

# Performance of leader-follower networks in directed trees and lattices

Fu Lin, Makan Fardad, and Mihailo R. Jovanović

**Abstract**—We study the performance of externally forced leader-follower networks in directed trees and lattices. By exploiting the lower triangular structure of Laplacian matrices of both classes of graphs, we derive explicit formulae for the transfer function from disturbances to the states of the nodes. For directed trees, we show that the worst-case componentwise amplification of disturbances is achieved at zero temporal frequency and that it is a convex function of edge weights. For directed 1D and 2D lattices, we study the steady-state variance distribution in networks with leaders placed on the boundary. We show that as one moves away from leaders, the variance of the followers scales as a square-root function of node indices in 1D lattices and as a logarithmic function along the diagonal nodes in 2D lattices.

**Index Terms**—Convex optimization, directed lattices, directed trees, Laplacian matrices, leader-follower networks, lower triangular matrices.

## I. INTRODUCTION

Systems over graphs arise in many emerging applications ranging from control of vehicular formations, to distributed estimation in sensor networks, to synchronization of networks of oscillators [1]–[11]. Recent studies have established fundamental performance limitations in the control of these multi-agent systems [3]–[11]. Even though most of these studies employ undirected graphs to model the interconnections between subsystems, several references have demonstrated that departing from undirected information patterns can significantly improve performance of dynamic networks [6]–[8], [11]. In spite of this appealing feature, analysis and design problems for systems on directed graphs are in general more challenging than those for systems on undirected graphs. Several research efforts have thus focused on identifying classes of directed networks that are more amenable to analysis and design [8], [12]. Our paper is a step in this direction.

We focus on two classes of directed graphs, namely, on directed trees and directed lattices. We exploit the lower triangular structure of Laplacian matrices arising in these graphs, and examine performance of the corresponding leader-follower networks. The major contributions of this paper are summarized as follows.

- For directed trees, we show that the sparsity structure of the inverse of reduced Laplacian matrices (obtained by removing rows and columns from graph Laplacian) is characterized by the existence of directed paths between two nodes. In particular, the  $ij$ th entry of the inverse

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matrix is nonzero if and only if there is a directed path from node  $j$  to node  $i$ .

- For directed trees, we obtain explicit formulae for the transfer function from disturbances to the states of the leader-follower network. We show that the worst-case componentwise amplification of disturbances is achieved at zero temporal frequency and that it is a convex function of edge weights.
- Finally, for 1D and 2D directed lattices, we study the steady-state variance distribution in networks with leaders placed on the boundary. For 1D lattices (i.e., directed paths) with the root being the leader, we show that the variance of the follower nodes grows asymptotically as a *square-root* function of the distance from the leader. For 2D lattices with nodes on the first row and the first column assigned as leaders, we show that the variance along the lattice diagonal grows as a *logarithmic* function of node indices.

Related work on leader-follower consensus problems has examined how controllability of the network can be influenced by assigning a number of agents as leaders [13], [14]. The problem of selecting a fixed number of leaders to minimize the steady-state variance of deviation from consensus has been studied recently by several authors [15]–[18]. All these studies considered undirected graphs as the underlying interconnected network. Recently, several authors have also exploited lower triangular structure in the design of optimal distributed controllers; see [19]–[21]

The paper is organized as follows. In Section II, we consider Laplacian matrices of directed trees and determine the inverse of reduced Laplacian matrices. In Section III, we apply these results to the leader-follower network and study the worst-case amplification from disturbances to the states of the network. In Section IV, we establish the asymptotic scaling of the variance of followers in directed 1D and 2D lattices with boundary nodes being assigned as leaders. We summarize our results in Section V and relegate the proofs to the appendices.

## II. LAPLACIAN MATRICES OF DIRECTED TREES

In this section, we examine structure of Laplacian matrices associated with directed trees. We also provide an example of leader-follower network to motivate the study of the inverse of reduced Laplacian matrices.

A directed graph is a *rooted tree* if it does not contain a cycle and if it has a node that is connected to every other node via a directed path. Such a node is called the root of the tree. Let us assign index 1 to the root and let us enumerate the remaining  $N - 1$  nodes according to the path-length (i.e., the number of edges in the path) between the root and the node; for an illustration, see Fig. 1.

