



Design of optimal controllers for spatially invariant systems with finite communication speed[☆]

Makan Fardad^a, Mihailo R. Jovanović^{b,*}

^a Department of Electrical Engineering and Computer Science, Syracuse University, Syracuse, NY 13244, United States

^b Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455, United States

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ABSTRACT

We consider the problem of designing optimal distributed controllers whose impulse response has limited propagation speed. We introduce a state-space framework in which all spatially invariant systems with this property can be characterized. After establishing the closure of such systems under linear fractional transformations, we formulate the \mathcal{H}_2 optimal control problem using the model-matching framework. We demonstrate that, even though the optimal control problem is non-convex with respect to some state-space design parameters, a variety of numerical optimization algorithms can be employed to relax the original problem, thereby rendering suboptimal controllers. In particular, for the case in which every subsystem has scalar input disturbance, scalar measurement, and scalar actuation signal, we investigate the application of the Steiglitz–McBride, Gauss–Newton, and Newton iterative schemes to the optimal distributed controller design problem. We apply this framework to examples previously considered in the literature to demonstrate that, by designing structured controllers with infinite impulse response, superior performance can be achieved compared to finite impulse response structured controllers of the same temporal degree.

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1. Introduction

The synthesis problem of distributed control has received considerable attention in recent years (Bamieh, Paganini, & Dahleh, 2002; Bamieh & Voulgaris, 2005; Borrelli & Keviczky, 2008; D’Andrea & Dullerud, 2003; de Castro & Paganini, 2002; Dullerud & D’Andrea, 2004; Jovanović, 2010; Langbort, Chandra, & D’Andrea, 2004; Motee & Jadbabaie, 2008; Rantzer, 2006a,b, 2009; Rotkowitz & Lall, 2006; Voulgaris, Bianchini, & Bamieh, 2003). Standard optimal control design methods, when applied to distributed systems, yield *centralized* controllers (Bamieh et al., 2002). In this scenario the controller of each subsystem demands information about the state of the entire system. Such solutions are undesirable from a practical point of view, as they are expensive in hardware and computation requirements and

demand excessive communication between different subsystems. In the control of distributed systems a desired scenario is to have each subsystem possess its own controller and each controller exchange information only within an *a priori* assigned *local* architecture.

For optimal distributed controllers of spatially invariant systems the dependence of a controller on information coming from other parts of the system decays exponentially as one moves away from that controller (Bamieh et al., 2002). This motivates the search for inherently *localized* controllers. For example, one could search for optimal controllers that are subject to the condition that they communicate only to their nearest neighbors or to other controllers within a certain radius.

Optimal control problems are often reformulated using the model-matching framework, which allows for a closed-loop transfer function that is affine in the Youla parameter (Francis, 1987; Youla, Jabr, & Bongiorno, 1976). However, in the distributed setup this generally comes at the expense of losing convexity of the constraint set to which the design parameter belongs. This is due to the nonlinearity of the mapping from the controller to the Youla parameter.

Recently, certain subspaces of localized systems which remain invariant under this nonlinear mapping have been characterized. Bamieh and Voulgaris (2005) and Voulgaris et al. (2003) introduce the subspaces of *cone causal* and *funnel causal* systems, respectively. These subspaces describe how information from every

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* Corresponding author. Tel.: +1 612 625 7870; fax: +1 612 625 4583.
E-mail addresses: makan@syr.edu (M. Fardad), mihailo@umn.edu (M.R. Jovanović).

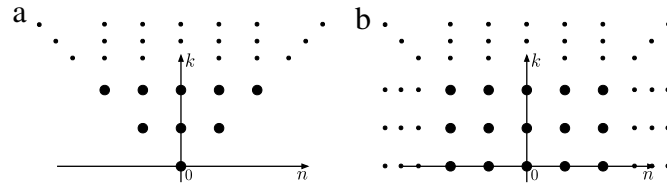


Fig. 1. The vertical axis denotes time and the horizontal axis denotes space. (a) The support of the spatio-temporal impulse response of a *cone causal* system. (b) The support of the spatio-temporal impulse response of a *centralized* system.

controller propagates through the distributed system. A similar but more general characterization, termed *quadratic invariance*, was introduced in Rotkowitz and Lall (2006). It is important to note that constructs such as cone and funnel causality lead to optimal control problems that are convex in the Markov (i.e., impulse response) parameters of the Youla variable, but *not* in the state-space parameters. Therefore, one is still faced with solving a realization problem for a distributed system.

In this paper we consider the problem of designing optimal distributed controllers for spatially invariant systems with finite communication speed. Our approach is close in spirit to Voulgaris et al. (2003), in that we consider controllers whose spatio-temporal impulse response belongs to a cone in the spatio-temporal plane. However, we depart from Voulgaris et al. (2003) by considering a state-space description of such controllers, which gives a new physical interpretation to the framework of (Voulgaris et al., 2003) in terms of nearest neighbor interactions. We show that although the design problem is non-convex with respect to some state-space design parameters, relaxations and numerical optimization algorithms can often be effectively employed to obtain suboptimal controllers. Such numerical schemes include Steiglitz–McBride (SM) (Dumitrescu & Niemistö, 2004; Steiglitz & McBride, 1965), Gauss–Newton (GN) (Nocedal & Wright, 2006), gradient descent, and Newton’s method (Boyd & Vandenberghe, 2004).

The paper is organized as follows. In Section 2 we describe the class of \mathcal{C} -causal discrete time systems. In Section 3 we use the model-matching framework to formulate the \mathcal{H}_2 optimal design problem. In Section 4 we present numerical algorithms for the design of structured distributed controllers. We apply our results to illustrative examples in Section 5, and conclude our presentation in Section 6.

1.1. Preliminaries

We consider discrete time systems on a discrete one-dimensional spatial lattice. All systems are linear time invariant and spatially invariant, and all signals are functions of both space and time. λ denotes the temporal (one-sided) transform variable corresponding to the time variable k ,

$$\psi_n(\lambda) = \sum_{k=0}^{\infty} \psi_{n,k} \lambda^k,$$

and ζ denotes the spatial (two-sided) transform variable corresponding to the spatial variable n ,

$$\psi_k(\zeta) = \sum_{n=-\infty}^{\infty} \psi_{n,k} \zeta^n.$$

When evaluated on the unit circle, λ and ζ are denoted by $e^{j\omega}$ and $e^{j\theta}$, respectively. $U^* = \bar{U}^T$ if U is a constant matrix and $U(\zeta, \lambda)^* = \bar{U}(\zeta^{-1}, \lambda^{-1})^T$ if U is a spatio-temporal transfer function, where the overbar denotes complex conjugation and $(\cdot)^T$ denotes transposition. If U is a matrix, U^\dagger denotes the pseudo-inverse of U ; if U is a system, then its pseudo-inverse is given by the formula in Proposition 1. The distinction between inverses of matrices and systems will be clear from the context.

2. Cone causal and \mathcal{C} -causal systems

Recently, Voulgaris et al. (2003) introduced a class of distributed spatially invariant systems which have finite propagation speed and are closed under the operations of addition, composition, and inversion. Through use of these closure properties, it was demonstrated in Voulgaris et al. (2003) that the optimal control problem can be effectively formulated using the model-matching framework. In this section we briefly review the results of Voulgaris et al. (2003), and then introduce a new state-space representation of such systems.

2.1. Cone causal systems

We begin by defining the class of cone causal systems introduced in Voulgaris et al. (2003).

Definition 1. A linear spatially invariant discrete time system defined on a discrete spatial lattice is called cone causal if its spatio-temporal transfer function is of the form

$$G(\zeta, \lambda) = \sum_{k=0}^{\infty} g_k(\zeta) \lambda^k, \quad (1)$$

$$g_k(\zeta) = \sum_{n=-k}^k g_{n,k} \zeta^n, \quad g_0(\zeta) = g_{0,0},$$

where $g_{n,k}$ can be matrices in general.

