

# Leader selection in directed networks

Neil K. Dhingra, Marcello Colombino, and Mihailo R. Jovanović

**Abstract**—We study the problem of leader selection in directed consensus networks. In this problem, certain ‘leader’ nodes in a consensus network are equipped with absolute information about their state. This corresponds to diagonally strengthening a dynamical generator given by the negative of a directed graph Laplacian. We provide a necessary and sufficient condition for the stabilization of directed consensus networks via leader selection and form regularized  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  optimal problem leader selection problems. We draw on recent results that establish the convexity of the  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norms for structured decentralized control of positive systems and identify sparse sets of leaders by imposing an  $\ell_1$  penalty on the vector of leader weights. This allows us to develop a method that simultaneously assigns leader weights and selects a limited number of leaders. We use proximal gradient and subgradient method to solve the optimization problems and provide examples to illustrate our developments.

## I. INTRODUCTION

Consensus networks have attracted much interest for problems dealing with collective decision-making and collective sensing [1], [2]. These networks are seen in applications as diverse as modeling animal group dynamics, formation flying of spacecraft, and data fusion in sensor networks [3], [4].

The leader selection problem poses the question of how to equip a network obeying consensus dynamics, which describes exclusively relative information exchange, with absolute information at a limited subset of nodes [5]–[15]. In this paradigm, the nodes within a network are partitioned into a set of leaders and a set of followers. Both sets of nodes update their states by averaging their states with those of their neighbors in the network, but leader nodes also assign a weight to their own state.

In [9], the authors develop a greedy algorithm for leader selection in undirected networks and use convex relaxations to quantify performance bounds. In [8], [11], the authors derive an explicit expression for the set of optimal leaders with a fixed weight in undirected networks in terms of graph theoretic properties. In [13], the authors characterize bounds on the convergence rate based on the distance between leaders and followers. Most of these results focus on undirected networks with the exception of [10]. In this work, the authors derive the optimal leaders for one-dimensional directed path networks.

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Leader selection amounts to strengthening the dynamical generator, the negative of a graph Laplacian, with a sparse diagonal matrix that specifies the leaders. Since for undirected networks the graph Laplacian is symmetric, it is well known that regularized versions of the leader selection problem are convex [9]. However, since the negative of the graph Laplacian is Metzler, there is also an interesting connection to a related class of decentralized control problems for positive systems [16]–[19]. A recent result on convexity of structured decentralized control [20] can be used to show convexity of regularized leader selection problems even for directed networks.

Our contributions are as follows. First, we provide a necessary and sufficient condition for closed-loop stability for sets of leader nodes. Then, we explore several properties for leader selection in balanced graphs. Finally, we develop our algorithm and illustrate its utility with examples.

The rest of the paper is organized as follows. In Section II, we describe the problem formulation and performance metrics that we employ. In Section III, we discuss stability of consensus networks with leaders and properties of balanced graphs. In Section IV, we develop a proximal algorithm for leader selection and illustrate its utility in Section V. Finally, in Section VI we offer concluding remarks.

## Notation and basic results

The set  $\mathbb{R}_+$  ( $\mathbb{R}_{++}$ ) denotes the nonnegative (positive) reals. The set of  $n \times n$  Metzler matrices (matrices with nonnegative off diagonal elements) is denoted by  $\mathbb{M}^n$ . The set of (nonnegative) integers is denoted by  $\mathbb{Z}$  ( $\mathbb{Z}_+$ ). Given a matrix  $A$ ,  $A^T$  denotes its transpose. We use  $\bar{\sigma}(A)$  to indicate the largest singular value of  $A$ ,  $\text{trace}(A)$  to denote its trace, and  $\|A\|_F^2 := \text{trace}(A^T A)$  to denote its Frobenius norm. We write  $A \geq 0$  ( $A > 0$ ) if  $A$  has nonnegative (positive) entries and  $A \succeq 0$  ( $A \succ 0$ ) to denote that  $A$  is symmetric and positive semidefinite (definite). The vector inner product is given by  $\langle x, y \rangle := x^T y$  and the matricial inner product is given by  $\langle X, Y \rangle := \text{trace}(X^T Y)$ . Given a set  $\mathcal{C}$  we define the indicator function

$$I_{\mathcal{C}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

We define the sparsity pattern  $\text{sp}(u)$  of a vector  $u$  to be the set of indices for which  $u_i$  is nonzero. The  $\ell_1$  norm of the vector  $u$  is given by,  $\|u\|_1 := \sum_i |u_i|$ .