The Laplacian of a rooted tree with  $N$  nodes is a *lower triangular* matrix  $L \in \mathbb{R}^{N \times N}$ . Since the first row of  $L$

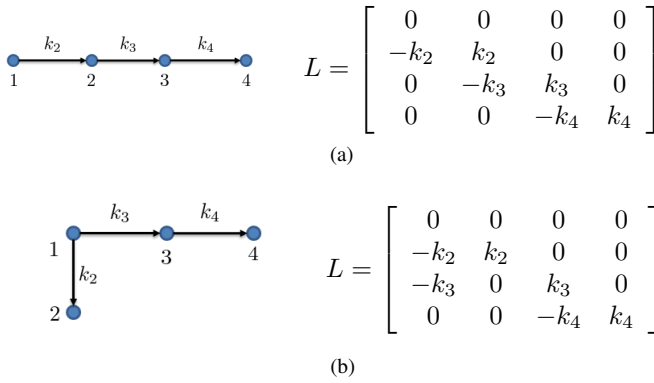


Fig. 1: Examples of directed trees with the corresponding Laplacian matrices. Node 1 corresponds to the root.

corresponds to the root, it is identically equal to zero. On the other hand, the nonzero elements of the  $i$ th row of  $L$  are determined by

$$L_{ii} = k_i, \quad L_{ij} = -k_i, \quad \text{for } i = 2, \dots, N \quad (1)$$

where  $k_i$  is the weight of the edge pointing from node  $j$  to node  $i$ ; e.g., see Figs. 1a and 1b.

Let  $\tilde{L}$  be the *reduced* Laplacian obtained by removing the first row and the first column from  $L$ . Since  $\tilde{L}$  is a lower triangular matrix with nonzero entries on its diagonal, its inverse exists and it is also a lower triangular matrix. The inverse of the reduced Laplacian arises in several applications, including the leader-follower network described in Example 1.

*Example 1:* In the leader-follower network, the objective for all nodes is to follow a desired constant trajectory. Let the root be the leader whose state does not deviate from the desired trajectory. Thus, in coordinates that determine the deviation from the desired trajectory, we have

$$x_1 \equiv 0.$$

We also assume that the remaining nodes of the directed are followers and that they update their states using relative information from its neighbor

$$\dot{x}_i = -k_i(x_i - x_j) + d_i \quad \text{for } i = 2, \dots, N.$$

Here, node  $j$  is a neighbor of node  $i$  if there is an edge pointing from  $j$  to  $i$ . Therefore, the network dynamics are governed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \tilde{L} \mathbf{1} & -\tilde{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_f \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix}$$

where  $x_f = [x_2 \dots x_N]^T \in \mathbb{R}^{N-1}$  is the state of the follower nodes,  $d \in \mathbb{R}^{N-1}$  is the white stochastic disturbance with zero-mean and unit-variance, and  $\mathbf{1} \in \mathbb{R}^{N-1}$  is the vector of all ones. Since  $x_1 \equiv 0$ , the transfer function from  $d$  to  $x_f$  is determined by

$$H(s) = (sI + \tilde{L})^{-1}. \quad (2)$$

As we show in Section III, each component of  $H(j\omega)$  achieves its largest magnitude at zero temporal frequency,  $\omega = 0$ . Consequently, the power spectral density of the

transfer function from  $d$  to  $x_f$ ,

$$\|H(j\omega)\|_F^2 = \text{trace}(H^*(j\omega)H(j\omega)) = \sum_{i,j} |H_{ij}(j\omega)|^2$$

peaks at  $\omega = 0$ . It is thus of interest to examine properties of  $H(0)$ , which is determined by the inverse of the reduced Laplacian

$$H(0) = \tilde{L}^{-1}.$$

#### A. Inverse of reduced Laplacian matrices

We next show that the sparsity structure of the inverse of reduced Laplacian  $\tilde{L}$  of directed trees is characterized by the existence of directed paths between follower nodes. We also demonstrate that the transfer function from the disturbance entering node  $j$  to the state of node  $i$  is determined by a cascade connection of first-order transfer functions whose cutoff frequencies are given by the edge weights along the path from  $j$  to  $i$ .

