State and Noise Covariance Estimation in Power Grids using Limited Nodal PMUs

Deepjyoti Deka*, Armin Zare†, Andrey Lokhov*, Mihailo Jovanović†, and Michael Chertkov*
*Los Alamos National Laboratory, USA, †University of Southern California, USA

***Abstract*—The statistics of critical state variables and nodal fluctuations in power grids are useful for control of its dynamics and procurement of regulation resources. This paper studies estimation of the statistics of grid state variables and ambient fluctuations using time-stamped nodal measurements collected from limited PMUs in the grid. We show that in the presence of time-stamped observations that enable computation of delayed covariances, PMUs located at half of the number of grid buses are sufficient to reconstruct all state and noise statistics exactly. For lower number of available PMUs, we provide a convex optimization framework for estimating the covariance matrices that outshines standard schemes that do not utilize the availability of time-stamped measurements.**

***Index Terms*—Covariance completion, Delayed covariance, Lyapunov equation, Power grid, PMU, Swing dynamics**

I. Introduction

Stable operation of the grid in real-time is necessary for reliable delivery of electricity. Over short time scales, stochastic perturbations in nodal injections (generation and load) lead to dynamics in the voltages and frequencies at grid nodes. These are represented mathematically by coupled swing equations [1], [2]. In recent years, due to the proliferation of stochastic renewable resources and active loads, the fluctuations in operating frequency has increased and led to concerns about the stability of the grid [3] and increased procurement of fast regulation resources for control [4], [5]. Past research has shown that statistics of nodal states deviate from their steady state as the system moves towards instability [6]–[8]. Such statistics and their deviation from steady state can be used as early 'warning sings' by the system operator to improve the readiness of the emergency control and regulation resources in the grid [9], [10]. Accurate estimation of the steady state covariance of the grid variables is thus an important benchmark with use in early detection and subsequent response. The amount of regulation resources (like MWs of primary and secondary response) that the grid operator procures is dependent on the knowledge of the fluctuations in nodal injections. Accurate estimation of the nodal injection statistics is thus necessary for correct resource availability and allocation. In this paper, we are interested in the problem of estimating both statistics in state covariances and nodal fluctuations in the ambient regime, based on the swing equation based model for grid dynamics.

In prior work, researchers have analyzed different aspects of estimation and utilization of the steady state covariance matrix in power grids. [10]–[12] identifies several indicators of grid instability using state covariance matrices. [13], [14] look at the inverse problems of estimating the Jacobian matrix and system parameters respectively using the estimated full state covariance matrix. Note that in reality, the estimation problem in the grid is rendered difficult by the scarcity of high fidelity real-time meters like PMUs (Phasor Measurement Units) [15], micro-PMUs [16], FNET (Frequency Monitoring Network) devices. Expansion efforts of such devices, despite recent trends, still have not been able to bring observability to large sections of the network. In work for general linear dynamical systems outside of power grids, covariance completion from partial observations of statistics has been studied under different noise models, both colored and uncorrelated [17]–[19]. This has applications for noise modeling, system identification and minimal filter design. Learning algorithms in this area utilize a convex formulation with constraints arising from the Lyapunov relation between state and noise statistics as well as matching the known covariances.

A. Contribution

In this paper, we analyze estimation of the state and noise covariance matrices for ambient fluctuations in power grids in the incomplete regime, where state measurements pertaining to only a limited number of nodal PMUs are available for reconstruction. The chief distinguishing feature of our work is that we utilize the time-stamps in the available measurements to improve the covariance estimation. In particular, this is applicable for measurements collected by PMUs as they have GPS synchronized time stamps. The time-stamps enable us to compute 'delayed' covariances in observed state variables that can be included as linear constraints in the estimation algorithm. We show that in the presence of time-stamps, nodal PMU measurements of frequency and phase angle at half the grid nodes are sufficient to estimate the complete state covariance matrix and noise statistics directly. For scenarios where the number of PMUs available are much less than that number, we present a convex optimization based estimation algorithm to learn the necessary statistics of state and noise using linear constraints that include delayed covariance of observed states. Simulation results on test IEEE 39 bus dynamic network demonstrate the improvement in estimation for both state and noise covariances over schemes that do not use knowledge of time-stamps.