Finally, we provide some basic definitions and lemmas.

**Definition 1 (Weakly connected graph):** A graph  $(\mathcal{V}, \mathcal{E})$  is weakly connected if there is a path, not necessarily directed, from every node in  $\mathcal{V}$  to every other node.

*Definition 2 (Strongly connected graph):* A graph  $(\mathcal{V}, \mathcal{E})$  is strongly connected if there is a *directed* path linking from every node in  $\mathcal{V}$  to every other node.

*Definition 3 (Balanced graph):* A graph  $(\mathcal{V}, \mathcal{E})$  is balanced if the weighted in-degree of each node is the same as the weighted out-degree.

*Lemma 1:* Let  $A \in \mathbb{M}^n$ . Then  $e^A \geq 0$ .

*Proof:* The matrix  $A \in \mathbb{M}^n$  can be written as  $\tilde{A} - \alpha I$  with  $\tilde{A} \geq 0$  and  $\alpha > 0$ . Then  $e^A = e^{-\alpha} e^{\tilde{A}} \geq 0$  because  $e^{\tilde{A}} \geq 0$ . ■

*Lemma 2:* Let  $A \in \mathbb{M}^n$  be Hurwitz and  $Q \succ 0$  be a nonnegative matrix. Then the solution  $X$  to the Lyapunov equation

$$AX + XA^T + Q = 0$$

is elementwise nonnegative.

*Proof:* This follows directly from Lemma 1 and the fact that  $X = \int_0^\infty e^{At} Q e^{A^T t} dt$ . ■

*Lemma 3:* [21, proposition 1.34] Let  $A \in \mathbb{R}^{n \times n}$  be such that  $\mathcal{G}(A)$  has self-edges at each node. Then, the following statements are equivalent:

- $\mathcal{G}(A)$  is strongly connected.
- $A^{n-1}$  is positive.

## II. PROBLEM FORMULATION

Given a directed network  $\mathcal{G}$  obeying consensus dynamics, we consider the problem of selecting an optimal leader or set of leaders to minimize some performance metric of the closed-loop system.

### A. Consensus dynamics

We consider a directed network  $\mathcal{G}$  with  $n$  nodes governed by consensus dynamics. Each node  $i$  updates its state,  $x_i$  by averaging it with the states of other nodes in its set of neighbors  $\mathcal{N}_i$ ,

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j - x_i) + b_i^T d,$$

where  $w_{ij} \geq 0$  is an edge weight that quantifies the importance of the link from node  $j$  to node  $i$ ,  $d$  is a stochastic disturbance, and  $b_i \geq 0$  describes how  $d$  affects the  $i$ th node. The aggregate dynamics can be written as,

$$\dot{x} = -Lx + Bd$$

where  $L$  is the weighted directed graph Laplacian [22] associated with  $\mathcal{G}$  and  $B \geq 0$ . In this work, we assume that the network  $\mathcal{G}$  is weakly connected.

The graph Laplacian has an eigenvalue at 0 corresponding to a right eigenvector of all ones, i.e., it is row stochastic,  $L\mathbf{1} = 0$ . If this eigenvalue is simple, the nodes  $x_i$  converge to a constant vector  $\bar{x}\mathbf{1}$  in the absence of external forcing. When  $\mathcal{G}$  is balanced,  $\bar{x} = \frac{1}{n}\mathbf{1}^T x(0)$  is the average of the initial node values. In general,  $\bar{x} = w^T x(0)$  where  $w$  is the left eigenvector of  $L$  corresponding to 0, i.e.  $w^T L = 0$ .

### B. Leader selection

In consensus networks, the dynamics are governed exclusively by relative information exchange between the nodes. In the leader selection paradigm, certain leader are additionally equipped with *absolute* information.

An example application is given by a kinematic model of vehicles where  $x$  represents the positions of a formation of autonomous vehicles. Relative information exchange over the consensus network corresponds to maintaining constant distances between neighboring vehicles in the network  $\mathcal{G}$ . In this scenario, leader nodes may have access to absolute information in the form of GPS units.