Let us augment  $\tilde{L}$  with a zero row and a zero column

$$M = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{L}^{-1} \end{bmatrix} \in \mathbb{R}^{N \times N}. \quad (3)$$

As shown below, for  $i \geq j \geq 2$ , the  $ij$ th entry of the matrix  $M$  is nonzero if and only if there is a directed path from node  $j$  to node  $i$ . Note that working with  $M$  instead of working directly with  $\tilde{L}^{-1}$  has proved useful in the computation of effective resistance in undirected networks [22].

For the path graph shown in Fig. 1a, we have

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 1/k_2 & 1/k_3 & 0 \\ 0 & 1/k_2 & 1/k_3 & 1/k_4 \end{bmatrix}.$$

Note that the  $j$ th column of  $M$  for  $j \geq 2$  only consists of  $\{0, 1/k_j\}$ . In fact, we next show that this result holds for any directed tree.

*Proposition 1:* The matrix  $M$  in (3) is a lower triangular matrix with its main diagonal determined by  $\{0, 1/k_2, \dots, 1/k_N\}$ . Furthermore, the  $j$ th column of  $M$  for  $j \geq 2$  only consists of  $\{0, 1/k_j\}$ .

*Proof:* See Appendix A. ■

Since  $M_{ij}$  is either 0 or  $1/k_j$ , we next examine under what conditions we have  $M_{ij} = 1/k_j$ . For example, for the tree shown in Fig. 1b, it is readily verified that

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 1/k_3 & 1/k_4 \end{bmatrix}.$$

Note that  $M_{43} = 1/k_3$  and that there exists a directed path from 3 to 4. On the other hand, there is no directed path from 2 to either 3 or 4, and  $M_{32} = M_{42} = 0$ .

The following Proposition establishes relation between  $M_{ij}$  (with  $i \geq j \geq 2$ ) and the existence of a directed path from  $j$  to  $i$ .

*Proposition 2:* For  $i \geq j \geq 2$ , the  $ij$ th entry of  $M$  in (3) is equal to  $1/k_j$  if and only if there is a directed path from node  $j$  to node  $i$ .

*Proof:* See Appendix B. ■

We next augment the transfer function from the disturbance  $d$  to the state of the followers  $x_f$  with a zero row and

a zero column

$$T(s) = \begin{bmatrix} 0 & 0 \\ 0 & H(s) \end{bmatrix}$$

and provide the formula for the  $ij$ th entry of  $T(s)$  for  $i \geq j \geq 2$ .

*Proposition 3:* Let  $i \geq j \geq 2$  be two follower nodes in a directed tree and let  $\{j, v_1, \dots, v_n, i\}$  be a directed path from  $j$  to  $i$  with  $j \leq v_1 \leq \dots \leq v_n \leq i$ . Then the transfer function  $T_{ij}(s)$  from the disturbance entering node  $j$  to the state of node  $i$  is given by

$$T_{ij}(s) = \frac{k_i}{s + k_i} \times \left( \prod_{l=1}^n \frac{k_{v_l}}{s + k_{v_l}} \right) \times \frac{1}{s + k_j}, \quad i > j \geq 2$$

$$T_{ii}(s) = \frac{1}{s + k_j}, \quad i = j \geq 2.$$

*Proof:* See Appendix C.  $\blacksquare$

Note that  $T_{ij}(s)$  depends on the edge weights of the path from  $j$  to  $i$ . For example, the transfer function from  $d_2$  to  $x_4$  in the path graph in Fig. 1a is given by

$$T_{42}(s) = \frac{k_4}{s + k_4} \times \frac{k_3}{s + k_3} \times \frac{1}{s + k_2}.$$

### III. PERFORMANCE OF LEADER-FOLLOWER NETWORKS IN DIRECTED TREES

In this section, we study the performance of leader-follower networks in directed trees. We show that the magnitude of each component of the transfer function  $H(s)$  from  $d$  to  $x_f$  achieves its largest value at zero temporal frequency. We then consider the problem of minimizing the power spectral density at  $\omega = 0$  subject to a fixed budget on the edge weights. For this strictly convex problem, we provide analytical expression for the optimal edge weights. We also examine how the largest value of power spectral density scales with the size of the network for path and star graphs.