The rest of the paper is organized as follows. Section II includes the mathematical formulation for the topology and swing equations in the grid. Section III describes the properties of the steady state covariance matrices in the system under

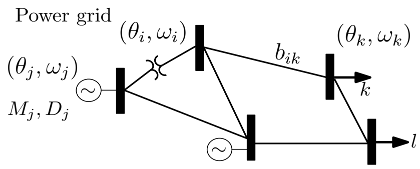


Fig. 1. Power grid and its associated state variables (nodal phase angle θ and frequency ω) and parameters (nodal inertia M, damping D, line susceptance b).

stochastic nodal fluctuations. Section IV includes our result on number of PMUs necessary for exact recovery using delayed statistics. Section V presents the convex optimization based estimation algorithm for covariance completion. We provide numerical experiments and a comparison with existing work in Section VI. Finally, Section VII details directions of future work and concludes the paper.

II. Mathematical formulation

We represent the power grid mathematically by a connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{0, 2, ..N\}$ is the set of $N+1$ buses/nodes of the graph and $\mathcal{E} = \{e_{ij}\}$ is the set of undirected lines/edges (see Fig. 1). Let $y_{ij} = g_{ij} - \hat{j}b_{ij}$ denote the complex admittance of line (ij) in the grid ($\hat{j}^2 = -1$) with conductance $g_{ij} > 0$ and susceptance $b_{ij} > 0$. Each node i is associated with a time varying complex voltage V_i of magnitude $|V_i|$ and phase angle θ_i. The frequency at node i is denoted by ω_i where $\omega_i = \frac{\delta \theta_i}{\delta t}$. Under stable operating conditions, all nodal frequencies are maintained at a constant value $\omega^0 = 60$Hz (in U.S.A.). The temporal dynamics of the frequency at each node is governed by the following swing equation [20]:

$$M_i \dot{\omega}_i + D_i(\omega_i - \omega^0) = P_i^m - \sum_{(ij)\in\mathcal{E}} P_{ij} + u_i \tag{1}$$

Here, M_i denotes the inertia of the rotating mass at node i, which primarily stems from inertia of generators. D_i represents the damping at node i. P_i^m represents the net real power injected at node i. P_{ij} is the real power flowing out of node i through lines connected to its neighbors. u_i is the external stochastic disturbance affecting the node. Therefore the terms on the right hand side represent the net power imbalance at node i. As we consider ambient disturbances of moderate/small amplitude, we make the linearized DC approximation [1] of the flow equation by assuming constant voltage magnitude ($|V_i| = 1$) and small angle differences at neighboring nodes ($\theta_i - \theta_j << 1$). This yields

$$P_{ij} = b_{ij}(\theta_i - \theta_j) \tag{2}$$

It is worth mentioning that in the absence of external disturbances, the system of equations has a stable operating point ($\omega_i = \omega^0, \dot{\omega}_i = 0$). As the swing equation with DC power flow is linear, all nodal frequencies can be measured as deviations from the stable operating point. Further we take phase angle at one node (node 0) as reference and express phase angles at all other nodes in terms of deviations from the reference node. Abusing notation, from this point we use ω_i, θ_i to denote the deviations from the reference node at node i and restrict our analysis to the reduced system with N non-reference nodes $(1, 2, ...N)$). In vector form, the linearized swing equation for the reduced system is given by:

$$\dot{\theta} = \omega, \quad [M]\dot{\omega} + [D]\omega = -L_B\theta + u \tag{3}$$

