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A primal-dual Laplacian gradient flow dynamics for distributed resource allocation problems

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***Abstract*— We employ the proximal augmented Lagrangian method to solve a class of convex resource allocation problems over a connected undirected network of n agents. The agents are coupled by a linear resource equality constraint and their states are confined to a nonnegative orthant. By introducing the indicator function associated with a nonnegative orthant, we bring the problem into a composite form with a non-smooth objective and linear equality constraints. A primal-dual Laplacian gradient flow dynamics based on the proximal augmented Lagrangian is proposed to solve the problem in a distributed way. These dynamics conserve the sum of the agent states and the corresponding equilibrium points are the Karush-Kuhn-Tucker points of the original problem. We combine a Lyapunov-based argument with LaSalle's invariance principle to establish global asymptotic stability and use an economic dispatch case study to demonstrate the effectiveness of the proposed algorithm.**

I. Introduction

The resource allocation problem considers how to allocate the resource to individuals by minimizing the cost of the allocation. It arises in many areas, including communication networks, power grids, and production management in economics [1]. Due to the emergence of the next-generation communication networks and the cyber-physical systems, the resource allocation problem seeks to allocate the given resource in a distributed way, using local information exchange in a network. Particular instances of resource allocation are given by the network utility maximization [2]–[5] and the economic dispatch problems [6]–[10].

A class of resource allocation problems can be formulated as follows

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & \mathbb{1}^T x - b = 0 \\ & x_i \in \Omega_i, \; i = 1, \ldots, n \end{array} \qquad (1)$$

where $f(x) = \sum_{i=1}^{n} f_i(x_i)$ is a separable convex objective function, $x = [x_1 \cdots x_n]^T \in \mathbb{R}^n$ is the optimization variable, $\mathbb{1}$ is the vector of all-ones, $b \in \mathbb{R}$ is the given resource, and Ω_i is a convex set. All x_i's are coupled by the resource equality constraint and each x_i has a set constraint.

Distributed resource allocation problems have attracted significant attention in recent years. In this setup, each node i in a network has a local objective function f_i and a decision variable x_i. The nodes can exchange information with their neighbors with the goal of computing solution of the resource allocation problem in a distributed manner. Many distributed algorithms have been proposed for (1) or have been applied to (1) as a case study. Two main classes include the consensus-based algorithms and the primal-dual dynamics. Some recent works in the first class include [7]–[11], where consensus algorithms were incorporated into the centralized gradient descent to enable distributed computation. In [8], a robust distributed algorithm was presented for the problem with box constraints. In [9], two distributed projected algorithms were proposed. While [8] and [9] do not require any special initialization, algorithms in [7], [10], [11] do need particular initialization. For (1) without the set constraint, the exponential convergence of consensus-based algorithms was established in [9]–[11].

The second class is based on primal-dual dynamics [12]–[15], where the set constraint can be inequality constraints. In [12], [13], for a non-smooth augmented Lagrangian, it was shown that its primal-dual dynamics can converge exponentially for (1) with either the resource equality constraint or the set constraint. In [14], [15], global asymptotic stability of the projected primal-dual dynamics was established by utilizing local convexity-concavity of the saddle function. The asymptotic convergence of the primal-dual dynamics was further discussed in [16]–[18]. Although the primal-dual dynamics are asymptotically stable, the set constraint may cause the difficulty in establishing the exponential convergence and the resource allocation constraint can impede the distributed implementation.

Motivated by the above results, we are interested in developing a distributed primal-dual algorithm to solve (1) with non-negative constraints. To deal with such constraints, we utilize the proximal augmented Lagrangian method [19] which yields a non-smooth composite optimization problem with linear equality constraints. For this class of problems, the gradient flow dynamics associated with the proximal Lagrangian are not convenient for distributed implementation. Instead, we propose a primal-dual Laplacian gradient flow dynamics to compute the solution via in-network optimization and prove that these dynamics converge to the optima globally and asymptotically. Finally, we use the economic dispatch problem to illustrate the performance of the proposed algorithm.

Our presentation is organized as follows. In Section II, we formulate the problem and provide background on the proximal augmented Lagrangian method. In Section III, we propose primal-dual Laplacian gradient flow dynamics for

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in-network resource allocation and combine a Lyapunov-based argument with LaSalle's invariance principle to establish global asymptotic stability. In Section IV, we use computational experiments on an economic dispatch problem to demonstrate merits and effectiveness of our approach. We conclude the paper and highlight future directions in Section V.