#### A. Maximum value of power spectral density

As shown in Proposition 3, for  $i > j \geq 2$ ,  $T_{ij}(s)$  is a cascade connection of first-order transfer functions with the cutoff frequencies equal to the edge weights along the path from node  $j$  to node  $i$ . Since

$$|T_{ij}(j\omega)|^2 = \frac{k_i^2}{\omega^2 + k_i^2} \times \left( \prod_{l=1}^n \frac{k_{v_l}^2}{\omega^2 + k_{v_l}^2} \right) \times \frac{1}{\omega^2 + k_j^2}$$

is a strictly decreasing function of  $\omega^2$ , we conclude that the largest magnitude of  $T_{ij}(j\omega)$  is achieved at  $\omega = 0$ ,

$$\max_{\omega} |T_{ij}(j\omega)| = |T_{ij}(0)| = 1/k_j.$$

A similar argument shows that for  $i \geq 2$ ,

$$\max_{\omega} |T_{ii}(j\omega)| = |T_{ii}(0)| = 1/k_i.$$

Thus, the componentwise worst-case amplification is a convex function of positive edge weights  $k_i$ 's.

Recall that the power spectral density of the transfer function from  $d$  to  $x_f$  is given by

$$\Pi(j\omega) := \text{trace}(H^*(j\omega)H(j\omega)) = \sum_{i,j} |H_{ij}(j\omega)|^2.$$

Since the maximum value of each component  $|H_{ij}(j\omega)|$  is attained at  $\omega = 0$ , it follows that the maximum value of  $\Pi(j\omega)$  is also attained at  $\omega = 0$

$$\Pi_{\max} := \max_{\omega} \Pi(j\omega) = \sum_{i,j} |H_{ij}(0)|^2 = \sum_{i,j} (\tilde{L}_{ij}^{-1})^2.$$

From Proposition 2, it follows that

$$\Pi_{\max} = \sum_{i=2}^N \frac{z_i}{k_i^2}$$

where  $k_i$  is the weight on the edge pointing to node  $i$  and  $z_i$  is the number of directed paths that contain this edge. For example, for the path graph in Fig. 1a, we have

$$\Pi_{\max} = \frac{3}{k_2^2} + \frac{2}{k_3^2} + \frac{1}{k_4^2}.$$

Therefore, the problem of minimizing the maximum value of the power spectral density with the total edge weights equal to 1 can be formulated as

$$\begin{aligned} & \underset{k}{\text{minimize}} \quad \Pi_{\max} = \sum_{i=2}^N \frac{z_i}{k_i^2} \\ & \text{subject to} \quad \sum_{i=2}^N k_i = 1, \quad k_i > 0. \end{aligned} \quad (4)$$

Note that  $\Pi_{\max}$  is a strictly convex function of positive  $k_i$ 's and that the constraints in (4) are strictly feasible (e.g.,  $k_i = 1/(N-1)$  for  $i = 2, \dots, N$ ). It follows that the unique solution to (4) can be determined by solving the KKT conditions [23, Section 5.5.3] associated with problem (4). Thus, the optimal edge weights are given by

$$k_i^* = \left( \frac{2z_i}{\nu^*} \right)^{1/3}$$

where

$$\nu^* = \left( \sum_{i=2}^N (2z_i)^{1/3} \right)^3$$

and the optimal value of (4) is determined by

$$\Pi_{\max}^* = \left( \sum_{i=2}^N z_i^{1/3} \right)^3. \quad (5)$$

#### B. Examples

We next examine how the largest value of power spectral density scales with the size of networks. In particular, we consider two simple examples, namely, path and star graphs.

1) *Path:* Since the number of directed paths that contain the edge pointing to node  $i$  is  $z_i = N + 1 - i$ , it follows that

$$k_i^* = \frac{(2(N+1-i))^{1/3}}{\sum_{l=1}^{N-1} (2l)^{1/3}}, \quad \Pi_{\max}^* = \left( \sum_{l=1}^{N-1} l^{1/3} \right)^3.$$

Note that the optimal feedback gain  $k_i^*$  is monotonically decreasing as one moves away from the root. By approximating the summation in  $\Pi_{\max}^*$  using integral formula for large  $N$ , it can be shown that  $\Pi_{\max}^*$  scales asymptotically as a 4th order polynomial of  $N$ , i.e.,  $\Pi_{\max}^* \sim O(N^4)$ .