Here, ω, θ, u are the $N\times 1$ vectors. $[M]$ and $[D]$ are diagonal matrices representing the nodal inertia and damping. L_B is the $N\times N$ susceptance weighted reduced graph Laplacian (without node 0) in grid $\mathcal{G}$ with the following structure:

$$\forall 1 \le i,j \le N, L_B(i,j) = \begin{cases} -b_{ij} & \text{if } (ij) \in \mathcal{E} \\ \sum_{(ik)\in\mathcal{E}} b_{ik} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The state of each node in this system thus comprises of the voltage phase angle and the frequency. For the rest of the paper we assume the following:

Assumption 1: At each node i, $M_i, D_i \neq 0$

Assumption 2: The disturbance vector u has mean zero and is uncorrelated in time and space with diagonal covariance matrix $\mathbb{E}[uu^T] = \Sigma_u$.

The assumption of non-zero inertia and damping is made for convenience of presentation. The analysis in its absence follows directly by modifying the swing equations for nodes without inertia and damping. The assumption of 'delta- correlated' noise can be argued at short times scales where ambient disturbances arise from fluctuations of loads or noise at generators [14], [21], [22].

Writing Eq. (3) in standard Linear Time Invariant (LTI) system form ($\dot{x} = Ax + Bu$) [23], we have

$$\begin{bmatrix}\dot{\theta}\\ \dot{\omega}\end{bmatrix} = \begin{bmatrix} 0 & \mathbb{I} \\ -M^{-1}L_B & -M^{-1}D\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix} + \begin{bmatrix}0\\ M^{-1}\end{bmatrix} u \tag{5}$$

where $\mathbb{I}$ is the identity matrix of size $N \times N$, $x = [\theta^T \omega^T]^T$ is the system state, and (A, B) are the matrices $\begin{bmatrix} 0 & I \\ -M^{-1}L_B & -M^{-1}D\end{bmatrix}$, $\begin{bmatrix}0\\ M^{-1}\end{bmatrix}$, respectively. We assume that parameters M, D, L_B and hence system matrices A, B are known to the observer. Additionally, under normal operation we assume that the matrix A is Hurwitz [20] and system (A, B) is controllable. In the next section we discuss properties of the steady state covariance matrix of x and u in Eq. (5) that is of interest for our reconstruction problem.

III. Steady State statistics of Linear Dynamical Systems

For a general controllable linear dynamical system $\dot{x} = Ax + Bu$, there is a rich literature on linear relations involving steady state covariances of system states x when u is a zero-mean input with bounded covariance. As the ambient disturbance is assumed to be uncorrelated across time and space, we have $\mathbb{E}[(u_i(t)u_j(t+\tau)] = \sigma_i\delta_{i=j}\delta_{\tau=0}$. Using this the following continuous time Lyapunov equation [23] holds for the steady state covariance $R_x = \mathbb{E}[x(t)x(t)^T]$,

$$AR_x + R_xA^T = -B\Sigma_uB^T \tag{6}$$

where Σ_u is the diagonal covariance matrix of the ambient disturbance under Assumption 2. Further, by stability of A and controllability of (A, B) it follows that R_x is positive definite and unique for a given Σ_u [24]. Note that x is a vector Ornstein-Uhlenbeck process [23]. As $u(t)$ is 'delta'-correlated noise, it can be shown that the following holds for the 'delayed' covariance $R_x^\tau = \mathbb{E}[x(t+\tau)x(t)^T]$

$$R_x^\tau = e^{\tau A} R_x \tag{7}$$

where $e^{\tau A}$ is a matrix exponential. Note that both Eqs. (6,7) are linear relations involving the state and noise covariance matrices. We use these relations in the next section to analyze recovery of state and noise covariance matrices in the regime of incomplete observability where measurements are limited to a few PMUs in the system.

IV. Exact Covariance Completion using PMUs

If covariance matrix Σ_u of noise is known, then Eq. (6) can be solved (using numeric solvers) to derive the state covariance matrix R_x. We are interested in estimating Σ_u along with R_x when time-stamped nodal PMU measurements at a set O of $k \leq N$ nodes are available. The set of nodes H denotes the set of 'hidden' nodes with no resident PMUs. Note that the total number of states for the system is $2N$ (each node has frequency and phase angle as its state), while $2k$ states are observed directly by the PMUs in set O.