The dynamics of a node in this network is given by,

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j - x_i) - k_i x_i + b_i^T d,$$

where  $k_i \geq 0$  is the weight that node  $i$  places on its absolute position. If  $k_i > 0$ , it is a leader and if  $k_i = 0$  it is a follower. The aggregate dynamics can be compactly written as,

$$\begin{aligned} \dot{x} &= -(L + K)x + Bd \\ z &= Cx \end{aligned} \quad (1)$$

where  $C \geq 0$ ,  $z$  is a regulated output and  $K := \text{diag}(k)$ . We evaluate the performance of a leader vector  $k$  by the  $\mathcal{H}_2$  or  $\mathcal{H}_\infty$  norm of the associated closed-loop system (1) from the disturbances  $d$  to the regulated output  $z$ . We note that system (1) is marginally stable in the absence of leaders and much work on consensus networks focuses on driving the *deviations* from the average node value to zero [23]. Here, we focus on driving the node values themselves to zero.

The  $\mathcal{H}_2$  performance of (1) quantifies the variance amplification from  $d$  to  $z$  and can be computed by

$$\begin{aligned} J_2(k) &= \text{trace}(C^T X_c C) = \text{trace}(B X_o B^T) \\ 0 &= -(L + K) X_c - X_c (L + K)^T + B B^T \\ 0 &= -X_o (L + K) - (L + K)^T X_o + C^T C \end{aligned}$$

where  $X_c$  and  $X_o$  are the controllability and observability gramians respectively. The  $\mathcal{H}_\infty$  performance of (1) quantifies the worst case disturbance amplification from  $d$  to  $z$ ,

$$J_\infty(k) = \sup_{\omega} \bar{\sigma}(C(j\omega I + L + K)^{-1} B).$$

For positive systems the supremum over  $\omega$  is achieved at  $\omega = 0$  [16], [24], so

$$J_\infty(k) = \bar{\sigma}(C(L + K)^{-1} B).$$

### C. Leader selection problems

The problem of adding leaders to a consensus network involves both selecting the optimal set of leaders and designing the optimal weights for the selected leaders. We introduce two distinct problems to address these challenges.

*Problem 1:* Given a network  $\mathcal{G}$  with a set  $\mathcal{V}$  of  $n$  vertices, a number of leaders  $N \in \mathbb{Z}_+ \leq n$  and a leader weight  $\kappa$ , find the optimal set  $\mathcal{V}_L \subset \mathcal{V}$  of  $N$  leaders that solves

$$\begin{aligned} &\text{minimize } J(k) \\ &\text{subject to } k = \sum_{i \in \mathcal{V}_L} \kappa e_i \end{aligned}$$

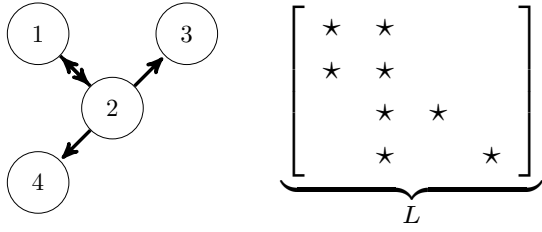


Fig. 1: A directed network and the sparsity pattern of the corresponding graph Laplacian. This network is stabilized if and only if either node 1 or node 2 are made leaders.

where  $J$  is one of the performance metrics described in section II-B.

Problem 1 is a combinatorial problem which assumes a fixed leader weight  $\kappa$ . By introducing a regularized version of the leader selection problem, we jointly tackle the problems of selecting leaders and designing their weights.

*Problem 2:* Given a network  $\mathcal{G}$ , a quadratic leader penalty  $R \succeq 0$  and an  $\ell_1$  penalty parameter  $\gamma$ , design a vector of leader weights  $k$  to solve,

$$\text{minimize } J(k) + \gamma \|k\|_1 + k^T R k$$

where  $J$  is a performance metric, and  $\|k\|_1 = \sum |k_i|$ .

The matrix  $R \succeq 0$  imposes a quadratic penalty on the leader weights and thereby limits their magnitude. The  $\ell_1$  norm is a commonly used proxy for the cardinality function and the associated parameter,  $\gamma$ , specifies the importance on sparsity. We note that Problem 2 can be augmented with arbitrary convex regularizers, such as a budget constraints or box constraints on the leader weights.