Fig. 2: A directed path with its root being assigned as a leader (red).

2) *Star*: Since for the star graph  $z_i = 1$  for  $i = 2, \dots, N$ , we have

$$k_i^* = 1/(N-1), \quad \Pi_{\max}^* = (N-1)^3.$$

Note that  $\Pi_{\max}^*$  scales as a cubic polynomial  $O(N^3)$  of the number of nodes  $N$ , as opposed to the 4th order polynomial  $O(N^4)$  for the path graph.

#### IV. VARIANCE DISTRIBUTION IN DIRECTED LATTICES

Undirected lattices play an important role in understanding the fundamental performance limitations of local feedback design in several distributed control and estimation problems [3], [4], [10], [11]. On the other hand, several authors have shown that departing from undirected networks results in significant improvement for achievable performance in large directed networks [6]–[8], [11].

Motivated by these results, we next examine performance of leader-follower networks in *directed* lattices. Following [3], [4], [8], [10], we use the steady-state variance of the network as a performance measure. However, in contrast to these studies [3], [4], [8], [10] that focused on the *total* variance of undirected lattices, we investigate how the variance distributed in directed lattices. In particular, we place leaders on the boundary of lattices and study how the variance of followers scales as one moves away from the leaders.

In the simplest scenario, namely, a path graph with the root being a leader as illustrated in Fig. 2, it was shown in [11] that as one moves downstream, the variance of followers scales asymptotically as a *square-root* function of distance from the leader at the root. This result was established by exploiting the *Toeplitz* lower triangular structure of the corresponding modified Laplacian matrix given by (6).

For a 2D lattice with leaders placed on the first row and the first column as illustrated in Fig. 3, the modified Laplacian given by (8) has *block* Toeplitz lower triangular structure. Exploiting this result we determine an analytical expression for the variance of each individual node in the 2D lattice. Furthermore, we show that the variance of the followers grows as a *logarithmic* function of node indices along the diagonal of the lattice.

##### A. 1D lattice

A node is called a *noise-corrupted leader* if, in addition to relative information from its neighbors, it also has access to its own state

$$\dot{x}_i = - \sum_{j \in \mathcal{N}_i} (x_i - x_j) - \alpha_i x_i + d_i$$

where  $\alpha_i$  is a positive number. We consider the directed path in which the root is assigned as a noise-corrupted leader with  $\alpha_1 = 1$  and all edge weights are equal to 1. The dynamics

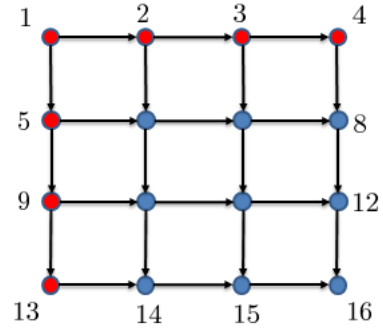


Fig. 3: A directed 2D lattice with nodes on the first row and the first column being assigned as leaders (red).

of the leader-follower network are governed by

$$\dot{x} = -\bar{L}x + d$$

where  $x = [x_1 \dots x_N]^T \in \mathbb{R}^N$  and, with a slight abuse of notation, the disturbance  $d$  is a vector of length  $N$ .

Note that  $\bar{L}$  is the matrix obtained by modifying the Laplacian of the path graph with  $\bar{L}_{11} = \alpha_1 = 1$ . Therefore,  $\bar{L}$  is a Toeplitz matrix with 1 on the main diagonal,  $-1$  on the first sub-diagonal, and zero everywhere else. For example, for  $N = 4$ , we have

$$\bar{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \quad (6)$$

Let  $P$  be the steady-state covariance of  $x$

$$P := \text{trace} \left( \lim_{t \rightarrow \infty} \mathcal{E}(x(t)x^T(t)) \right) = \int_0^\infty e^{-\bar{L}t} e^{-\bar{L}^T t} dt.$$

By exploiting the Toeplitz lower triangular structure of  $\bar{L}$  in (6), it can be shown that [11]

$$P_{nn} = \frac{n(2n)!}{2^{2n} n! n!}$$

and the total variance normalized by the number of nodes is

$$V(N) = \frac{1}{N} \text{trace}(P) = \frac{2N+1}{3N} \times \frac{N(2N)!}{2^{2N} N! N!}.$$