Consider the division of matrices R_x and R_x^τ as follows:

$$R_x = \begin{bmatrix} R_x(H,H) & R_x(H,O) \\ R_x(H,O)^T & R_x(O,O) \end{bmatrix}, R_x^\tau = \begin{bmatrix} R_x^\tau(H,H) & R_x^\tau(H,O) \\ R_x^\tau(H,O)^T & R_x^\tau(O,O) \end{bmatrix}$$

where $R_x(O,O)$ and $R_x^\tau(O,O)$ denote the $2k \times 2k$ submatrices in the steady state and delayed covariance matrices respectively that correspond to the phase angles and frequency at the observed nodes. Similarly, other submatrices refer to covariances between states at hidden nodes and/or observed nodes. As time-stamped measurements are available, the observer can empirically compute $R_x(O,O)$ and $R_x^\tau(O,O)$ in R_x and R_x^τ respectively.

We consider the case where $k \geq N/2$, i.e., **PMUs are located at at least half the number of grid nodes**. Thus $|O| \geq |H|$. The following theorem demonstrates that direct computation of state and error covariances is possible in this setting.

Theorem 1. *If time-stamped PMUs measuring voltage phase angle and frequency at half the grid nodes are available, then the steady state covariance matrix and noise covariance matrix can be reconstructed.*

Proof. Consider a division of matrices A and $e^{A\tau}$ into blocks corresponding to observed and hidden states like R_x and R_x^τ. Using Eq. (7), we can derive

$$R_x^\tau(O,O) - e^{\tau A}(O,O)R_x(O,O) = e^{\tau A}(O,H)R_x(H,O)$$

As $|O| \geq |H|$ and A has full-rank, select τ such that $e^{\tau A}(O,H)$ has full column rank. The preceding equation can hence be used to derive unknown submatrix $R_x(H,O)$. From Eq. (5) and Assumption 2, it follows that $B\Sigma_u B^T$ is a diagonal matrix with first N entries of its diagonal being 0. The equality relation for the principal $k \times k$ submatrix of $B\Sigma_u B^T$ in Eq. (6) gives

$$\begin{aligned} &A(H,H)R_x(H,O) + A(H,O)R_x(H,H) + \\ &R_x(H,H)A(H,O)^T + R_x(H,O)A(O,O)^T = 0 \end{aligned} \tag{8}$$

which can be used to derive $R_x(H,H)$. Once all missing entries in R_x are estimated, Eq. (6) can be used to compute Σ_u. □

In the next section, we discuss covariance estimation when PMUs are placed at less than half the grid nodes.

V. Convex Optimization based Covariance Estimation

Here we consider the problem where the system matrices A and B are known, but the input time-series comprise of phase angles and frequencies in set O of size less than $N/2$. In this setting, we present the following convex optimization formulation to estimate both Σ_u and R_x.

$$\arg\min_{R,\Sigma} \|AR + RA^T + B\Sigma B^T\|_F \tag{P-1}$$

$$\text{s.t. } R \succ 0 \tag{9}$$

$$\Sigma_{ij} = \begin{cases} \geq 0 & N < i = j \leq 2N \\ = 0 & \text{otherwise} \end{cases} \tag{10}$$

$$R(O,O) = R_x(O,O), [e^{\tau A}R](O,O) = R_x^\tau(O,O) \tag{11}$$

Note that the optimization function in Problem (P-1) minimizes the Lyapunov equality (6). Eq. (9) follows from the positive definiteness of the steady state covariance matrix as the system is controllable. The noise covariance matrix, by Assumption 2, is diagonal with sparsity pattern constrained by Eq. (10). The linear constraints of Eq. (11) match the values in the optimizer R and the delayed covariance matrix (see Eq. (7)) at the observed nodes with true values in R_x and R_x^τ. Note that positive semi-definiteness of Σ ensures a unique positive-definite covariance R [24], and makes constraint (9) redundant. In practice, we also observe that the true covariance matrix R_x, though positive-definite, is close to being singular (rank deficient) for several IEEE test cases. We thus ignore constraint (9). Note that closeness to singularity creates numerical issues for functions like $\log\det$ that can be used as surrogate for positive definiteness [19]. Next, we discuss the performance of this formulation in estimating steady state covariance matrix of system states and noise.