Note that by the above definition, a spatio-temporal system can be cone causal without being stable. Cone causality is only a condition on the support of the impulse response in the spatio-temporal domain. Fig. 1(a) illustrates the support of the spatio-temporal impulse response of a cone causal system. A system described by (1) in which the support of the impulse response is not restricted to any cone in the spatio-temporal plane is said to be *centralized*. In other words, a centralized system is one which has infinite communication speed; for an illustration, see Fig. 1(b).

Example 1. Consider the following discrete time system

$$x_{n,k+1} = a_1 x_{n-1,k} + a_0 x_{n,k} + a_{-1} x_{n+1,k} + b u_{n,k},$$

$$y_{n,k} = c x_{n,k} + d u_{n,k},$$

where $n \in \mathbb{Z}$ and $\{a_i, b, c, d\}$ are constant parameters. This nearest-neighbor-interaction system is spatially invariant and its spatio-temporal transfer function is determined by

$$G(\zeta, \lambda) = \frac{\lambda c b}{1 - \lambda a(\zeta)} + d, \quad a(\zeta) = a_{-1} \zeta^{-1} + a_0 + a_1 \zeta.$$

The functions $g_k(\zeta)$ in (1), which denote the ζ -parameterized Markov parameters of this system, are determined by

$$g_0 = d, \quad g_k(\zeta) = c a^{k-1}(\zeta) b, \quad k \geq 1.$$

Based on the definition of $a(\zeta)$ we see that $G(\zeta, \lambda)$ and $g_k(\zeta)$ satisfy Definition 1, which implies cone causality of this system.

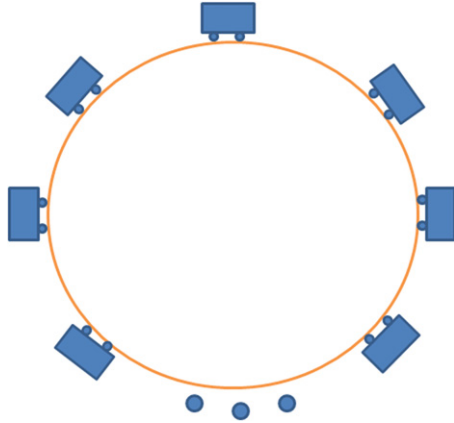


Fig. 2. A formation of identical vehicles arranged in a circle.

2.2. \mathcal{C} -causal systems

Consider a linear system G with state-space representation

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = D + \lambda C (I - \lambda A)^{-1} B. \tag{2}$$

Definition 2. We denote by \mathcal{C} the set of spatially invariant discrete time systems, defined on a discrete spatial lattice, that satisfy the following assumptions.

- (i) Matrices B and D are independent of ζ .
- (ii) Matrices A and C are of the form

$$A(\zeta) = A_{-1} \zeta^{-1} + A_0 + A_1 \zeta,$$

$$C(\zeta) = C_{-1} \zeta^{-1} + C_0 + C_1 \zeta,$$

with $A_n, C_n, n = -1, 0, 1$ independent of ζ .

We refer to systems that belong to the set \mathcal{C} as \mathcal{C} -causal.

Loosely speaking, systems that belong to the set \mathcal{C} are systems in which effects propagate at most one unit in space for every unit in time. Clearly, the above definition includes systems for which either or both of the matrices A and C are ζ -independent. We denote the dimension of the matrix A by μ and refer to it as the *temporal order* or *temporal degree* of G .

Example 2. Consider a formation of identical vehicles arranged in a circle, as in Fig. 2. This is an idealization of the case of equally-spaced vehicles on a closed track. The dynamics of each vehicle can be modeled as the discrete time equivalent of a double integrator, and the input to the n th vehicle is its control signal u_n . The objective is to keep all vehicles at equal distances from each other.

The control signal for each vehicle is computed using measurements of its distance with neighboring vehicles. This can be done in a number of ways. If every vehicle only measures its distance with the vehicle ahead, then $y_n = x_n - x_{n-1}$ and thus $C(\zeta) = 1 - \zeta$. Another possibility is for every vehicle to measure the sum of its distances with its two immediate neighbors, in which case $y_n = (x_n - x_{n+1}) + (x_n - x_{n-1})$ and thus $C(\zeta) = -\zeta^{-1} + 2 - \zeta$. A more general scenario is for every vehicle to independently measure its distance with both the vehicle ahead and the one behind, so that $y_n = [x_n - x_{n-1} \quad x_n - x_{n+1}]^T$ and thus $C(\zeta) = [1 - \zeta \quad 1 - \zeta^{-1}]^T$. In all of these scenarios, both the dynamics and the actuation of each vehicle is independent of other vehicles and therefore A and B are ζ -independent.

Example 3. Consider a large array of tightly-packed capacitively-actuated micro-cantilevers, as in Fig. 3, used in applications such as highly parallel scanning probe microscopy or high density

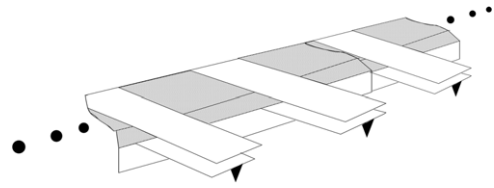


Fig. 3. An array of capacitively-actuated micro-cantilevers.

data storage. Each cantilever is the movable plate of a capacitor and the displacement of the cantilever varies with the voltage applied across the capacitor plates. Measurement of the displacement is performed by observing the current through the capacitive cantilever (Napoli, 2004).

Due to the small distance between micro-cantilevers, there is coupling between neighboring devices as a result of the capacitive fringe fields. Thus $A(\zeta) = A_{-1} \zeta^{-1} + A_0 + A_1 \zeta$, where A_{-1} and A_1 capture the effect of electrostatic coupling with neighboring cantilevers. However, since each cantilever has independent actuation and measurement capabilities, B and C are ζ -independent.

As we next show, \mathcal{C} is closed under addition, composition, and inversion of systems. Thus it is closed under feedback and linear fractional transformations (LFTs) (Zhou, Doyle, & Glover, 1996).

Ref. Voulgaris et al. (2003) demonstrates closure results for cone causal systems using Markov parameter descriptions. The following proposition proves closure results for \mathcal{C} -causal systems using state-space descriptions. Let G^\dagger and D^\dagger denote the right (left) inverses of G and D , respectively.

Proposition 1. Let

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], \quad \tilde{G} = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right],$$

and assume that D^\dagger exists. If G and \tilde{G} belong to \mathcal{C} then $G + \tilde{G}$, $G\tilde{G}$, and G^\dagger belong to \mathcal{C} .

Proof. We have (Zhou et al., 1996)

$$G + \tilde{G} = \left[\begin{array}{c|c|c} A & 0 & B \\ 0 & \tilde{A} & \tilde{B} \\ \hline C & \tilde{C} & D + \tilde{D} \end{array} \right], \quad G\tilde{G} = \left[\begin{array}{c|c|c} A & B\tilde{C} & B\tilde{D} \\ 0 & \tilde{A} & \tilde{B} \\ \hline C & D\tilde{C} & D\tilde{D} \end{array} \right],$$

$$G^\dagger = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array} \right].$$

It is clear from Definition 2 and the state-space representations of $G + \tilde{G}$, $G\tilde{G}$, and G^\dagger that they all belong to \mathcal{C} , and the proof is complete. \square

Proposition 2. The set \mathcal{C} is equal to the set of cone causal systems.