II. Problem formulation and background

In this section, we formulate the resource allocation problem as a non-smooth composite optimization problem and introduce the proximal augmented Lagrangian.

We consider the resource allocation problem,

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & \mathbb{1}^T x \,-\, b \,=\, 0 \\ & x \,\geq\, 0 \end{array} \tag{2}$$

where f is a convex objective function, $x \in \mathbb{R}^n$ is the optimization variable, and b is the given resource. For $b > 0$, optimization problem (2) is feasible. By introducing an auxiliary variable $z := [\, z_1 \, \cdots \, z_n \,]^T$, (2) can be cast as a non-smooth composite optimization with equality constraints,

$$\begin{array}{ll} \underset{x,\,z}{\text{minimize}} & f(x) \,+\, g(z) \\ \text{subject to} & \mathbb{1}^T x \,-\, b \,=\, 0 \\ & x - z = 0 \end{array} \tag{3}$$

where $g(z) = \sum_{i=1}^{n} g_i(z_i)$ and each g_i is an indicator function of the non-negative orthant,

$$g_i(z_i) \,:=\, \begin{cases} 0, & z_i \,\geq\, 0 \\ \infty, & \text{otherwise.} \end{cases} \tag{4}$$

Since the objective function in (3) is non-differentiable, the existing results (e.g., [13]) cannot be employed directly to establish exponential convergence. The proximal augmented Lagrangian method was recently proposed in [19] to deal with non-smooth composite optimization problems. The proposed approach exploits the proximal operator associated with the non-smooth part of the objective function to eliminate the auxiliary variable z from the augmented Lagrangian.

The proximal operator associated with the function g is the minimizer of the following optimization problem

$$\mathbf{prox}_{\mu g}(v) \,:=\, \underset{z}{\text{argmin}} \;\, g(z) \,+\, \frac{1}{2\mu}\,\|z \,-\, v\|^2.$$

Here, v is a given vector, μ is a positive parameter, and the associated value function specifies the Moreau envelope,

$$M_{\mu g}(v) \,=\, g(\mathbf{prox}_{\mu g}(v)) \,+\, \frac{1}{2\mu}\,\|\mathbf{prox}_{\mu g}(v) \,-\, v\|^2.$$

Notably, the Moreau envelope is continuously differentiable, even when g is not, and its gradient is determined by

$$\nabla M_{\mu g}(v) \,=\, \frac{1}{\mu}\left(v \,-\, \mathbf{prox}_{\mu g}(v)\right).$$

For the indicator function of the non-negative orthant (4), we have

$$\begin{aligned} \mathbf{prox}_{\mu g}(v_i) \,&=\, [v_i]_+ \,=\, \begin{cases} v_i, & v_i \,\geq\, 0 \\ 0, & v_i \,<\, 0 \end{cases} \\ M_{\mu g}(v_i) \,&=\, \tfrac{1}{2\mu}\,[v_i]_-^2, \\ \nabla M_{\mu g}(v_i) \,&=\, \tfrac{1}{\mu}\,[v_i]_-. \end{aligned}$$

The Lagrangian associated with (3) is given by

$$\mathcal{L}(x,z;\lambda,y) \,=\, f(x) + g(z) + \lambda(\mathbb{1}^T x - b) + y^T(x - z) \tag{5}$$

where $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$ are the Lagrange multipliers. The augmented Lagrangian contains additional terms that introduce quadratic penalties on the violation of linear constraints,

$$\begin{aligned} \mathcal{L}_\mu(x,z;\lambda,y) \,=\, &\,\mathcal{L}(x,z;\lambda,y) \,+ \\ &\tfrac{1}{2\mu}\left((\mathbb{1}^T x - b)^2 \,+\, \|x - z\|^2\right) \end{aligned} \tag{6}$$

where μ is a positive parameter.