It follows from recent results that Problem 2 is a convex problem [20] in the leader vector  $k$ . Although decentralized control of positive systems can be solved with convex programming, standard methods [16], [17], require a change of variables that does not us to impose arbitrary convex regularization on  $k$  for the purpose of leader selection.

### III. LEADER SELECTION IN DIRECTED NETWORKS

#### A. Stability

For a vector of leader weights  $k$  to be feasible for Problems 1 or 2, it must stabilize system (1); i.e.,  $-(L+K)$  must be Hurwitz. In a connected undirected consensus network, any  $k \geq 0$  with at least one nonzero entry is stabilizing.

This is not the case for directed networks. In Fig. 1, making node 1 or 2 a leader will stabilize the network but making nodes 3 or 4 a leader will not. Our first result is a condition for the stabilization of a directed consensus network by a set of leaders. This will allow us to impose closed-loop stability as an linear constraint on  $k$ . First, we restate a well-known result from matrix theory.

*Theorem 4:* Let  $A \in \mathbb{C}^{n \times n}$  have entries  $a_{ij}$ . Let  $D(c, R)$  be the closed disc centered at  $c$  with radius  $R$ . Every eigenvalue of  $A$  lies within at least one of the Gershgorin

discs,

$$D(a_{ii}, \sum_{i \neq j} |a_{ij}|).$$

*Proof:* See [25]. ■

*Theorem 5:* Let  $L$  be a weighted directed graph Laplacian and let  $K = \text{diag}(k)$ . The system,

$$\dot{x} = -(L + K)x$$

is stable if and only if  $w \circ k \neq 0$  for all nonzero  $w$  such that  $w^T L = 0$  where  $\circ$  is the elementwise (Hadamard) product.

*Proof:* ( $\Leftarrow$ ) If  $w \circ k = 0$ ,  $w^T K = 0$ . If in addition,  $w^T L = 0$ ,

$$w^T (L + K) = 0,$$

and therefore 0 is an eigenvalue of  $-(L + K)$ .

( $\Rightarrow$ ) Since the graph Laplacian  $L$  is row stochastic and  $K$  is diagonal and nonnegative, by the Gershgorin circle theorem 4, the eigenvalues of  $-(L + K)$  are at most 0. Therefore, to show that  $-(L + K)$  is Hurwitz, it suffices to show that it has no eigenvalue at 0. We show this by contradiction. Assume there exists a nonzero  $w$  such that

$$-w^T (L + K) = 0.$$

This implies that either  $w^T L = w^T K = 0$  or that  $w^T L = -w^T K$ . The first case is not possible because  $w^T K \neq 0$  for any  $w$  such that  $w^T L = 0$  since  $w \circ k \neq 0$  by assumption. If the second case is true, then  $w^T L v = -w^T K v$  must also hold for all  $v$ . However, if we take  $v = \mathbf{1}$ , then  $w^T L \mathbf{1} = 0$  but  $-w^T K \mathbf{1}$  is nonzero. This completes the proof. ■

*Remark 1:* Only the set of leader nodes is relevant to the question of stability. If  $k$  does not stabilize (1), no  $\alpha k$  with  $\alpha > 0$  will. Similarly if  $k$  stabilizes (1), every  $\alpha k$  will.

Intuitively, this condition requires that there is a path from the set of leader nodes to every node in the network.

*Corollary 6:* If  $\mathcal{G}$  is strongly connected, any choice of leader node will stabilize (1).

*Proof:* The graph Laplacian associated with a strongly connected graph is irreducible. By the Perron-Frobenius theorem for irreducible matrices [25], the left eigenvector associated with  $-L$  is positive. As such,  $w \circ k \neq 0$  for any nonzero  $k$  and therefore the system is stable by Theorem 5. ■

#### B. Balanced graphs

*Proposition 7:* Let  $\mathcal{G}$  be a balanced graph with graph Laplacian  $L$ , let  $\hat{\mathcal{G}}$  be the same graph with all edges reversed, and let  $B = C^T$ . Then, the  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norms associated with a given set of leader weights  $k$  are the same for both  $\mathcal{G}$  and  $\hat{\mathcal{G}}$ .

*Proof:* Since  $\mathcal{G}$  is balanced, the graph Laplacian associated with  $\hat{\mathcal{G}}$ ,  $\hat{L} = L^T$ . Since  $B = C^T$ , the controllability gramian of (1) associated with  $\mathcal{G}$  is the observability gramian of (1) associated with  $\hat{\mathcal{G}}$ . Equivalence of the  $\mathcal{H}_2$  norm follows from its expression. Equivalence of the  $\mathcal{H}_\infty$  norm follows from invariance of the singular values under transposition, i.e.,  $\sigma(C(L + K)^{-1}B) = \sigma(B^T(L^T + K)^{-1}C^T)$  ■

The space of balanced graphs is spanned by cycles. Interestingly therefore, the optimal set of leaders is invariant

under reversal of all cycles. In [26], the authors explored how cycles affect undirected consensus networks. This result suggests that cycles also play a fundamental role in the performance of directed consensus networks.

*Proposition 8:* Let  $\mathcal{G}$  be a balanced graph with graph Laplacian  $L$  and let  $B = C = I$ . Let  $\hat{\mathcal{G}}$  represent the graph corresponding to  $\hat{L} := \frac{1}{2}(L + L^T)$ . Then,  $\mathcal{H}_2$  or  $\mathcal{H}_\infty$  performance of any leader vector  $k$  with  $\hat{\mathcal{G}}$  is an upper bound on the performance of  $k$  with  $\mathcal{G}$ .

*Proof:* Since  $\mathcal{G}$  is balanced,  $\hat{L}$  is a graph Laplacian which corresponds to the symmetric component of  $L$ . By [27] and [28], the  $\mathcal{H}_2$  performance and the  $\mathcal{H}_\infty$  performance corresponding to the symmetric component of a dynamical system is an upper bound on the performance of original system. ■

This property is convenient since, by [8], [11], optimal sets of 1 or 2 leaders in undirected consensus networks can be determined explicitly from graph theoretic properties. In other words, the explicit solution to Problem 1 for an undirected network, given by [8], [11], provides an upper bound on Problem 1 for a balanced network.

#### IV. METHODS FOR LEADER SELECTION

In this section we develop methods for solving the leader selection problem 2. Problem 2 is convex for the performance metrics  $J$  [20] introduced in section II-B.

##### A. Regularized problem

Problem 2 can be used to simultaneously find the optimal leaders and design leader weights. We propose Algorithm 1 to select  $N$  leaders. Even though  $J_2$  and  $J_\infty$  are convex, it is not clear if they admit formulations which are amenable to standard solvers. Since  $\|k\|_1$  is not differentiable, we employ a proximal gradient descent method to obtain a solution.

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##### Algorithm 1: Leader selection algorithm

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Set  $\gamma > 0, N \leq n \in \mathbb{N}, R \succ 0$ ;
Identify topology
while  $\text{card}(k_\gamma) > N$  do
     $k_\gamma = \text{argmin}_u J(k) + k^T Rk + \gamma \|k\|_1$ ;
    increase  $\gamma$ ;
end
Design optimal edge weights
 $u^* = \text{argmin} J(k) + k^T Rk$ 
    subject to  $\text{sp}(k) \subseteq \text{sp}(k_\gamma)$ . ;

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*Proximal (sub)gradient descent:* Proximal (sub)gradient descent provides a generalization of standard gradient descent which can be applied to nonsmooth optimization problems [29].

$$k^{j+1} = \mathcal{S}_{\alpha\gamma}(k^j - \alpha(\nabla J(k^j) + 2Rk^j))$$

where  $\mathcal{S}_\mu$  is the soft-thresholding operator,

$$\mathcal{S}(v_i) = \begin{cases} 0 & |v_i| \leq \mu \\ v_i - \mu & v_i \geq \mu \\ v_i + \mu & v_i \leq -\mu \end{cases}$$

We employ standard backtracking and termination criteria [30] for proximal gradient methods. When  $J$  is nonsmooth, as is the case for  $\mathcal{H}_\infty$  control of certain networks, we use proximal subgradient methods to obtain a solution [31].