Furthermore,

$$\lim_{n \rightarrow \infty} \frac{P_{nn}}{\sqrt{n}} = \frac{1}{\sqrt{\pi}}, \quad \lim_{N \rightarrow \infty} \frac{V(N)}{\sqrt{N}} = \frac{2}{3\sqrt{\pi}}. \quad (7)$$

*Remark 1:* Equation (7) implies that nodes farther away from the leader have larger steady-state variance; in particular, the variance at the  $n$ th node asymptotically scales as a square-root function of  $n$ . Furthermore, the total variance normalized by the number of nodes  $N$  scales as a square-root function of  $N$ .

##### B. 2D lattice

We next consider a directed 2D lattice of size  $N \times N$  with nodes on the first row and the first column being assigned as noise-corrupted leaders; see Fig. 3. Let  $x_i = [x_{i1} \dots x_{iN}]^T \in \mathbb{R}^N$  be the state of the nodes at the  $i$ th

row of the 2D lattice and let  $x = [x_1^T \cdots x_N^T]^T \in \mathbb{R}^{N^2}$  be the state of all nodes in the lattice.

Consider the 2D lattice with all edge weights being equal to 1 and the absolute feedback gain for the leader at the  $(1, 1)$  node being  $\alpha_i = 2$  and for all the other leaders being  $\alpha_i = 1$ . Then the modified Laplacian  $\bar{L}$  has the *block* Toeplitz lower triangular structure. For example, for  $N = 4$ , we have

$$\bar{L} = \begin{bmatrix} K & 0 & 0 & 0 \\ -I & K & 0 & 0 \\ 0 & -I & K & 0 \\ 0 & 0 & -I & K \end{bmatrix} \in \mathbb{R}^{N^2 \times N^2} \quad (8)$$

where  $I \in \mathbb{R}^{N \times N}$  is the identity matrix and

$$K = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

We next provide the main result in Proposition 4 and omit its proof due to space limitation.

*Proposition 4:* Let  $P_{nn} \in \mathbb{R}^{N \times N}$  be the  $n$ th diagonal block of the matrix

$$P := \int_0^\infty e^{-\bar{L}t} e^{-\bar{L}^T t} dt \in \mathbb{R}^{N^2 \times N^2}$$

where  $\bar{L}$  is given by (8). Then the  $m$ th diagonal entry of  $P_{nn}$  has the following explicit expression

$$P_{nn}^{mm} = \sum_{i=1}^n \sum_{j=1}^m \frac{(2(i+j)-4)!}{4^{2(i+j)-3}((i-1)!(j-1)!)^2}.$$

Furthermore, the  $n$ th diagonal entry of  $P_{nn}$ , denoted by  $P_{nn}^{nn}$ , scales asymptotically as a logarithmic function of  $n$ .

*Remark 2:* Let the nodes on the diagonal of the 2D lattice be denoted by  $(1, 1), \dots, (n, n), \dots, (N, N)$ . Then the steady-state variance of the  $n$ th diagonal node of the 2D lattice is determined by  $P_{nn}^{nn}$ . Therefore, Proposition 4 implies that nodes farther away from the leaders have larger steady-state variance; in particular, the variance at the  $n$ th diagonal node asymptotically scales as a logarithmic function of  $n$ .

## V. CONCLUDING REMARKS

In this paper, we characterize the sparsity structure of the inverse of reduced Laplacian matrices in directed trees in terms of the existence of directed path between a pair of nodes. Based on this characterization, we obtain explicit formulae for the transfer function from disturbances to the state of the leader-follower network. We show that the maximum value of the power spectral density of this transfer function is a convex function of the edge weights. Furthermore, we study the steady-state variance distribution in directed lattices with leaders being placed along the boundary of the lattice. In 2D lattices, we show that the variance of followers along the diagonal scales as a logarithmic function of node indices.

