional descriptor formulation in § II-A. Therefore, in § V-A, we showed how the differentiation operators and their adjoints are approximated numerically. In order to compute the $\mathcal{H}_2$ norm of the infinite dimensional system (17), we need to compute solutions of generalized projected Lyapunov equations that involve numerical approximation of operator adjoints. However, we saw that numerical approximation of the adjoint operators are not equal to complex conjugate transpose of numerically approximated operators. Rather, to obtain the adjoint operators, one should pre and post multiply the operators by symmetric positive definite matrices S_i^{-1} and S_i of appropriate size, $i = 1, 2$.

In order to be able to utilize the tools developed for solving the finite dimensional generalized Lyapunov equations in (6), we introduce the following change of variables

$$\begin{aligned} E_s \;&:=\; S_2^{\frac{1}{2}}\,E\,S_2^{-\frac{1}{2}}, \quad A_s \;:=\; S_2^{\frac{1}{2}}\,A\,S_2^{-\frac{1}{2}}, \\ B_s \;&:=\; S_2^{\frac{1}{2}}\,B\,S_1^{-\frac{1}{2}}, \quad C_s \;:=\; S_1^{\frac{1}{2}}\,C\,S_2^{-\frac{1}{2}}, \end{aligned} \tag{22}$$

where $S_i^{\frac{1}{2}}$ denotes matrix square root for a positive definite

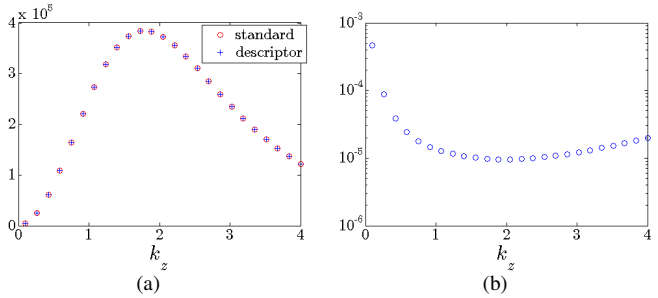

Fig. 2. (a) $\mathcal{H}_2$ norm computed from descriptor and standard state-space formulations for $R = 2000$ and $k_x = 0$, (b) relative error.

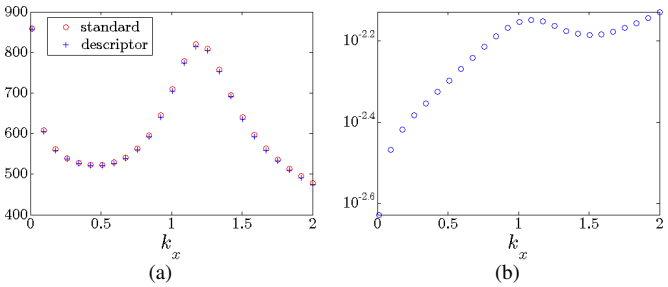

Fig. 3. (a) $\mathcal{H}_2$ norm computed from descriptor and standard state-space formulations for $R = 2000$ and $k_z = 0$, (b) relative error.

matrix S_i, i.e., $(S_i^{\frac{1}{2}})^2 = S_i$. The same type of change of variable as that for A_s and E_s is done on projection operators like P_r.

The change of variables in (22), effectively connects the solution of the following generalized Lyapunov equations

$$\begin{cases} E\,G_c\;A^+ + A\,G_c\;E^+ &= -P_l\,B\,B^+\,P_l^+, \\ E\,G_{nc}\,E^+ - A\,G_{nc}\,A^+ &= \quad Q_l\,B\,B^+\,Q_l^+, \end{cases} \tag{23}$$

$$\begin{cases} E_s\,G_{cs}\;A_s^* + A_s\,G_{cs}\;E_s^* &= -P_{ls}\,B_s\,B_s^*\,P_{ls}^*, \\ E_s\,G_{ncs}\,E_s^* - A_s\,G_{ncs}\,A_s^* &= \quad Q_{ls}\,B_s\,B_s^*\,Q_{ls}^*, \end{cases} \tag{24}$$

via the following formulae

$$G_c = S_2^{-\frac{1}{2}}\,G_{cs}\,S_2^{\frac{1}{2}}, \qquad G_{nc} = S_2^{-\frac{1}{2}}\,G_{ncs}\,S_2^{\frac{1}{2}}. \tag{25}$$

Therefore, to solve the generalized Lyapunov equations involving adjoint matrices (23), we first solve the generalized Lyapunov equations (24) involving complex conjugate transpose of operators using the method discussed in § III-B, and then we obtain the desired solutions using (25).

VI. Results and Concluding Remarks

We compute the $\mathcal{H}_2$ norm of plane channel flow both from descriptor and standard state-space formulations, equations (17) and (18), respectively. We set $R = 2000$ and $N = 30$ in all computations.

Fig. 2 shows the $\mathcal{H}_2$ norm computed from both formulations as a function of k_z for $k_x = 0$. We see the the results match with less than 10^{-3} relative error.

Remark 4: It turns out that the $\mathcal{H}_2$ norm associated with the non-causal or impulsive response of system (17) is very small ($\mathcal{O}(10^{-4})$) and is negligible in comparison with the $\mathcal{H}_2$ norm associated with the causal part. Therefore, only the summation of the two is shown and we have not shown separate plots for the causal and noncausal parts.

Fig. 3 shows the $\mathcal{H}_2$ norm computed from both formulations as a function of k_x for $k_z = 0$. The results are very close with less than 10^{-2} relative error.

From the above results, we see that the $\mathcal{H}_2$ norm computed from the descriptor formulation compares very well with the $\mathcal{H}_2$ norm computed from the standard state-space formulation.

We note that size of the two standard Lyapunov equations (11) and (14) are determined by the number of finite and infinite generalized eigenvalues of (A, E), respectively. In our problem with $N = 30$, these numbers turn out to be 58 and 64. Size of the Lyapunov equation that needs to be solved in standard formulation is equal to 60. Thus, cost of solving the required Lyapunov equations in both formulations is almost the same. Therefore, the difference in computational cost of obtaining the $\mathcal{H}_2$ norm from the two formulations is mainly determined by the cost of solving the additional Lyapunov equation plus cost of the QZ algorithms required for obtaining the Weierstrass transformation and solving the system of generalized Sylvester equations (9).

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