##### B. Gradients of the performance metrics

###### 1) $\mathcal{H}_2$ norm:

*Proposition 9:* Let  $-(L + K)$  be Hurwitz and let  $K = \text{diag}(k)$ . Then,

$$\nabla J_2(k) = 2\text{diag}(X_c X_o) \quad (2)$$

where  $X_c$  and  $X_o$  are the controllability and observability Gramians of the closed-loop system (1),

$$\begin{aligned} -(L + K)X_c - X_c(L + K)^T + BB^T &= 0 \\ -(L + K)^T X_o - X_o(L + K) + C^T C &= 0. \end{aligned}$$

*Proof:* See [20]. ■

*Remark 2:* Lemma 2 implies that  $X_c$  and  $X_o$  are nonnegative. Therefore, the diagonal of  $X_c X_o$  is positive and, since the gradient is a weighted sum these diagonal elements,  $J_2$  is a monotone function of  $k$ . This implies that increasing the leader weight of any leader always decreases  $J_2$ .

*Remark 3:* The matrix  $X_c X_o$  appears often in model reduction. The gradient with respect to  $k_i$  corresponds to the inner product between the  $i$ th columns of  $X_c$  and  $X_o$ .

###### 2) $\mathcal{H}_\infty$ norm:

*Proposition 10:* Let  $A_{cl} := -(L + K)$  be Hurwitz and let  $K = \text{diag}(k)$ . Then,

$$\begin{aligned} \partial J_\infty(d) = \left\{ \sum_i \alpha_i \text{diag}(A_{cl}^{-1} v_i w_i^T A_{cl}^{-1}) \right. \\ \left. | w_i^T A_{cl}^{-1} v_i = J_\infty(u), \alpha_i \in \mathcal{P} \right\}. \end{aligned} \quad (3)$$

where  $\mathcal{P} := \{\alpha_i | \alpha_i \geq 0, \sum_i \alpha_i = 1\}$

*Proof:* See [20]. ■

*Proposition 11:* If  $\mathcal{G}$  is strongly connected,  $J_\infty$  is a continuously differentiable function of  $k$ .

*Proof:* See [20]. ■

*Remark 4:* The gradient at node  $i$  is given by,

$$\nabla_{u_i} J_\infty(u) = (e_i^T A_{cl}^{-1} v)(w^T A_{cl}^{-1} e_i).$$

Since  $-A_{cl}^{-1}$ ,  $w$ , and  $v$  are nonnegative, the gradient is always nonnegative so  $J_\infty$  is a monotone function of  $k$ .

*Remark 5:* The quantity  $e_i^T A_{cl}^{-1} v$  is a measure of how much node  $i$  is affected by the input forcing which causes the largest overall response of system (1). In contrast,  $w^T A_{cl}^{-1} e_i$  measures how much forcing at node  $i$  affects the direction of the largest output response.

#### V. NUMERICAL EXAMPLES

Here, we perform simultaneous leader selection and leader weight design for two simple directed networks and a real world network.

### A. Synthetic examples

Consider the directed graph with unit edge weights whose topology is shown in Figure 2. We choose the input matrix as  $B = \text{diag}(1, 1, 1, 1, 5, 1)$  to make node 5 most affected by the disturbance. The output matrix is  $C = I$  and the quadratic control penalty is given by  $R = I$ . For the  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  performance metrics, we employ Algorithm 1 to identify a single leader. Since the  $\mathcal{H}_\infty$  performance index is concerned with the worst case disturbance amplification, it identifies node 5 as the most important. In contrast, the  $\mathcal{H}_2$  performance index measures average performance and identifies node 3 as the best single leader.

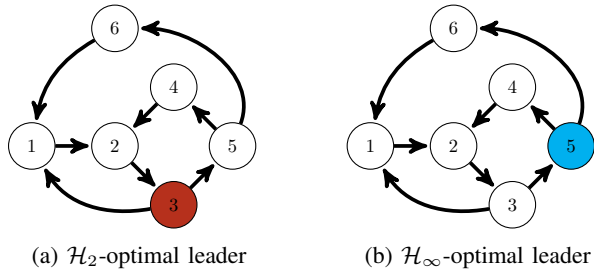


Fig. 2:  $\mathcal{H}_2$  vs.  $\mathcal{H}_\infty$  leader selection.

We next consider the directed network in Figure 3 with unit edge weights and employ Algorithm 1 for  $\mathcal{H}_2$  optimal leader selection. As  $\gamma$  is increased, fewer leaders are selected. Note that although node 7 has the largest out-degree, it is not identified as one of the three most important leaders. Table I shows the optimal leader weights for different  $N$ .