Proof. We first prove that \mathcal{C} is a subset of cone causal systems. Let G belong to \mathcal{C} . Then its state-space parameters satisfy the conditions in Definition 2. Thus, from Definition 1, the systems with transfer functions $A\lambda, B, C\lambda, D$ are all cone causal. Therefore, by closure of cone causality under addition, composition, and inversion (Voulgaris et al., 2003), the system

$$G = D + \lambda C (I - \lambda A)^{-1} B$$

is cone causal.

We now prove that the set of cone causal systems is a subset of \mathcal{C} . Let G be a cone causal system. Then from Definition 1 its transfer function can be written as

$$G(\zeta, \lambda) = \sum_{n,k} g_{n,k} \zeta^n \lambda^k, \quad g_{n,k} = 0 \quad \text{for } |n| > k.$$

If we show that each of the terms in the above sum belongs to \mathcal{C} , then $G \in \mathcal{C}$ by the closure of \mathcal{C} under addition.

Define the transfer functions

$$\begin{aligned} H_o(\zeta, \lambda) &= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array} \right] = I, \\ H_\uparrow(\zeta, \lambda) &= \left[\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array} \right] = I\lambda, \\ H_\nearrow(\zeta, \lambda) &= \left[\begin{array}{c|c} 0 & I \\ \hline I\zeta & 0 \end{array} \right] = I\zeta\lambda, \\ H_\nwarrow(\zeta, \lambda) &= \left[\begin{array}{c|c} 0 & I \\ \hline I\zeta^{-1} & 0 \end{array} \right] = I\zeta^{-1}\lambda, \end{aligned}$$

all of which satisfy the conditions of Definition 2 and thus belong to \mathcal{C} . We demonstrate that any cone causal system with transfer function $g_{n,k} \zeta^n \lambda^k$, $|n| \leq k$ can be written as a combination of $H_o, H_\uparrow, H_\nearrow, H_\nwarrow$.

Assume that $n \geq 0$. From the cone causality of $g_{n,k} \zeta^n \lambda^k$ it follows that $k \geq n$. We compose the two transfer functions $H_{n,n}(\zeta, \lambda) = I \zeta^n \lambda^n$ and $H_{0,k-n}(\zeta, \lambda) = I \lambda^{(k-n)}$; the transfer function $H_{n,n}(\zeta, \lambda)$ can be formed from the composition of n identical systems $H_\nearrow = I \zeta \lambda$,

$$H_{n,n}(\zeta, \lambda) = H_\nearrow H_\nearrow \dots H_\nearrow \quad (n \text{ times}),$$

and the transfer function $H_{0,k-n}(\zeta, \lambda)$ can be formed from the composition of $k - n$ identical systems $H_\uparrow = I \lambda$,

$$H_{0,k-n}(\zeta, \lambda) = H_\uparrow H_\uparrow \dots H_\uparrow \quad (k - n \text{ times}).$$

Finally, $g_{n,k} \zeta^n \lambda^k$ is obtained from the composition of $H_{n,n}(\zeta, \lambda)$, $H_{0,k-n}(\zeta, \lambda)$, and $H_{0,0}(\zeta, \lambda) = g_{n,k} H_o = g_{n,k}$. Since \mathcal{C} is closed under composition then $g_{n,k} \zeta^n \lambda^k \in \mathcal{C}$.

The steps for $n \leq 0$ are similar, with H_\nearrow replaced by H_\nwarrow . The proof is now complete. \square

3. The structured \mathcal{H}_2 optimal control problem

In this section we use Youla parameterization (Youla et al., 1976) and model-matching (Francis, 1987) to formulate the distributed \mathcal{H}_2 optimal control problem. In this framework, the set of \mathcal{C} -causal systems defined in the previous section plays a critical role. The closure properties of \mathcal{C} allow us to go back and forth between the Youla parameter Q and the controller K without ever leaving this set. In other words, an optimal $Q \in \mathcal{C}$ immediately results in an optimal $K \in \mathcal{C}$, even though the map that relates these two systems is nonlinear.

Consider the system $G \in \mathcal{C}$,

$$G = \left[\begin{array}{cc|cc} G_{zw} & G_{zu} & A & B_w & B_u \\ G_{yw} & G_{yu} & C_z & 0 & D_{zu} \\ \hline & & C_y & D_{yw} & 0 \end{array} \right]. \quad (3)$$

Note that since $G \in \mathcal{C}$ then B_w, B_u, D_{zu}, D_{yw} are independent of ζ , and A, C_z, C_y have ζ -dependence of the form described in Definition 2. We also make the following simplifying assumption.

Assumption 1. For any given value of ζ , the state-space parameters in system (3) have the following dimensions

$$\begin{aligned} A : n_\psi \times n_\psi, & \quad B_w : n_\psi \times 1, & \quad B_u : n_\psi \times 1, \\ C_z : n_z \times n_\psi, & & \quad D_{zu} : n_z \times 1, \\ C_y : 1 \times n_\psi, & \quad D_{yw} : 1 \times 1, \end{aligned}$$

where n_ψ and n_z are the dimensions of the state and performance output, respectively, after the application of a spatial Fourier transform. Note that $n_w = n_y = n_u = 1$, where n_w, n_y , and n_u are the dimensions of the disturbance, the measurement, and the control signal, respectively, after the application of a spatial Fourier transform.

Assumption 1 implies that in the transform domain, for any given value of ζ , the transfer functions G_{yw} and G_{yu} from disturbance w and control u to measurement y are both SISO (single input single output). In the spatial domain, this means that every subsystem has scalar input disturbance, measurement, and control signal.

The closed-loop interconnection of G with a SISO controller K results in the closed-loop transfer function

$$G_{cl} = G_{zw} + G_{zu} K (I - G_{yu} K)^{-1} G_{yw}. \quad (4)$$

Before we discuss the optimal control problem of interest, we have to define the system norm we will be using.

Definition 3. Let G_{cl} be a stable system. Then the spatio-temporal \mathcal{H}_2 norm of G_{cl} is defined by (Bamieh et al., 2002)

$$\|G_{cl}\|_{\mathcal{H}_2}^2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \text{tr} [G_{cl}(e^{j\theta}, e^{j\omega}) G_{cl}(e^{j\theta}, e^{j\omega})^*] d\theta d\omega,$$

where θ and ω denote the spatial and temporal frequencies, respectively.

The problem of interest in this paper is the following.

- Given system $G \in \mathcal{C}$ find a stabilizing controller $K \in \mathcal{C}$ such that the closed-loop norm $\|G_{cl}\|_{\mathcal{H}_2}^2$ is minimized.

Remark 1. The condition $G \in \mathcal{C}$ can be relaxed to $G_{yu} \in \mathcal{C}$, i.e., it suffices for the state-space parameters (C_y, A, B_u) to satisfy the conditions in Definition 2.

Remark 2. Structured optimal control problems such as the one posed above are hard to solve because of the nonlinear way in which the design parameter K appears in the expression for G_{cl} in (4). As we show below, a change of variables allows for a new design parameter Q to appear affinely in G_{cl} , thus forming a convex objective function. However, the mapping from K to Q is nonlinear, and therefore a convex constraint set for K is not always mapped to a convex constraint set for Q . This underlines the importance of subspaces characterizing cone causal (Voulgaris et al., 2003), funnel causal (Bamieh & Voulgaris, 2005), quadratically invariant (Rotkowitz & Lall, 2006), and \mathcal{C} -causal systems: they remain invariant under the map $K \mapsto Q$. Since every subspace is a convex set, we thus end up with optimizing a convex objective over a convex set, which is a desired scenario. This remark is summarized in Theorem 3 below.