### APPENDIX

#### A. Proof of Proposition 1

To compute the  $j$ th column of  $\bar{L}^{-1}$  for  $j = 1, \dots, N-1$ , we solve the linear equation

$$\bar{L} \tilde{x} = e_j \quad (9)$$

for  $\tilde{x} \in \mathbb{R}^{N-1}$ , where  $e_j$  is the  $j$ th unit vector of  $\mathbb{R}^{N-1}$ . Since  $\bar{L}$  is a lower triangular matrix with nonzero diagonal elements and since the  $i$ th entry of  $e_j$  is zero for  $i = 1, \dots, j-1$ , it follows that

$$\tilde{x}_i = 0, \quad i = 1, \dots, j-1$$

and

$$\tilde{x}_j = 1/\bar{L}_{jj} = 1/L_{(j+1)(j+1)} = 1/k_{j+1}.$$

Thus, the  $(j+1)$ th row of equation (9) is given by

$$\bar{L}_{(j+1)j} \tilde{x}_j + \bar{L}_{(j+1)(j+1)} \tilde{x}_{j+1} = 0.$$

From the definition (1) of the Laplacian matrix of the rooted tree, it follows that  $\bar{L}_{(j+1)j}$  is equal to  $-\bar{L}_{(j+1)(j+1)}$  if there is an edge pointing from  $j$  to  $j+1$  or it is equal to 0 otherwise. Thus,

$$\begin{aligned} \tilde{x}_{j+1} &= -(\bar{L}_{(j+1)j} \tilde{x}_j) / \bar{L}_{(j+1)(j+1)} \\ &= \begin{cases} 0, & \bar{L}_{(j+1)j} = 0 \\ \tilde{x}_j, & \bar{L}_{(j+1)j} = -\bar{L}_{(j+1)(j+1)}. \end{cases} \end{aligned} \quad (10)$$

Similarly, the  $(j+2)$ th row of equation (9) is given by

$$\bar{L}_{(j+2)j} \tilde{x}_j + \bar{L}_{(j+2)(j+1)} \tilde{x}_{j+1} + \bar{L}_{(j+2)(j+2)} \tilde{x}_{j+2} = 0.$$

We have

$$\tilde{x}_{j+2} = \begin{cases} 0, & \bar{L}_{(j+2)j} = \bar{L}_{(j+2)(j+1)} = 0 \\ \tilde{x}_j, & \bar{L}_{(j+2)j} = -\bar{L}_{(j+2)(j+2)} \\ \tilde{x}_{j+1}, & \bar{L}_{(j+2)(j+1)} = -\bar{L}_{(j+2)(j+2)} \end{cases}$$

which, in conjunction with (10), yields that

$$\tilde{x}_{j+2} = 0 \quad \text{or} \quad \tilde{x}_{j+2} = \tilde{x}_j.$$

In particular,  $\tilde{x}_{j+2} = \tilde{x}_j$  if there is an edge pointing directly from  $j$  to  $j+2$  or if there is a path from  $j$  to  $j+1$  to  $j+2$ .

Using induction we conclude that for  $i = j+3, \dots, N-1$ ,  $\tilde{x}_i$  is either equal to 0 or equal to  $\tilde{x}_j$ . In other words, the vector  $\tilde{x}$  only consists of  $\{0, 1/k_{j+1}\}$ ; or equivalently, the  $(j+1)$ th column of  $M$  only consists of  $\{0, 1/k_{j+1}\}$ .

#### B. Proof of Proposition 2

Suppose that a directed path exists from  $j$  to  $i$  for  $i \geq j \geq 2$ , i.e., there is a sequence of nodes  $\{j, v_1, \dots, v_n, i\}$  that are connected by edges  $\{(j, v_1), \dots, (v_n, i)\}$ . Consider the  $(j-1)$ th column of  $\bar{L}^{-1}$ , which is determined by the solution to the linear equation

$$\bar{L} \tilde{x} = e_{j-1}. \quad (11)$$

Using the same argument in the proof of Proposition 1 in Appendix A, it follows that

$$\tilde{x}_{j-1} = \tilde{x}_{v_1-1} = \cdots = \tilde{x}_{v_n-1} = \tilde{x}_{i-1} = 1/k_j.$$

Therefore, the existence of a directed path from  $j$  to  $i$  is a sufficient condition for  $(\bar{L}^{-1})_{(i-1)(j-1)} = M_{ij} = 1/k_j$ .