Without loss of generality, we have selected the same penalty on the violation of both linear constraints in (3). The completion of squares can be used to rewrite $\mathcal{L}_\mu$ as

$$\begin{aligned} \mathcal{L}_\mu(x,z;\lambda,y) \,=\, &\,f(x) \,+\, \lambda(\mathbb{1}^T x - b) \,+\, \tfrac{1}{2\mu}\,(\mathbb{1}^T x - b)^2 \,+ \\ &g(z) \,+\, \tfrac{1}{2\mu}\,\|z - (x + \mu y)\|^2 \,-\, \tfrac{\mu}{2}\|y\|^2 \end{aligned} \tag{7}$$

and the explicit minimizer of (7) with respect to z is given by

$$z_\mu^\star(x,y) \,=\, \mathbf{prox}_{\mu g}(x \,+\, \mu y).$$

As demonstrated in [19], the proximal augmented Lagrangian is obtained by restricting the augmented Lagrangian on the manifold that results from explicit minimization of $\mathcal{L}_\mu$ over z. This eliminates the non-smooth term from $\mathcal{L}_\mu$ and casts it in terms of the Moreau envelope,

$$\begin{aligned} \mathcal{L}_\mu(x;\lambda,y) \,=\, &\,f(x) \,+\, \lambda(\mathbb{1}^T x - b) \,+\, \tfrac{1}{2\mu}\,(\mathbb{1}^T x - b)^2 \,+ \\ &M_{\mu g}(x + \mu y) \,-\, \tfrac{\mu}{2}\|y\|^2 \end{aligned} \tag{8}$$

The Arrow-Hurwicz-Uzawa gradient flow dynamics based on proximal augmented Lagrangian (8) is given by

$$\begin{aligned} \dot{x} \,&=\, -\,\nabla\mathcal{L}_\mu(x;\lambda,y) \\ \dot{\lambda} \,&=\, +\,\nabla\mathcal{L}_\mu(x;\lambda,y) \\ \dot{y} \,&=\, +\,\nabla\mathcal{L}_\mu(x;\lambda,y) \end{aligned}$$

or, equivalently,

$$\begin{aligned} \dot{x} \,&=\, -(\nabla f(x) + \nabla M_{\mu g}(x + \mu y) + \mathbb{1}\lambda + \tfrac{1}{\mu}\mathbb{1}(\mathbb{1}^T x - b)) \\ \dot{\lambda} \,&=\, \mathbb{1}^T x - b \\ \dot{y} \,&=\, \mu\,(\nabla M_{\mu g}(x + \mu y) \,-\, y). \end{aligned} \tag{9}$$

In [19], global asymptotic stability of the gradient flow dynamics for problems of the form (3), but without the resource allocation constraint, was established. The above gradient flow dynamics accounts for this constraint; however, due to the appearance of $\mathbb{1}^T x$, the update of x_i in (9) requires global information which impedes distributed implementation. In what follows, we modify (9) in order to solve the resource allocation problem (2) over a connected undirected network and establish global asymptotic stability of the resulting primal-dual Laplacian gradient flow dynamics.

III. Primal-dual Laplacian gradient flow dynamics

In this section, we propose a modification of the Arrow-Hurwicz-Uzawa gradient flow dynamics that satisfies the resource allocation constraint for all times. We show that the equilibrium points of the resulting Laplacian gradient flow dynamics correspond to the KKT points in (3) and prove global asymptotic stability.

It is easy to show that the following primal-dual Laplacian gradient flow dynamics

$$\begin{aligned}\dot{x} \;&=\; -L\,(\nabla f(x) \,+\, \nabla M_{\mu g}(x+\mu y)) \\ \dot{y} \;&=\; \mu\,(\nabla M_{\mu g}(x+\mu y) \,-\, y)\end{aligned} \tag{10}$$

satisfies $\mathbb{1}^T x(t) - b = 0$ for all $t \geq 0$ if $\mathbb{1}^T x(0) - b = 0$. This is because $\mathbb{1}^T \dot{x} = 0$ implies that $\mathbb{1}^T x(t)$ is a conserved quantity of (10). Furthermore, since $f(x)$ and $\nabla M_{\mu g}(x+\mu y)$ are sums of n individual functions, primal-dual Laplacian gradient flow dynamics (10) is a distributed algorithm.

The set of KKT points for (3) is given by

$$\Omega^\star = \left\{ (x,z,\lambda,y) \,\middle|\, \begin{array}{l} \nabla f(x) + \mathbb{1}\lambda + y = 0,\ x - z = 0 \\ y \in \partial g(z),\ \mathbb{1}^T x - b = 0 \end{array} \right\}$$

where $\partial g(z)$ is the sub-gradient set of g at z. On the other hand, if $\mathbb{1}^T x(0) - b = 0$, the equilibrium points $(\bar{x},\bar{y})$ of (10) are given by

$$\bar{\Omega} = \left\{ (x,y) \,\middle|\, \begin{array}{l} L\,(\nabla f(x) + \nabla M_{\mu g}(x+\mu y)) = 0 \\ \nabla M_{\mu g}(x+\mu y) - y = 0,\ \mathbb{1}^T x - b = 0 \end{array} \right\}$$

The following lemma establishes correspondence between the sets $\Omega^\star$ and $\bar{\Omega}$.

Lemma 1: For any $(\bar{x},\bar{y}) \in \bar{\Omega}$, there exists $z^\star$ and $\lambda^\star$ such that $(\bar{x},z^\star,\lambda^\star,\bar{y}) \in \Omega^\star$. Similarly, for any $(x^\star,z^\star,\lambda^\star,y^\star) \in \Omega^\star$, $(x^\star,\lambda^\star) \in \bar{\Omega}$.

Proof: For any $(\bar{x},\bar{y}) \in \bar{\Omega}$, $\nabla f(\bar{x}) + \nabla M_{\mu g}(\bar{x}+\mu\bar{y})$ is in the null space of the Laplacian L and $\nabla M_{\mu g}(\bar{x}+\mu\bar{y}) = \bar{y}$. Thus, there is a non-zero λ such that $\nabla f(\bar{x}) + \bar{y} + \lambda\mathbb{1} = 0$. Furthermore,

$$\nabla M_{\mu g}(\bar{x}+\mu\bar{y}) = \frac{1}{\mu}\big(\bar{x}+\mu\bar{y} - \mathbf{prox}_{\mu g}(\bar{x}+\mu\bar{y})\big) = \bar{y}$$

implies $\bar{x} = \mathbf{prox}_{\mu g}(\bar{x}+\mu\bar{y}) = \bar{z}$ and, consequently, $\bar{y} = -(\nabla f(\bar{x}) + \lambda\mathbb{1}) \in \partial g(\bar{x})$. Finally, since the equality constraint $\mathbb{1}^T\bar{x} = b$ is, by construction, satisfied for all times, by selecting $x^\star = z^\star = \bar{x}$, $\lambda^\star = \lambda$, and $y^\star = \bar{y}$, we conclude that $(x^\star,z^\star,\lambda^\star,y^\star) \in \Omega^\star$.

Conversely, for any $(x^\star,z^\star,\lambda^\star,y^\star) \in \Omega^\star$, we have $L(\nabla f(x^\star) + y^\star) = 0$, $\mathbb{1}^T x^\star = b$, and $y^\star \in \partial g(x^\star)$. Furthemore, since $x^\star = z^\star = \mathbf{prox}_{\mu g}(x^\star + \mu y^\star)$, from the definition of the gradient of the Moreau envelope, we have $\nabla M_{\mu g}(x^\star + \mu y^\star) = y^\star$. By selecting $\bar{x} = x^\star$, $\bar{y} = y^\star$, we conclude that $(\bar{x},\bar{y}) \in \bar{\Omega}$. ■

In what follows, we conduct stability analysis of primal-dual Laplacian gradient flow dynamics (10) with the initial condition $\mathbb{1}^T x(0) = b$. This is done for objective functions f in (2) that are strongly convex and have Lipschitz continuous gradients.

Assumption 1: The objective function f in (2) is strongly convex and its gradient is Lipschitz continuous.

A. Transformed primal-dual Laplacian gradient flow dynamics

The eigenvalue decomposition of the graph Laplacian L of a connected undirected network is given by

$$L = V\Lambda V^T = \begin{bmatrix} U & \tfrac{1}{n}\mathbb{1} \end{bmatrix} \begin{bmatrix} \Lambda_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U^T \\ \tfrac{1}{n}\mathbb{1}^T \end{bmatrix} = U\Lambda_0 U^T$$

where Λ_0 is a diagonal matrix of non-zero eigenvalues of L and the matrix $U \in \mathbb{R}^{n\times(n-1)}$ and satisfies the following properties:

(i) $U^T U = I$;
(ii) $UU^T = I - \frac{1}{n}\mathbb{1}\mathbb{1}^T$;
(iii) $\mathbb{1}^T U = 0$ and $U^T\mathbb{1} = 0$.

We introduce an affine coordinate transformation

$$x = U\xi + \mathbb{1}\theta \tag{11}$$

where $\xi \in \mathbb{R}^{n-1}$ and $\theta = \frac{1}{n}\mathbb{1}^T x(0)$. From the above properties of U, we have $\xi = U^T x$, and (10) can be transformed into the following form

$$\begin{aligned}\dot{\xi} \;&=\; -\Lambda_0 U^T\,(\nabla f(x) + \nabla M_{\mu g}(x+\mu y)\,) \\ \dot{y} \;&=\; \mu\,(\nabla M_{\mu g}(x+\mu y) \,-\, y)\end{aligned} \tag{12}$$

where $\xi(0) \in \mathbb{R}^{n-1}$ and $y(0) \in \mathbb{R}^n$ are arbitrary initial conditions.

B. Global asymptotic stability

We first state a lemma about proximal operators which is used in the proof of Theorem 3.

Lemma 2: Let $g : \mathbb{R}^n \to \mathbb{R}$ be a proper, lower semicontinuous, convex function and let $\mathbf{prox}_{\mu g} : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding proximal operator. Then, for any $a, b \in \mathbb{R}^n$, we can write

$$\mathbf{prox}_{\mu g}(a) \,-\, \mathbf{prox}_{\mu g}(b) \,=\, D(a-b) \tag{13}$$

where D is a symmetric matrix satisfying $0 \preceq D \preceq I$.

Proof: See [19, Lemma 2]. ■

We next establish global asymptotic stability of transformed primal-dual Laplacian gradient flow dynamics (12).

Theorem 3: Let Assumption 1 hold. Then, (12) is globally asymptotically stable.

Proof: Let $(\tilde{\xi},\tilde{y})$ denote perturbations around the equilibrium points $(\bar{\xi},\bar{y})$ of (12). Then,

$$\begin{aligned}\dot{\tilde{\xi}} \;&=\; -\Lambda_0 U^T\,(\nabla f(x) - \nabla f(\bar{x}) + \tfrac{1}{\mu}\,\tilde{m}\,) \\ \dot{\tilde{y}} \;&=\; \tilde{m} - \mu\tilde{y}\end{aligned} \tag{14}$$

where $\tilde{m} = \mu(\nabla M_{\mu g}(x+\mu y) - \nabla M_{\mu g}(\bar{x}+\mu\bar{y}))$, $x = U\xi + \mathbb{1}\theta$, and $\bar{x} = U\bar{\xi} + \mathbb{1}\theta$.

Derivative of the Lyapunov function candidate

$$V(\tilde{\xi},\tilde{y}) \,=\, \frac{1}{2}\,\tilde{\xi}^T\Lambda_0^{-1}\tilde{\xi} \,+\, \frac{1}{2}\|\tilde{y}\|^2$$

along the solutions of (14) is determined by

$$\begin{aligned}\dot{V} \,&=\, \tilde{\xi}^T\Lambda_0^{-1}\dot{\tilde{\xi}} + \tilde{y}^T\dot{\tilde{y}} \\ &=\, -\tilde{\xi}^T U^T(\nabla f(U\xi + \mathbb{1}\theta) - \nabla f(U\bar{\xi} + \mathbb{1}\theta)) - \tfrac{1}{\mu}\tilde{\xi}^T U^T\tilde{m} \\ &\quad + \tilde{y}^T\tilde{m} \,-\, \mu\|\tilde{y}\|^2\end{aligned}$$

From Lemma 2, we have $\tilde{m} = (I - D)(U\tilde{\xi} + \mu\tilde{y})$, where $0 \preceq D \preceq I$, and

$$\begin{aligned}\dot{V} \,=&\, -\tilde{\xi}^T U^T(\nabla f(U\xi + \mathbb{1}\theta) - \nabla f(U\bar{\xi} + \mathbb{1}\theta)) \\ &- \tfrac{1}{\mu}\tilde{\xi}^T U^T(I - D)U\tilde{\xi} \,-\, \mu\tilde{y}^T D\tilde{y}.\end{aligned} \tag{15}$$

Strong convexity of f implies $\dot{V} \leq 0$. Furthermore, $\dot{V} = 0$ is equivalent to $U\tilde{\xi} = 0$ and $D\tilde{y} = 0$.

We next show that the largest invariant set of $\dot{V} = 0$ is contained in $\bar{\Omega}$. First, from $\tilde{x} = U\tilde{\xi}$, we have $x = \bar{x}$. Second, from the property of the matrix D in Lemma 2, we have $\mathbf{prox}_{\mu g}(\bar{x} + \mu y) = \mathbf{prox}_{\mu g}(\bar{x} + \mu\bar{y})$. Third, $\bar{x} = \mathbf{prox}_{\mu g}(\bar{x}+\mu\bar{y})$ implies $\frac{1}{\mu}(\bar{x}+\mu y-\mathbf{prox}_{\mu g}(\bar{x}+\mu y))-y = \nabla M_{\mu g}(\bar{x} + \mu y) - y = 0$. Thus, (x, y) satisfies the second and third conditions in $\bar{\Omega}$.

Furthermore, since $L(\nabla f(\bar{x}) + \nabla M_{\mu g}(\bar{x} + \mu y)) = L(\nabla M_{\mu g}(\bar{x} + \mu y) - \nabla M_{\mu g}(\bar{x} + \mu\bar{y})) = L(I - D)\tilde{y} = L\tilde{y}$, substitution of $U\tilde{\xi} = 0, D\tilde{y} = 0$, and $\tilde{x} = 0$ into (14) implies that the stationary point for $\tilde{y}$ is $\tilde{y} = c\mathbb{1}$ where c is a non-zero scalar. Therefore, we have $L(\nabla f(\bar{x}) + \nabla M_{\mu g}(\bar{x} + \mu y)) = 0$ and the first condition in $\bar{\Omega}$ holds for (x, y).

By combing the above expressions, we conclude that $(x, y) = (\bar{x}, y)$ is in $\bar{\Omega}$. Thus, since V is radially unbounded LaSalle's invariance principle implies that (12) is globally asymptotically stable. ■

IV. An example: economic dispatch problem

In this section, we provide an example of the economic dispatch problem to illustrate the performance of the proposed primal-dual Laplacian gradient flow dynamics.

An IEEE 118-bus benchmark problem [20] has 54 generators and each generator has a quadratic cost function, $f_i(x_i) = a_i + b_i x_i + c_i x_i^2$, where x_i is the power injection at bus i, $a_i \in [6.78, 74.33]$, $b_i \in [8.3391, 37.6968]$, and $c_i \in [0.0024, 0.0697]$. The load is $b = 4200$. The communication network has a ring topology with several additional edges $(1, 11)$, $(11, 21)$, $(21, 31)$, $(31, 41)$, and $(41, 51)$. To show the robustness to the fluctuations in electricity price, at $t = 2000$ we increase the parameters in the objective function by 20%.

We use ODE45 in MATLAB to simulate primal-dual Laplacian gradient flow dynamics (10), with $n = 54$ and $\mu = 0.15$. In all simulations, the relative and absolute error tolerances are set to 10^{-10} and 10^{-15}, respectively, the initial condition for each generator is set to $x_i(0) = 4200/54$, and $y(0) = 0$.

The simulation results are shown in Figs. 1 and 2. Figures 1a and 2a demonstrate that all power injections x_i stay in the feasible region and converge to the optimal solution computed by CVX [21]. The following relative error

$$\sqrt{\frac{\|x - \bar{x}\|^2 + \|y - \bar{y}\|^2}{\|x(0) - \bar{x}\|^2 + \|y(0) - \bar{y}\|^2}}$$

is used to show the exponential convergence in right plots in Figs. 1 and 2, where vertical axis is shown in the logarithmic scale. In both cases, logarithmic relative errors decrease linearly except for the time instant at which the objective function has been perturbed.

V. Concluding remarks

In this paper, we employ the proximal augmented Lagrangian method to solve a class of convex resource allocation problems over a connected undirected network with n agents. We show that the resource allocation problem can be formulated as a composite optimization problem. To perform in-network computations, we propose a primal-dual Laplacian gradient flow dynamics based on the proximal augmented Lagrangian. We demonstrate that the equilibrium points correspond to KKT points of the original problem. A Lyapunov-based argument is used to establish global asymptotic stability and demonstrate that the proposed gradient flow dynamics globally converge to the optimal solution. Finally, we apply the proposed algorithm to an economic dispatch problem to illustrates its effectiveness.