VI. Numerical Experiments

In this section, we demonstrate the effectiveness of solving optimization problem (P-1) on the IEEE 39 bus test system [25], [26] shown in Fig. 2. The test case originally has 10 generators. We insert small additional damping and inertia at the non-generator nodes following Assumption 1. In each

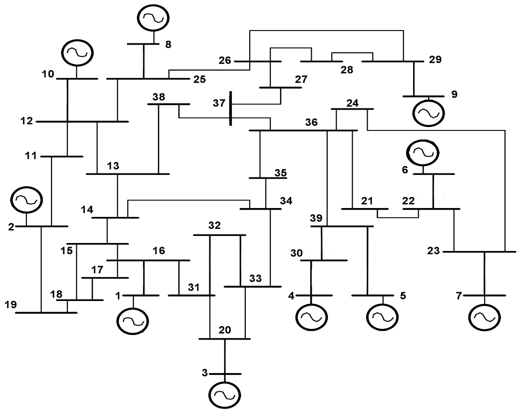
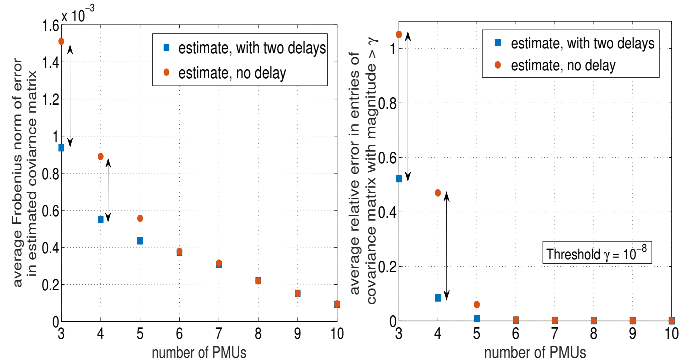


Fig. 2. IEEE 39 bus system

simulation, we take a random noise covariance matrix and generate the true steady state and two delayed covariance matrices (with τ's taken as .45 and .4) using *lyap* function in Matlab R2015. We then randomly select a fixed number of nodes as PMU locations and separate the covariances corresponding to those nodal states as available information. We estimate the full state covariance matrix and noise statistics by solving problem (P-1) using the CVX package [27]. For comparison, we also run problem (P-1) without the delayed covariance constraints. The error (averaged over multiple simulations) in steady state covariance matrix estimation is presented in Fig. 3 using two metrics. Fig. 3(a) includes the Frobenius norm of the difference between the true and estimated covariance matrices. Fig. 3(b) shows the relative error per entry of the covariance matrix. As relative errors diverge for entries extremely close to zero, we consider matrix entries with magnitude greater than 10^{-8} (holds for $> 98\%$ of all entries). Unlike Frobenius norm, the relative error per entry does not depend on the size of the matrix. Note that errors for either metric decrease with an increase in the number of available PMUs. The performance due to inclusion of delayed covariances outshines the traditional case with no temporal statistics, the effect being more prominent at low PMU installation. In Fig. 4, we present an instance of the estimation of nodal noise statistics with or without inclusion of delayed covariances with 3 PMUs. Observe that the inclusion of delayed covariance constraints leads to almost exact matching between the estimated and true values. For the same instance, Figs. 5(a) and 5(b) provide the heat maps of true and estimated covariance matrices (with two delayed constraints) of phase angles that compare well on inspection.