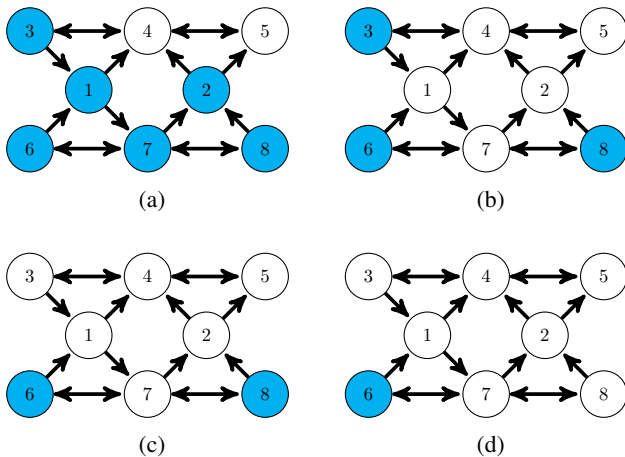


Fig. 3:  $\mathcal{H}_2$  leader selection for  $N = 6, 3, 2, 1$  corresponding to  $\gamma = 0.65, 1.97, 3.43, 125.9$

### B. College Football Ranking

Leader selection has also been used as a proxy for identifying important nodes in a network. Inspired by the recent use of graph theoretic tools for ranking athletic teams and to illustrate the utility of our algorithm on real data, we consider the problem of ranking college football teams.

Due to the number of teams in the top division of college football (128) and the relative scarcity of games between

$N = 6$	$N = 3$	$N = 2$	$N = 1$
$\begin{bmatrix} 0.385 \\ 0.361 \\ 0.606 \\ 0 \\ 0 \\ 0.676 \\ 0.451 \\ 0.650 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.693 \\ 0 \\ 0 \\ 0.832 \\ 0 \\ 0.787 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.950 \\ 0 \\ 0.898 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.364 \\ 0 \\ 0 \end{bmatrix}$

TABLE I: Optimal leader weights

them (around 13 per team), ranking these teams is an underdetermined problem. The current practice of ranking by a committee is clearly subject to bias. Recently, graph theoretic measures, such as average path length from each node, have been explored for the purpose of objectively ranking teams or athletes [32], [33].

We used the scores of college football games from the 2015 – 2016 season collected from [34] to generate a network. If team A beat team B, a edge was placed from A to B with a weight equal to the score difference in the game. There were 203 teams (nodes) and 863 games (edges) in our data set. We use Algorithm 1 to select the top  $N$  teams by identifying  $N$   $\mathcal{H}_2$  optimal leaders. Interestingly, the metrics we use are biased against selecting leaders which are close in the network. In this context, such proximity would correspond to teams who have many common opponents.

Fig. 4 shows the network generated. The large connected component in the center represents the teams in the top division. Since our dataset included games played between teams from the top division and lower divisions but not from games played between teams of the lower divisions, there are a number of topographically isolated nodes.

Table II shows sets of 2, 4, 6, and 8 leaders with the corresponding end-of-season rankings from the Associated Press (AP) [35]. Our algorithm selected teams which agree well with the AP rankings with the notable exception of Southern Illinois. This team played only one game in the our set – a close loss to a poorly ranked team. We ascribe this anomaly to the the topological isolation of this team.

$\gamma$	0.225	0.6	1	10
Teams	8	6	4	2
AP #1	Alabama	Alabama	Alabama	Alabama
AP #4	Ohio State	Ohio State	Ohio State	Ohio State
AP #2	Clemson	Clemson	Clemson	
AP #8	Houston	Houston	Houston	
AP #3	Stanford	Stanford		
AP NR	S. Illinois	S. Illinois		
AP #12	Michigan			
AP #10	Mississippi			

TABLE II: Leaders selected for different values of  $\gamma$ .

## VI. CONCLUDING REMARKS

We study the leader selection problem for directed consensus networks and provide a necessary and sufficient condition



Fig. 4: Network of games in the 2015–2016 College Football season. The central connected component corresponds to the top division, and the distal nodes are lower division teams for whom there are less data.

for stability of the system with a given set of leaders. We explore the properties of balanced graphs with regards to leader selection. Using recent results on the convexity of positive systems, we develop an algorithm for simultaneously selecting leaders and designing their edge weights. Finally, we illustrate the utility of our algorithm on two synthetic examples and one real-world application.

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