Now suppose that  $M_{ij} = 1/k_j$ , i.e., the  $(i-1)$ th entry of the solution  $\tilde{x}$  to (11) is given by  $\tilde{x}_{i-1} = 1/k_j$ . Consider the  $(i-1)$ th row of (11)

$$\bar{L}_{(i-1)(j-1)} \tilde{x}_{j-1} + \bar{L}_{(i-1)j} \tilde{x}_j + \cdots + \bar{L}_{(i-1)(i-1)} \tilde{x}_{i-1} = 0.$$

Since  $\bar{L}_{(i-1)(i-1)} \tilde{x}_{i-1} = k_i/k_j \neq 0$  and since there is at

most one nonzero entry in the  $(i-1)$ th row of  $\tilde{L}$  from the definition (1) of Laplacian of rooted tree, it follows that one of the entries  $\{\tilde{L}_{(i-1)(v_n-1)}\}$  for  $v_n = j, \dots, i-1$  must be nonzero. Thus, there exists an edge pointing to node  $i$  from node  $v_n \in \{j, \dots, i-1\}$ . Consequently,

$$\tilde{x}_{v_n-1} = \tilde{x}_{i-1} = 1/k_j.$$

Note that  $v_n$  is strictly less than  $i$ . Using the same argument for the  $(v_n-1)$ th row of equation (11), we conclude that there exists an edge pointing to node  $v_n$  from node  $v_{n-1} \in \{j, \dots, v_n-1\}$ . Thus, induction argument implies that there is a sequence of nodes  $\{j, v_1, \dots, v_{n-1}, v_n, i\}$  that form a directed path from node  $j$  to node  $i$ .

### C. Proof of Proposition 3

The proof is similar to the proof of Proposition 1. We consider the  $(j-1)$ th column of  $(sI + \tilde{L})^{-1}$ , which is the solution to the linear equation

$$(sI + \tilde{L})\tilde{x} = e_{j-1}. \quad (12)$$

Since  $sI + \tilde{L}$  is a lower triangular matrix and its  $(j-1)$ th main diagonal entry is given by  $s + k_j$ , it follows that

$$\tilde{x}_r = 0 \quad \text{for } r < j-1 \quad (13)$$

and

$$\tilde{x}_r = (s + k_j)^{-1} \quad \text{for } r = j-1. \quad (14)$$

Thus, the  $jj$ th entry of  $T$  is given by

$$T_{jj}(s) = \tilde{x}_r = (s + k_j)^{-1}.$$

To compute  $\tilde{x}_r$  for  $r > j-1$ , we write explicitly the  $r$ th row of equation (12)

$$\tilde{L}_{r1}\tilde{x}_1 + \dots + \tilde{L}_{r(r-1)}\tilde{x}_{r-1} + (s + \tilde{L}_{rr})\tilde{x}_r = 0. \quad (15)$$

For  $r = j$ , from (13), it follows that equation (15) becomes

$$\tilde{L}_{r(r-1)}\tilde{x}_{r-1} + (s + \tilde{L}_{rr})\tilde{x}_r = 0.$$

Thus, we have

$$\tilde{x}_r = \begin{cases} 0, & \tilde{L}_{r(r-1)} = 0 \\ -(s + \tilde{L}_{rr})^{-1}\tilde{L}_{r(r-1)}\tilde{x}_{r-1}, & \tilde{L}_{r(r-1)} \neq 0. \end{cases}$$

In other words, if there is an edge pointing from  $j$  to  $j+1$  with the edge weight  $k_{j+1}$ , then

$$\tilde{x}_r = \frac{k_{j+1}}{s + k_{j+1}}\tilde{x}_{r-1} \quad \text{for } r = j$$

otherwise,  $\tilde{x}_r = 0$ . Therefore, if there is a directed path  $\{j, v_1, \dots, v_n, i\}$  from  $j$  to  $i$ , then

$$\tilde{x}_{i-1} = \frac{k_i}{s + k_i} \times \left( \prod_{l=1}^n \frac{k_{v_l}}{s + k_{v_l}} \right) \tilde{x}_r. \quad (16)$$

Substituting (14) into (16) yields the desired result

$$T_{ij}(s) = \tilde{x}_{i-1} = \frac{k_i}{s + k_i} \times \left( \prod_{l=1}^n \frac{k_{v_l}}{s + k_{v_l}} \right) \frac{1}{s + k_j}.$$

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