VII. Conclusion and Future Work

This paper analyzes the problem of estimating the state covariance matrix and statistics of ambient disturbance in power grids using PMU measurements at a limited number of grid nodes. Using delayed covariances of nodal states, it is shown that PMUs placed at half the grid nodes ensures exact recovery. Further a convex optimization formulation is presented to estimate the statistics when the number of available PMUs is much less. Simulations show that inclusion



(a) (b)

Fig. 3. Average error metrics in estimation of state covariance matrix by Problem (P-1) vs number of available PMUs in IEEE 39 bus system; (a) Forbenius norm of difference in estimate and true covariance matrices (b) Relative error per entry of true covariance matrix with magnitude greater than $\gamma = 10^{-8}$

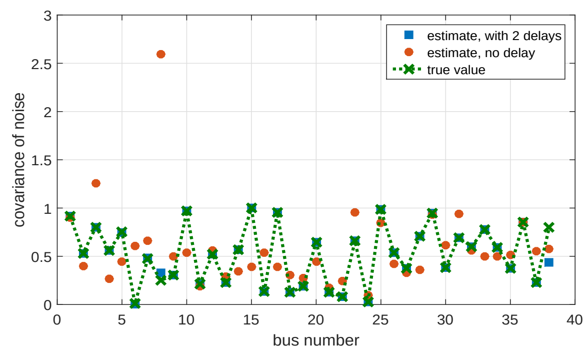
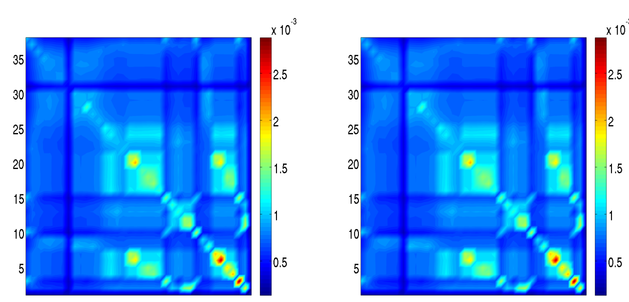


Fig. 4. One instance of estimation of noise covariances in IEEE 39 bus system by Problem (P-1) with 3 PMUs

of delayed covariance in the optimization framework leads to significant improvement in estimating unknown statistics. This demonstrates the benefit of using 'time-stamped' PMU measurements in estimation problems on power grids.

The use of delayed covariances leads to several future research directions. For the estimation problems considered in this paper, we have assumed accurate estimate of covariances at observed nodes. Covariance completion in the presence of measurement errors is a direction we plan to pursue in the future. We also plan to analyze the problem of optimal PMU placement to improve the performance of our algorithms.



(a) (b)

Fig. 5. One instance of heat map of phase covariances in IEEE 39 bus system; (a) true value (b) estimated by Problem (P-1) with 3 PMUs

REFERENCES

[1] P. Kundur, *Power system stability and control*, vol. 7.

[2] C. Nwankpa, S. Shahidehpour, and Z. Schuss, "A stochastic approach to small disturbance stability analysis," *IEEE Transactions on Power Systems*, vol. 7, no. 4, pp. 1519–1528, 1992.

[3] A. Ulbig, T. S. Borsche, and G. Andersson, "Impact of low rotational inertia on power system stability and operation," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 7290–7297, 2014.

[4] J. Matevosyan, S. Sharma, S.-H. Huang, D. Woodfin, K. Ragsdale, S. Moorty, P. Wattles, and W. Li, "Proposed future ancillary services in electric reliability council of texas," in *PowerTech, 2015 IEEE Eindhoven*. IEEE, 2015, pp. 1–6.

[5] F. O. No, "755,frequency regulation compensation in the organized wholesale power markets, issued october 20, 2011."

[6] E. Cotilla-Sanchez, P. D. Hines, and C. M. Danforth, "Predicting critical transitions from time series synchrophasor data," *IEEE Transactions on Smart Grid*, vol. 3, no. 4, pp. 1832–1840, 2012.