Several future directions are of interest. First, as indicated in our simulations, the proposed gradient flow dynamics appear to be exponentially converging. The exponential convergence of the primal-dual Laplacian gradient flow dynamics is an open problem that requires further investigation. Second, the robustness of the proposed algorithm to various uncertainty sources is worth exploring. Third, to deal with a broader class of problems, it is of interest to extend the proposed gradient flow dynamics to more general constraints.

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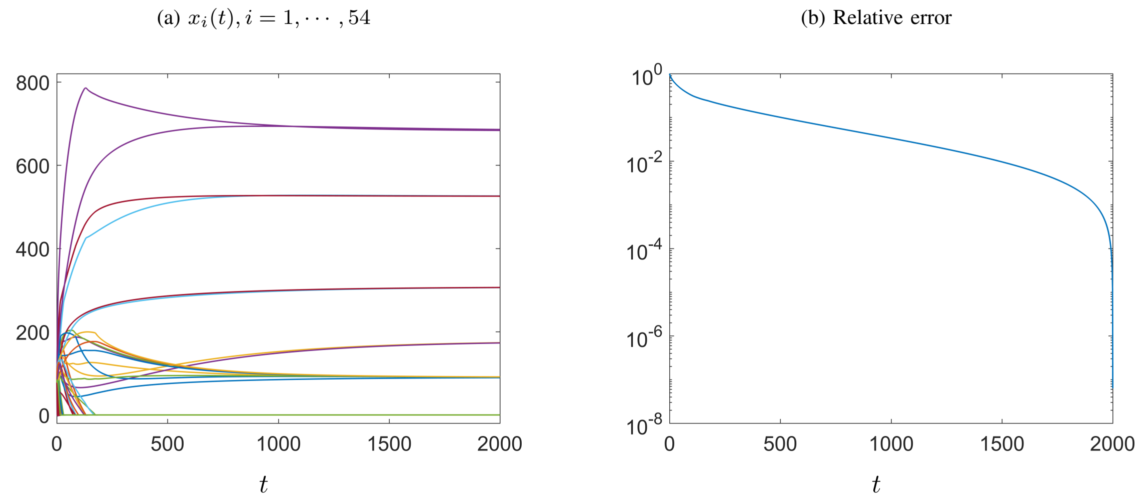

Fig. 1: (a) Power injections from 54 generators; and (b) relative error to the optima.

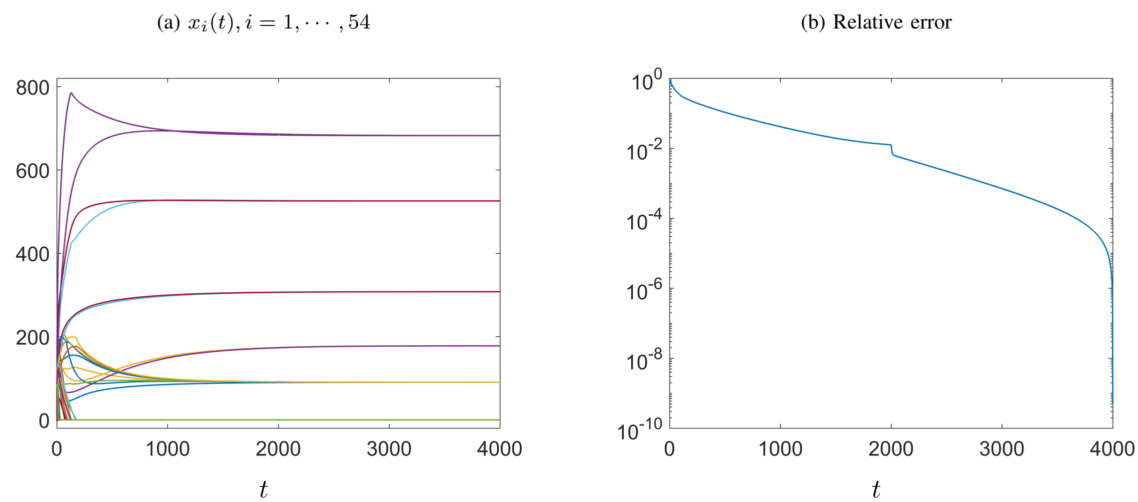

Fig. 2: (a) Power injections from 54 generators; and (b) relative error to the optima.

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