Using Youla parameterization (Youla et al., 1976), the transfer function of the closed-loop system (4) can be recast in the model-matching framework (Francis, 1987, Chap. 3) as $G_{cl} = T_1 - T_2 Q T_3$. From Assumption 1 and the description (Francis, 1987, Chap. 4) of the transfer functions T_1, T_2, T_3 , it follows that T_3 and Q are both SISO and thus commute. Defining $T := T_1$ and $U := T_2 T_3$, we obtain

$$G_{cl} = T - U Q. \quad (5)$$

Thus the problem of minimizing $\|G_{cl}\|_{\mathcal{H}_2}^2$ can be rewritten as

$$\text{minimize } \|T - U Q\|_{\mathcal{H}_2}^2, \quad (6)$$

where Q , referred to as the Youla parameter, is the optimization variable. The model-matching parameters T, U , and Q are all stable transfer functions; T and U have known state-space representations and can be found using only knowledge of the open-loop system G (i.e., they are independent of Q). Once problem (6) is solved and the optimal Youla parameter Q^{opt} is found, the optimal controller K^{opt} is obtained from Q^{opt} as discussed in Francis (1987) (see also proof of Theorem 3).

From Francis (1987, Chap. 4) it follows that

$$T = \left[\begin{array}{cc|c} A + B_u F & -B_u F & B_w \\ \hline 0 & A + H C_y & B_w + H D_{yw} \\ C_z + D_{zu} F & -D_{zu} F & 0 \end{array} \right], \quad (7)$$

$$U = \left[\begin{array}{cc|c} A + B_u F & B_u C_y & B_u D_{yw} \\ \hline 0 & A + H C_y & B_w + H D_{yw} \\ C_z + D_{zu} F & D_{zu} C_y & D_{zu} D_{yw} \end{array} \right], \quad (8)$$

where F and H are chosen such that $A + B_u F$ and $A + H C_z$ are Hurwitz at all spatial frequencies, i.e., the matrices $[A + B_u F](e^{j\theta})$ and $[A + H C_z](e^{j\theta})$ have eigenvalues with strictly negative real parts for every $\theta \in [0, 2\pi]$. Additionally, we make the following assumptions on H and F , which can be thought of as stabilizability/detectability-type conditions that respect the particular structure of \mathcal{C} -causal systems.

Assumption 2. For system (3) that satisfies Assumption 1,

- (i) There exists an $n_\psi \times 1$ vector H independent of ζ such that $A(e^{j\theta}) + H C_y(e^{j\theta})$ is Hurwitz for every $\theta \in [0, 2\pi]$.
- (ii) There exists a vector F of the form

$$F(\zeta) = F_{-1} \zeta^{-1} + F_0 + F_1 \zeta$$

with $1 \times n_\psi$ vectors F_{-1}, F_0, F_1 independent of ζ such that $A(e^{j\theta}) + B_u F(e^{j\theta})$ is Hurwitz for every $\theta \in [0, 2\pi]$.

Remark 3. The above conditions on the stability of $A + H C_y$ and $A + B_u F$ are a generalization of the conditions given in Bamieh and Voulgaris (2005), where it is assumed that H and F are ζ -independent.

We now state the main result of this section. Theorem 3 below demonstrates that once the optimal $Q \in \mathcal{C}$ is found, it uniquely determines the optimal controller K . Furthermore, K is guaranteed to both be stabilizing and belong to the set \mathcal{C} .

Theorem 3. Let system $G \in \mathcal{C}$ with state-space representation (3) satisfy Assumption 2. Then the map $Q \mapsto K$ is a bijection from \mathcal{C} to itself. In particular, K is stabilizing and belongs to \mathcal{C} if and only if Q is stable and belongs to \mathcal{C} .

Proof. The basic idea of the proof can be found in Bamieh and Voulgaris (2005). By Assumption 2 there exist H and F such that $A + H C_y$ and $A + B_u F$ are stable. From Chen and Francis (1995, Thm. 5.4.1) and Zhou et al. (1996, Thm. 12.8) all stabilizing controllers (\mathcal{C} -causal or not) can be parameterized by

$$K = J_{11} + J_{12} Q (I - J_{22} Q)^{-1} J_{21},$$

$$J = \left[\begin{array}{cc|cc} J_{11} & J_{12} & -H & B_u \\ J_{21} & J_{22} & 0 & I \\ \hline A + B_u F + H C_y & F & -H & B_u \\ -C_y & -C_y & I & 0 \end{array} \right],$$

Q stable,

and any K found from the above relation is stabilizing if and only if its corresponding Q is stable.

Next we bring into consideration the spatial structure of K and Q , and show that the map $Q \mapsto K$ is a bijection on \mathcal{C} .

From $G \in \mathcal{C}$, Assumption 2 on the matrices H and F , and the state-space representation of J , it follows that $J \in \mathcal{C}$. Now, assume $Q \in \mathcal{C}$. Since K is given by a linear fractional transformation of Q with coefficients $J_{ij} \in \mathcal{C}$, $i, j = 1, 2$ then $K \in \mathcal{C}$. Conversely, assume $K \in \mathcal{C}$. From Chen and Francis (1995, Thm. 5.4.1) we have

$$Q = J_{12}^{-1} (K - J_{11}) J_{21}^{-1} [I + J_{12}^{-1} (K - J_{11}) J_{21}^{-1} J_{22}]^{-1}.$$

Since $J_{ij} \in \mathcal{C}$, $i, j = 1, 2$ then $Q \in \mathcal{C}$. The proof is thus complete. \square

In light of Remark 1, it is clear that the condition $G \in \mathcal{C}$ in Theorem 3 can be relaxed to $G_{yu} \in \mathcal{C}$ without affecting the proof.

3.1. Model-matching problem

In this section we present the model-matching problem. We introduce an inner-outer factorization (Francis, 1987) of U , $U = U_{in} U_{out}$, where U_{out} is stable with a stable right inverse, and U_{in} satisfies $U_{in}^* U_{in} = I$. For a general inner-outer factorization, U_{in} and U_{out} are tall and wide matrices, respectively. In this paper, however, for any given ζ the transfer function U has dimension $n_z \times 1$ and thus U_{in} is an $n_z \times 1$ vector and U_{out} is a scalar.

Since the \mathcal{H}_2 norm is a quadratic norm, it satisfies (Francis, 1987, Chap. 8, Lem. 1)

$$\|E G\|_{\mathcal{H}_2}^2 = \|G\|_{\mathcal{H}_2}^2, \quad \text{where } E := \begin{bmatrix} U_{in}^* \\ I - U_{in} U_{in}^* \end{bmatrix}.$$

We thus have

$$\begin{aligned} \|T - U Q\|_{\mathcal{H}_2}^2 &= \|E (T - U_{in} U_{out} Q)\|_{\mathcal{H}_2}^2 \\ &= \left\| \begin{bmatrix} U_{in}^* T - U_{out} Q \\ (I - U_{in} U_{in}^*) T \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\ &= \|U_{in}^* T - U_{out} Q\|_{\mathcal{H}_2}^2 + \|(I - U_{in} U_{in}^*) T\|_{\mathcal{H}_2}^2 \\ &= \|[U_{in}^* T]_{st} + [U_{in}^* T]_{un} - U_{out} Q\|_{\mathcal{H}_2}^2 \\ &\quad + \|(I - U_{in} U_{in}^*) T\|_{\mathcal{H}_2}^2 \\ &= \|R - U_{out} Q\|_{\mathcal{H}_2}^2 + \|[U_{in}^* T]_{un}\|_{\mathcal{H}_2}^2 \\ &\quad + \|(I - U_{in} U_{in}^*) T\|_{\mathcal{H}_2}^2 \end{aligned}$$

where $R := [U_{in}^* T]_{st}$ and $[U_{in}^* T]_{un}$ correspond to the stable and unstable parts of $U_{in}^* T$, respectively. The last equality above follows from the orthogonality of the spaces of stable and unstable transfer functions in \mathcal{H}_2 ; see Chen and Francis (1995, Section 6.7) for details (for continuous time, see Doyle, Francis, and Tannenbaum (1990, Section 10.4)). The optimal solution, regardless of whether it belongs to \mathcal{C} or not, is given by

$$Q_c := U_{out}^{-1} R.$$

Note that Q_c is stable since U_{out}^{-1} is stable by construction.

The difficulty here is that once an inner-outer factorization of $U \in \mathcal{C}$ is performed, in general neither U_{in} nor U_{out} belongs to \mathcal{C} . In fact Q_c is a centralized system in general. This is because U_{in} and U_{out} contain parameters that are determined by the solution of an algebraic Riccati equation (ARE). In general, this solution cannot be expressed as a finite sum of powers of ζ and ζ^{-1} . In particular, the state-space realizations of U_{in} and U_{out} do not satisfy conditions (i) and (ii) of Definition 2. This is reminiscent of Youla parameterization in multidimensional systems theory (Youla & Gnani, 1979).

Henceforth, our aim is to find $Q \in \mathcal{C}$ that minimizes

$$J := \|R - U_{out} Q\|_{\mathcal{H}_2}^2 = \|U_{out} (Q_c - Q)\|_{\mathcal{H}_2}^2. \quad (9)$$

Thus we would like to find $Q \in \mathcal{C}$ that best approximates the centralized system Q_c in the sense of the weighted \mathcal{H}_2 norm; see Fig. 4. Note that in (9) there is no restriction on the temporal order of Q . However, one possibility is to choose the temporal order of Q equal to that of Q_c , so that Q imitates the temporal dynamics of Q_c . We emphasize that there is no reason to expect that such a choice of temporal order is optimal. The issue of temporal order will be further discussed in the examples of Section 5.

3.2. Literature overview

To the best of our knowledge, no exact solution to the problem posed at the end of the previous section is known in general, and to find $Q \in \mathcal{C}$ one has to resort to some form of approximation.

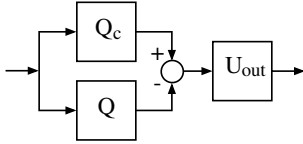


Fig. 4. The problem of finding the cone causal Q that best approximates the centralized Q_c in the sense of the U_{out} -weighted \mathcal{H}_2 norm.

Voulgaris et al. (2003) considered this problem in the Markov parameter setting using the projection theorem for Hilbert spaces. More specifically, Voulgaris et al. (2003) determined the spatio-temporal Markov parameters $q_{n,k}$, up to time $k = \kappa$, of an FIR (finite impulse response) cone causal system Q^κ

$$Q^\kappa(\zeta, \lambda) = \sum_{k=0}^{\kappa} q_k(\zeta) \lambda^k, \quad q_k(\zeta) = \sum_{n=-k}^k q_{n,k} \zeta^n,$$

such that Q^κ minimizes

$$\|R - U_{out} Q\|_{\mathcal{H}_2}^2, \quad Q \text{ cone causal.}$$

Here the temporal \mathcal{H}_2 norm is computed up to time $k = \kappa$. Furthermore, weak convergence of Q^κ to the unique optimal cone causal system Q^{opt} as $\kappa \rightarrow \infty$ was demonstrated in Voulgaris et al. (2003).

In a mathematical sense the optimal control problem was solved in Voulgaris et al. (2003). But the difficulty with this approach is with regards to the implementation of the resulting controller. The state-space realization of Q^κ is a deadbeat system¹ of order greater than or equal to κ . If κ is taken to be large to achieve a small closed-loop norm, Q^κ and thus the controller K^κ will have large temporal degrees.

It is well-known that generally an IIR (infinite impulse response) system can achieve better performance compared to an FIR system of the same temporal degree (Dumitrescu, 2007). This motivates the problem of solving the structured optimal control problem not with respect to the Markov parameters of Q , but with respect to its state-space or transfer function representation. This is the aim of Section 4, where for a given temporal order μ , we present numerical algorithms for computing $Q \in \mathcal{C}$ that minimize the objective function in (9).

4. Relaxations and numerical algorithms for computing suboptimal $Q \in \mathcal{C}$

In this section we search for a Youla parameter Q of temporal order less than or equal to μ that minimizes J in (9) subject to the constraints that $Q \in \mathcal{C}$ and that Q is SISO for any given value of ζ . The objective function J is a nonlinear function of the state-space parameters of Q . We use various relaxation schemes and numerical algorithms to find a suboptimal solution.

Consider the problem of minimizing the performance index in (9) with

$$Q = d + \frac{N(\zeta, \lambda)}{1 + M(\zeta, \lambda)} \quad (10)$$

$$= d + \frac{\lambda c_1(\zeta) + \lambda^2 c_2(\zeta) + \dots + \lambda^\mu c_\mu(\zeta)}{1 + \lambda a_1(\zeta) + \lambda^2 a_2(\zeta) + \dots + \lambda^\mu a_\mu(\zeta)}$$

$$= \left[\begin{array}{cccc|c} 0 & 1 & \dots & 0 & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ \hline -a_\mu(\zeta) & -a_{\mu-1}(\zeta) & \dots & -a_1(\zeta) & 1 \\ \hline c_\mu(\zeta) & c_{\mu-1}(\zeta) & \dots & c_1(\zeta) & d \end{array} \right]. \quad (11)$$

¹ A deadbeat system is a discrete time system whose A -matrix has only zero eigenvalues.

From Definition 2, it follows that Q belongs to \mathcal{C} if d is independent of ζ and

$$c_k(\zeta) = \sum_{n=-1}^1 c_{n,k} \zeta^n, \quad a_k(\zeta) = \sum_{n=-1}^1 a_{n,k} \zeta^n. \quad (12)$$

We assume that the temporal order μ of Q has been specified a priori (perhaps based on the temporal order of Q_c).

4.1. Steiglitz–McBride method

The Steiglitz–McBride (SM) algorithm is an iterative numerical optimization scheme originally used for the identification of linear systems (Steiglitz & McBride, 1965). Recently it has been further developed and coupled with other numerical methods for the purpose of designing IIR digital filters (Dumitrescu & Niemistö, 2004).

Let Q be as in (10), and consider

$$J = \|U_{out}(Q_c - Q)\|_{\mathcal{H}_2}^2$$

$$= \left\| U_{out} \left(Q_c - d - \frac{N}{1 + M} \right) \right\|_{\mathcal{H}_2}^2$$

$$= \left\| \frac{U_{out}}{1 + M} (Q_c - d + [Q_c - d]M - N) \right\|_{\mathcal{H}_2}^2. \quad (13)$$

Our objective is to find d and the coefficients of N and M so that Q belongs to \mathcal{C} and minimizes J .

The difficulty here is that J is not jointly convex in d and the coefficients of M and N . The SM algorithm circumvents this issue by relaxing the objective function (13) to

$$J_{SM} := \left\| \frac{U_{out}}{1 + \tilde{M}} (Q_c - d + [Q_c - \tilde{d}]M - N) \right\|_{\mathcal{H}_2}^2 \quad (14)$$

where \tilde{d} and \tilde{M} correspond to d and M obtained from the previous iteration or an initial guess. J_{SM} is now convex in the unknown coefficients since d , N , and M all appear affinely inside the norm and any norm is a convex function of its argument.

We rewrite the terms inside the norm in (14) as

$$Q_c - d - N + [Q_c - \tilde{d}]M = Q_c - \Lambda^T \chi_{sd}$$

$$= Q_c - \Lambda^T \Sigma \chi$$

where the vector χ_{sd} contains the (spatially dependent) coefficients d , c_k , a_k of Q , and the vector χ contains the (spatially independent) coefficients d , $c_{n,k}$, $a_{n,k}$ of Q after their spatial dependence has been stripped away. The matrix Σ contains all spatial transform variables and the vector Λ contains all remaining terms. For example, if

$$Q = d + \frac{\lambda c}{1 - \lambda a}$$

then

$$\Lambda = \begin{bmatrix} 1 \\ \lambda \\ \lambda(Q_c - \tilde{d}) \end{bmatrix}, \quad \chi_{sd} = \begin{bmatrix} d \\ c \\ a \end{bmatrix},$$

and

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1} & 1 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-1} & 1 & \zeta \end{bmatrix}, \quad \chi = \begin{bmatrix} d \\ c_{-1} \\ c_0 \\ c_1 \\ a_{-1} \\ a_0 \\ a_1 \end{bmatrix}.$$

In general, we have

$$J_{SM} = \left\| \frac{U_{out}}{1 + \tilde{M}} (Q_c - \Lambda^T \Sigma \chi) \right\|_{\mathcal{H}_2}^2$$

$$= \frac{1}{2} \chi^T \Gamma \chi + \rho^T \chi + \tau,$$

which is a standard quadratic program in χ with

$$\Gamma = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left| \frac{U_{out}}{1 + \tilde{M}} \right|^2 \Sigma^* \bar{\Lambda} \Lambda^T \Sigma \, d\theta \, d\omega,$$

$$\rho = \frac{-1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left| \frac{U_{out}}{1 + \tilde{M}} \right|^2 \text{Re}\{Q_c^* \Lambda^T \Sigma\} \, d\theta \, d\omega,$$

$$\tau = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left| \frac{U_{out}}{1 + \tilde{M}} \right|^2 |Q_c|^2 \, d\theta \, d\omega.$$

The new set of coefficients is therefore found by setting $\frac{\partial}{\partial \chi} J_{SM} = 0$, which yields

$$\chi = -\Gamma^{-1} \rho.$$

The vector of coefficients χ found above is the result of a single iteration and can now be used to form \tilde{M} and \tilde{d} to initialize the next iteration, and so on. We summarize the SM iterative algorithm as follows.

Steiglitz–McBride method.

given initial χ that yields a stable $Q \in \mathcal{E}$.

repeat

1. Set $\tilde{\chi} := \chi$.
2. Obtain \tilde{d} and \tilde{M} from $\tilde{\chi}$.
3. Compute the updated coefficients χ .

until stopping criterion $|\frac{J_{SM}(\chi) - J_{SM}(\tilde{\chi})}{J_{SM}(\chi)}| < \varepsilon$ is satisfied.

Remark 4. The \mathcal{H}_2 norm of a linear dynamical system can also be computed by solving an algebraic Lyapunov equation (Zhou et al., 1996). For the class of spatially invariant systems considered here, this would alleviate integration with respect to the temporal frequency variable ω . The Lyapunov equation was used in the SM algorithm in Fardad and Jovanović (2008).

Remark 5. The SM algorithm as described above does not guarantee the stability of Q at every step, although in many examples the resulting Q is indeed stable. In the absence of a spatial transform variable, Dumitrescu (2007) and Dumitrescu and Niemistö (2004) describe how at every step the SM optimization problem can be transformed to a semidefinite quadratic linear program (SQLP) for which reliable solution algorithms exist in the literature; see Dumitrescu (2007) for details.

4.2. Gauss–Newton method

The Gauss–Newton (GN) method is an iterative optimization scheme that decreases the value of the objective function J by incrementally improving the unknown coefficients χ of Q . The method is based on finding the descent direction δ at every iteration by replacing $Q(\chi + \delta)$ in the objective with its first order approximation, and then computing δ that minimizes the resulting $J(\chi + \delta)$. Thus, in the expression for J we substitute

$$Q(\chi + \delta) \approx Q(\chi) + [\nabla Q]^T(\chi) \delta_{sd}$$

$$= Q(\chi) + [\nabla Q]^T(\chi) \Sigma \delta,$$

where the column vector $[\nabla Q](\chi)$ is composed of the partial derivatives of Q with respect to each of its coefficients and then evaluated at the current value χ of the coefficients, the vector δ_{sd} contains the corresponding (spatially dependent) increments

of the coefficients of Q , and the vector δ contains the increments of the coefficients of Q after their spatial dependence has been stripped away. The matrix Σ contains all spatial transform variables. For ease of notation, we will henceforth drop the χ -dependence from the gradient of Q . For example, if

$$Q = d + \frac{\lambda c}{1 - \lambda a}$$

then

$$\nabla Q = \begin{bmatrix} \frac{\partial}{\partial d} Q \\ \frac{\partial}{\partial c} Q \\ \frac{\partial}{\partial a} Q \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \frac{1 - \lambda a}{\lambda^2 c} \\ \frac{1}{(1 - \lambda a)^2} \end{bmatrix}, \quad \delta_{sd} = \begin{bmatrix} \delta d \\ \delta c \\ \delta a \end{bmatrix},$$

and

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1} & 1 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-1} & 1 & \zeta \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta d \\ \delta c_{-1} \\ \delta c_0 \\ \delta c_1 \\ \delta a_{-1} \\ \delta a_0 \\ \delta a_1 \end{bmatrix}.$$

In general, we have

$$J = \|U_{out}(Q_c - Q(\chi + \delta))\|_{\mathcal{H}_2}^2$$

$$\approx \|U_{out}(Q_c - Q(\chi) - [\nabla Q]^T(\chi) \Sigma \delta)\|_{\mathcal{H}_2}^2$$

$$= \frac{1}{2} \delta^T \Gamma \delta + \rho^T \delta + \tau$$

$$=: J_{GN},$$

which is a standard quadratic program in δ with

$$\Gamma = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} |U_{out}|^2 \Sigma^* \nabla Q \nabla^T Q \Sigma \, d\theta \, d\omega,$$

$$\rho = \frac{-1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} |U_{out}|^2 \text{Re}\{(Q_c - Q)^* \nabla^T Q \Sigma\} \, d\theta \, d\omega,$$

$$\tau = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |U_{out}|^2 |Q_c - Q|^2 \, d\theta \, d\omega.$$

The optimal descent direction δ is therefore found by setting $\frac{\partial}{\partial \delta} J_{GN} = 0$, which yields

$$\delta = -\Gamma^{-1} \rho.$$

At any iteration, after having found the descent direction δ as above, we initiate a line search that updates the coefficients χ of Q such that *both* the objective function J decreases *and* the stability of Q is preserved. A comprehensive treatment of numerical optimization methods and line search algorithms can be found in Boyd and Vandenberghe (2004).

We summarize the GN iterative algorithm as follows.

Gauss–Newton method.

given initial χ that yields a stable $Q \in \mathcal{E}$.

repeat

1. Compute the descent direction δ .
2. Line search (see below) to choose the step size s .
3. Update $\chi := \chi + s \delta$.

until stopping criterion $\|\rho\|_2 < \varepsilon$ is satisfied.

Line search.

given a descent direction δ and $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$.

repeat $s := \beta s$, starting at $s = 1$.

until both conditions below are satisfied.

1. $Q(\chi + s \delta)$ is stable.
2. $J(\chi + s \delta) < J(\chi) + \alpha s \rho^T \delta$.

Remark 6. The vector ρ is nothing but the gradient ∇J of J evaluated at χ . The matrix Γ is an approximation of the Hessian $\nabla^2 J$ of J evaluated at χ .

Remark 7. The algorithm outlined above ensures stability of Q at every step. If the algorithm is initialized with coefficients χ that yield a stable $Q \in \mathcal{C}$, then the line search guarantees that stability is preserved through changing the coefficients by a sufficiently small increment.

4.3. Gradient descent and Newton's method

The gradient descent and Newton's method are variants of the GN optimization scheme described above, with the descent direction in each case, respectively, given by (Boyd & Vandenberghe, 2004)

$$\delta_{\text{grad}} = -\rho, \quad \delta_{\text{Newt}} = -\Gamma_{\text{Newt}}^{-1} \rho.$$

Here, ρ is the same as in the GN method and

$$\Gamma_{\text{Newt}} = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} |U_{\text{out}}|^2 (\Sigma^* \nabla Q \nabla^T Q \Sigma - \text{Re}\{(Q_c - Q)^* \Sigma^T \nabla^2 Q \Sigma\}) d\theta d\omega$$

is the Hessian of J evaluated at χ . Furthermore, the stopping criterion for Newton's method is often taken to be (Boyd & Vandenberghe, 2004)

$$\rho^T \Gamma_{\text{Newt}}^{-1} \rho \leq 2\varepsilon.$$

The square root of the term on the left hand side is called the Newton decrement.

4.4. Comparison of different optimization schemes

Each of the iterative schemes presented above has its advantages and disadvantages. In general, the SM algorithm is very efficient at avoiding local minima that reside close to the initialization point (Dumitrescu & Niemistö, 2004), but lacks proof of convergence and can be oscillatory as it approaches an optimal point. Classical descent algorithms such as gradient descent or Newton's method are guaranteed to converge to an optimum, however they may be trapped in local optima close to the initialization point (Dumitrescu & Niemistö, 2004). Furthermore, the convergence of gradient descent can be very slow, and Newton's method has to be modified for non-convex problems to guarantee a positive definite Hessian matrix at every iteration (Nocedal & Wright, 2006). For most problems the GN method constitutes a good compromise, in that it avoids local minima close to the initialization point, is guaranteed to converge to an optimum quickly, and automatically finds a positive definite approximation of the Hessian matrix.

It has been suggested in the digital filter design literature (Dumitrescu & Niemistö, 2004) that the best computational method is a multistage scheme where the SM, GN and classical descent are applied in this order, and each is initialized with the result of the previous algorithm. We refer the reader to Dumitrescu (2007) and Dumitrescu and Niemistö (2004) for a more detailed discussion.

5. Examples

This section contains numerical examples that illustrate the utility of the results of Section 4 in solving the structured optimal control problem characterized by (9).

Example 1. Consider the discrete time system

$$G = \left[\begin{array}{c|cc} a(\zeta) & 1 & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{array} \right].$$

Assuming that G is open-loop stable, i.e., $|a(e^{j\theta})| < 1$ for all $\theta \in [0, 2\pi]$, we have

$$T = \begin{bmatrix} \lambda \\ 1 - \lambda a \\ 0 \end{bmatrix}, \quad U = - \begin{bmatrix} \lambda \\ 1 - \lambda a \\ 1 \end{bmatrix} \frac{\lambda}{1 - \lambda a}.$$

Performing an inner-outer factorization on U and carrying out the steps described in Section 3, we arrive at

$$R = d_R + \frac{\lambda c_R}{1 - \lambda a_R}, \tag{15}$$

$$U_{\text{out}} = d_U + \frac{\lambda c_{1U} + \lambda^2 c_{2U}}{(1 - \lambda a_{1U})(1 - \lambda a_{2U})}, \tag{16}$$

where

$$\begin{aligned} a_R &= a, & c_R &= 1/(\gamma^* - \kappa^*/a), & d_R &= 1/\gamma^*, \\ a_{1U} &= a, & a_{2U} &= a, & d_U &= \kappa, \\ c_{1U} &= 2a\kappa - \gamma, & c_{2U} &= -a^2\kappa, \end{aligned}$$

and

$$\kappa = \sqrt{1 + a^*a/2} + \sqrt{1 + (a^*a)^2/4}, \quad \gamma = a/\kappa^*.$$

For concreteness, we assume that

$$a(\zeta) = \zeta^{-1}/4 + 1/2 + \zeta/8,$$

and consider $Q \in \mathcal{C}$ as described in (11) and (12). We compute the optimal Q for different temporal orders $\mu = 0, 1, 2, 3$, where a temporal order of zero corresponds to Q being just a constant gain, $Q = d$. The results are summarized in the table below, where the row indexed "o.l." corresponds to the open-loop scenario, $Q = 0$. These results were obtained as follows. For every μ the GN algorithm was initialized either with a random guess of the unknown parameters $d, c_{n,k}, a_{n,k}$ (under the condition that the resulting Q is stable) or with the result of an SM iteration. Although we make no claim for the global optimality of the obtained parameters, we point out that the solution of the numerical scheme always converged to the values that appear in Table 1 regardless of the initial guess. Furthermore, convergence was generally achieved in fewer than 20 iterations.

Fig. 5 shows the value of J versus the temporal order μ of Q . There is a large decrease in J from $\mu = 0$ to $\mu = 1$. There is a decrease of approximately 10% in J from $\mu = 1$ to $\mu = 2$. However, further increase of the temporal order beyond $\mu = 2$ does not result in significant decrease of J . It is interesting to note that $Q_c = U_{\text{out}}^{-1} R$ is a second order transfer function. Therefore it seems that taking Q to be of the same temporal order as Q_c is a good compromise between minimizing the value of J and keeping down the complexity of Q .

Example 2. We consider the discrete time example given in Voulgaris et al. (2003)

$$T = \frac{\lambda}{1 - \lambda r}, \quad U = \frac{\lambda^2}{(1 - \lambda \rho)(1 - \lambda r)},$$

with

$$\begin{aligned} \rho(\zeta) &= \zeta^{-1}/6 + 1/3 + \zeta/6, \\ r(\zeta) &= \zeta^{-1}/8 + 1/4 + \zeta/8. \end{aligned}$$

Table 1
Optimal values of the state-space parameters and the objective function for different temporal orders $\mu = 0, 1, 2, 3$ of Q . The rows indexed by “o.l.” and “ $\mu = 0$ ” correspond to $Q = 0$ and $Q = d$, respectively.

μ	d	$c_{n,k}$	$a_{n,k}$	J
o.l.				27.0917
0	0.1011			26.8364
1	2.5902	$c_{-11} = -1.1070$	$a_{-11} = 0.1769$	11.4055
		$c_{01} = -2.2121$	$a_{01} = 0.1256$	
		$c_{11} = -0.5559$	$a_{11} = 0.1302$	
2	2.7110	$c_{-11} = -1.2164$	$a_{-11} = 0.0865$	10.3247
		$c_{01} = -2.3907$	$a_{01} = -0.4849$	
		$c_{11} = -0.6380$	$a_{11} = 0.0570$	
		$c_{-12} = 0.6703$	$a_{-12} = 0.0351$	
		$c_{02} = 1.4290$	$a_{02} = -0.0494$	
		$c_{12} = 0.3320$	$a_{12} = 0.0119$	
3	2.7124	$c_{-11} = -1.2159$	$a_{-11} = 0.0715$	10.3122
		$c_{01} = -2.3939$	$a_{01} = -1.0535$	
		$c_{11} = -0.6368$	$a_{11} = 0.0344$	
		$c_{-12} = 1.4156$	$a_{-12} = -0.0091$	
		$c_{02} = 2.8086$	$a_{02} = 0.2252$	
		$c_{12} = 0.7527$	$a_{12} = -0.0055$	
		$c_{-13} = -0.4324$	$a_{-13} = -0.0059$	
		$c_{03} = -0.7869$	$a_{03} = 0.0060$	
		$c_{13} = -0.2302$	$a_{13} = -0.0030$	

Table 2
Optimal values of the state-space parameters and the objective function for different temporal orders $\mu = 0, 1, 2, 3$ of Q . The rows indexed by “o.l.” and “ $\mu = 0$ ” correspond to $Q = 0$ and $Q = d$, respectively.

μ	d	$c_{n,k}$	$a_{n,k}$	J
o.l.				0.1154
0	0.1869			0.0427
1	0.2515	$c_{-11} = -0.0139$	$a_{-11} = -0.0914$	0.0318
		$c_{01} = -0.0572$	$a_{01} = -0.1883$	
		$c_{11} = -0.0139$	$a_{11} = -0.0914$	
2	0.2500	$c_{-11} = -0.0104$	$a_{-11} = -0.1184$	0.0317
		$c_{01} = -0.0521$	$a_{01} = 0.0917$	
		$c_{11} = -0.0104$	$a_{11} = -0.1184$	
		$c_{-12} = -0.0084$	$a_{-12} = -0.0031$	
		$c_{02} = -0.0231$	$a_{02} = 0.0142$	
		$c_{12} = -0.0084$	$a_{12} = -0.0031$	
3	0.2500	$c_{-11} = -0.0104$	$a_{-11} = -0.1165$	0.0317
		$c_{01} = -0.0521$	$a_{01} = -0.3647$	
		$c_{11} = -0.0104$	$a_{11} = -0.1165$	
		$c_{-12} = -0.0037$	$a_{-12} = 0.0483$	
		$c_{02} = 0.0006$	$a_{02} = -0.0218$	
		$c_{12} = -0.0037$	$a_{12} = 0.0483$	
		$c_{-13} = 0.0039$	$a_{-13} = 0.0006$	
		$c_{03} = 0.0102$	$a_{03} = -0.0058$	
		$c_{13} = 0.0039$	$a_{13} = 0.0006$	

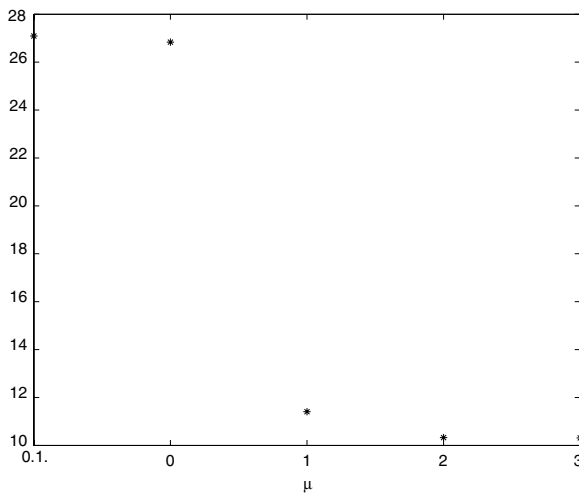


Fig. 5. Plot of J against μ corresponding to Table 1.

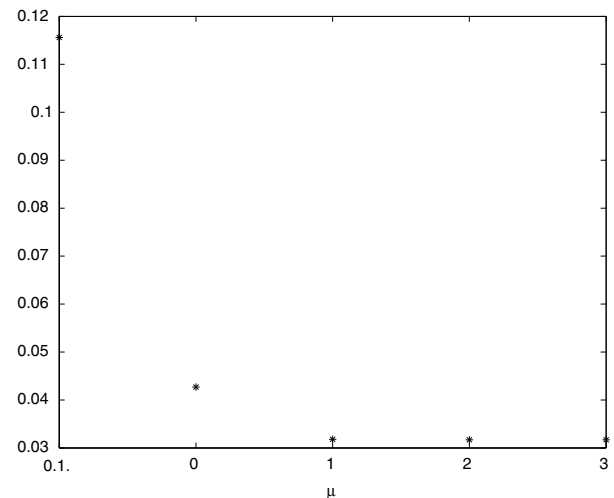


Fig. 6. Plot of J against μ corresponding to Table 2.

The transfer functions R and U_{out} for this problem have the same form as in (15) and (16) with

$$\begin{aligned}
 a_R &= r, & c_R &= r^2, & d_R &= r, \\
 a_{1U} &= \rho, & a_{2U} &= r, & d_U &= 1, \\
 c_{1U} &= \rho + r, & c_{2U} &= -\rho r.
 \end{aligned}$$

We consider $Q \in \mathcal{C}$ as described in (11) and (12), and proceed as in the previous example. We compute the optimal Q for different temporal orders $\mu = 0, 1, 2, 3$. The results are summarized in the table below. Just as in the previous example, for every μ the GN algorithm was initialized with a random guess of the unknown parameters $d, c_{n,k}, a_{n,k}$ (under the condition that the resulting Q is stable). Again the solution of the numerical scheme always converged to the values that appear in Table 2 regardless of the initial guess, and convergence was generally achieved in fewer than 20 iterations.

Fig. 6 shows the value of J versus the temporal order μ of Q . It is clear from the figure that increase of the temporal order beyond $\mu = 2$ does not result in significant decrease of J . Note that $Q_c = U_{out}^{-1}R$ is a second order transfer function.

Finally, we compare the results of the present work with those of Voulgaris et al. (2003). Let Q^1 denote the first order ($\mu = 1$) optimal Q obtained above, and let Q^{VBB} be the “truncated 2-relaxed” solution found in Voulgaris et al. (2003). Then

$$\|R - U_{out} Q^1\|_{\mathcal{H}_2}^2 = 0.0318,$$

$$\|R - U_{out} Q^{VBB}\|_{\mathcal{H}_2}^2 = 0.0659.$$

Note that both Q^1 and Q^{VBB} can be implemented with a single temporal delay element.

6. Conclusions

For spatially invariant systems, we consider the design of \mathcal{H}_2 optimal distributed controllers with finite communication speed. These are controllers whose impulse response has support inside a cone in the spatio-temporal domain. A state-space description of such systems, referred to as \mathcal{C} -causal, is given and closure properties under addition, composition, and inversion are proved. Using these closure properties and the model-matching framework, an

optimal control design problem is formulated for the case in which each subsystem has scalar input disturbance, measurement, and actuation signal. It is shown that the resulting optimization problem is non-convex in some state-space parameters. Different relaxations and numerical optimization techniques are then applied to the optimal design problem to obtain suboptimal controllers. Several examples suggest that the proposed numerical methods are indeed successful in finding a \mathcal{C} -causal controller whose performance is close to globally optimal.

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Makan Fardad received the B.S. and M.S. degrees in electrical engineering from Sharif University of Technology and Iran University of Science and Technology, respectively. He received the Ph.D. degree in mechanical engineering from the University of California, Santa Barbara, in 2006. He was a Postdoctoral Associate at the University of Minnesota before joining the Department of Electrical Engineering and Computer Science at Syracuse University, as an Assistant Professor, in August 2008. His research interests are in the analysis and control of distributed and large-scale interconnected systems.



Mihailo R. Jovanović received the Dipl. Ing. and M.S. degrees from the University of Belgrade, Serbia, in 1995 and 1998, respectively, and the Ph.D. degree from the University of California, Santa Barbara, in 2004. He was a Visiting Researcher in the Department of Mechanics, the Royal Institute of Technology, Stockholm, Sweden, from September to December 2004. He joined the University of Minnesota, Minneapolis, as an Assistant Professor of Electrical and Computer Engineering in December 2004.

Dr. Jovanović's research focuses on modeling, dynamics, and control of large-scale and distributed systems. He is a member of IEEE, SIAM, and APS and an Associate Editor of the IEEE Control Systems Society Conference Editorial Board. He received a CAREER Award from the National Science Foundation in 2007, and an Early Career Award from the University of Minnesota Initiative for Renewable Energy and the Environment in 2010.