[7] M. Chertkov, S. Backhaus, K. Turtisyn, V. Chernyak, and V. Lebedev, "Voltage collapse and ode approach to power flows: analysis of a feeder line with static disorder in consumption/production," *arXiv preprint arXiv:1106.5003*, 2011.

[8] D. Podolsky and K. Turitsyn, "Random load fluctuations and collapse probability of a power system operating near codimension 1 saddle-node bifurcation," in *Power and Energy Society General Meeting (PES), 2013 IEEE*. IEEE, 2013, pp. 1–5.

[9] X. Wang and K. Turitsyn, "Data-driven diagnostics of mechanism and source of sustained oscillations," *IEEE Transactions on Power Systems*, vol. 31, no. 5, pp. 4036–4046, 2016.

[10] G. Ghanavati, P. D. Hines, and T. I. Lakoba, "Identifying useful statistical indicators of proximity to instability in stochastic power systems," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1360–1368, 2016.

[11] B. Gao, G. Morison, and P. Kundur, "Voltage stability evaluation using modal analysis," *IEEE Transactions on Power Systems*, vol. 7, no. 4, pp. 1529–1542, 1992.

[12] P.-A. Lof, G. Andersson, and D. Hill, "Voltage stability indices for stressed power systems," *IEEE Transactions on Power Systems*, vol. 8, no. 1, pp. 326–335, 1993.

[13] Y. C. Chen, J. Wang, A. D. Domínguez-García, and P. W. Sauer, "Measurement-based estimation of the power flow jacobian matrix," *IEEE Transactions on Smart Grid*, vol. 7, no. 5, pp. 2507–2515, 2016.

[14] X. Wang and K. Turitsyn, "PMU-based estimation of dynamic state jacobian matrix," *arXiv preprint arXiv:1510.07603*, 2015.

[15] A. G. Phadke, "Synchronized phasor measurements in power systems," *IEEE Computer Applications in power*, vol. 6, no. 2, pp. 10–15, 1993.

[16] A. von Meier, D. Culler, A. McEachern, and R. Arghandeh, "Micro-synchrophasors for distribution systems," pp. 1–5, 2014.

[17] A. Zare, Y. Chen, M. Jovanović, and T. T. Georgiou, "Low-complexity modeling of partially available second-order statistics: Theory and an efficient matrix completion algorithm," *IEEE Transactions on Automatic Control*, 2016.

[18] A. Zare, M. R. Jovanović, and T. T. Georgiou, "Colour of turbulence," *Journal of Fluid Mechanics*, vol. 812, pp. 636–680, 2017.

[19] ——, "Perturbation of system dynamics and the covariance completion problem," in *Decision and Control (CDC), 2016 IEEE 55th Conference on*. IEEE, 2016, pp. 7036–7041.

[20] J. Machowski, J. Bialek, and J. R. Bumby, *Power system dynamics and stability*. John Wiley & Sons, 1997.

[21] G. B. Giannakis, V. Kekatos, N. Gatsis, S.-J. Kim, H. Zhu, and B. F. Wollenberg, "Monitoring and optimization for power grids: A signal processing perspective," *IEEE Signal Processing Magazine*, vol. 30, no. 5, pp. 107–128, 2013.

[22] P. Yang, Z. Tan, A. Wiesel, and A. Nehorai, "Power system state estimation using PMUs with imperfect synchronization," *IEEE Transactions on Power Systems*, vol. 28, no. 4, pp. 4162–4172, 2013.

[23] C. Gardiner, *Stochastic methods: a handbook for the natural and social sciences 4th ed.(2009)*. Springer.

[24] J. P. Hespanha, *Linear systems theory*. Princeton University Press, 2009.

[25] T. Athay, R. Podmore, and S. Virmani, "A practical method for the direct analysis of transient stability," *IEEE Transactions on Power Apparatus and Systems*, no. 2, pp. 573–584, 1979.

[26] P. Demetriou, M. Asprou, J. Quiros-Tortos, and E. Kyriakides, "Dynamic ieee test systems for transient analysis," *IEEE Systems Journal*, 2015.

[27] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming."