On the optimality of localised distributed controllers

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Abstract: Design of optimal distributed controllers with a priori assigned localisation constraints is a difficult problem. Alternatively, one can ask the following question: given a localised distributed exponentially stabilising controller, is it inversely optimal with respect to some cost functional? We study this problem for linear spatially invariant systems and establish a frequency domain criterion for inverse optimality (in the LQR sense). We utilise this criterion to separate localised controllers that are never optimal from localised controllers that are optimal. For the latter, we provide examples to demonstrate optimality with respect to physically appealing cost functionals. These are characterised by state penalties that are not fully decentralised and they provide insight about spatial extent of the LQR weights that lead to localised controllers.

Keywords: inverse optimality; localised control; spatially invariant systems.

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1 Introduction

Large arrays of spatially distributed dynamical systems are becoming prevalent in modern applications. These systems can range from the macroscopic – such as vehicular platoons (Varaiya, 1993; Raza and Ioannou, 1996; Swaroop and Hedrick, 1999; Seiler et al., 2004; Jovanović and Bamieh, 2005; Jovanović et al., 2008), Unmanned Aerial Vehicle (UAV) formations (Chichka and Speyer, 1998; Fowler and D'Andrea, 2003), and satellites constellations (Kapila et al., 2000; Beard et al., 2001; Wong et al., 2002) – to the microscopic, for example, arrays of micro-mirrors (Neilson, 2001) or micro-cantilevers (Napoli et al., 1999). Significant potential for research on these systems is due to the field of Micro-Electro-Mechanical Systems (MEMS) where the fabrication of very large arrays of sensors and actuators is now both feasible and economical. The key design issues in the control of these systems are architectural such as the choice of localised vs. centralised control.

Design of optimal distributed controllers with pre-specified localisation constraints is, in general, a difficult task (we refer the reader to Ayres and Paganini (2002), Voulgaris et al. (2003), D'Andrea and Dullerud (2003), Dullerud and D'Andrea (2004), Langbort et al. (2004), Bamieh and Voulgaris (2005), Rotkowitz and Lall (2005), Rantzer (2006a, 2006b), Motee and Jadbabaie (2007), and Motee et al. (2008) and references therein for recent efforts in this area). Alternatively, one can ask a following question:

- *Given a localised distributed exponentially stabilising controller, is it inversely optimal with respect to some meaningful performance index?*

We study this problem for Linear Spatially Invariant (LSI) systems (Bamieh et al., 2002) and derive a frequency domain condition for inverse optimality. This condition represents an extension of a well-known result for Linear Time Invariant (LTI) systems (Kalman, 1964) to a class of systems studied in this paper. We provide examples of localised distributed controllers that are *inversely optimal with respect to meaningful performance criteria*, and examples of localised distributed controllers that are *not optimal in any sense*. Our results can be used to motivate design of both optimal and inversely optimal distributed controllers for other classes of spatio-temporal systems (e.g., spatially varying).

Optimality of a closed-loop system is desirable because it guarantees, among other properties, favourable gain and phase margins. These margins provide robustness to different types of uncertainty (Anderson and Moore, 1990). In addition to this traditional motivation for optimality, our objective is to gain insight about spatial extent of the LQR weights that lead to distributed controllers with favourable localisation properties. We note that a judicious selection of weights can be also employed in design of optimal feedback controllers with degree constraints (Takyar et al., 2008).

Our presentation is organised as follows: in Section 2, we setup the problem, introduce necessary background material, and provide two examples of LSI systems to illustrate that LQR design with *fully decentralised performance indices* yields *centralised optimal controllers*. In Section 3, we establish frequency domain criterion for inverse optimality of spatially invariant controllers. For systems with a single input field, this criterion requires the absolute value of the corresponding return difference to be greater than or equal to one at all spatio-temporal frequencies. In Section 4, we provide examples of exponentially stabilising localised distributed controllers and utilise results of Section 3 to characterise control laws that are optimal (in the LQR sense). We show that optimality of localised distributed controllers can be guaranteed by departing from fully decentralised performance indices. In Section 4.3, we consider a vehicular platoon that is not spatially invariant and demonstrate how ideas of this paper can be used to identify conditions for inverse optimality in spatially varying distributed control problems. We end our presentation with some concluding remarks in Section 5.

2 Preliminaries

We consider distributed systems of the form

$$\partial_t \psi(t,\xi) = [\mathcal{A}\psi(t)](\xi) + [\mathcal{B}u(t)](\xi). \tag{1}$$

where operator $\mathcal{A}$ generates a *strongly continuous* (C_o) *semigroup* (Curtain and Zwart, 1995; Banks, 1983). We assume that spatial coordinate $\xi := [\xi_1 \cdots \xi_d]^*$ belongs to a commutative group $\mathbb{G}$, and that time independent operators $\mathcal{A}$ and $\mathcal{B}$ are invariant with respect to translations in this coordinate. These properties imply spatial invariance of equation (1). The analysis and design problems for LSI systems are greatly simplified by the application of the appropriate Fourier transform in the spatially invariant directions (Bamieh et al., 2002). By taking a (spatial) Fourier transform of equation (1), we obtain

$$\dot{\hat{\psi}}_\kappa(t) = \widehat{\mathcal{A}}_\kappa \hat{\psi}_\kappa(t) + \widehat{\mathcal{B}}_\kappa \hat{u}_\kappa(t), \tag{2}$$

where $\kappa := [\kappa_1 \cdots \kappa_d]^*$ denotes the vector of frequencies corresponding to the spatial coordinates $\xi = [\xi_1 \cdots \xi_d]^*$, $\hat{\psi}_\kappa(t) := \hat{\psi}(t,\kappa)$, $\hat{u}_\kappa(t) := \hat{u}(t,\kappa)$, whereas $\widehat{\mathcal{A}}_\kappa := \widehat{\mathcal{A}}(\kappa)$ and $\widehat{\mathcal{B}}_\kappa := \widehat{\mathcal{B}}(\kappa)$ denote multiplication operators (i.e., Fourier symbols of operators $\mathcal{A}$ and $\mathcal{B}$, respectively). We note that equation (2) represents *a finite dimensional family of systems parameterised by* $\kappa \in \widehat{\mathbb{G}}$: if $\psi(t,\xi)$ and $u(t,\xi)$ respectively denote fields with n and m components then, for any $\kappa \in \widehat{\mathbb{G}}$ and $t \in \mathbb{R}$, $\hat{\psi}_\kappa(t) \in \mathbb{C}^n$, $\hat{u}_\kappa(t) \in \mathbb{C}^m$, which implies that $\widehat{\mathcal{A}}_\kappa$ and $\widehat{\mathcal{B}}_\kappa$ respectively denote matrices that belong to $\mathbb{C}^{n\times n}$ and $\mathbb{C}^{n\times m}$. We refer to the systems with $m = 1$ as *single input systems*. It was established in Bamieh et al. (2002) that the dynamical properties of system (1) can be inferred by checking the same properties of system (2) for all $\kappa \in \widehat{\mathbb{G}}$. Similar holds for design problems: for example, the solution to the optimal control problems for system (1) can be obtained by solving the analogous problems for a κ-parameterised family of finite dimensional systems (2).

2.1 Distributed LQR

We associate a quadratic performance index

$$J = \frac{1}{2}\int_0^\infty (\langle \psi, \mathcal{Q}\psi\rangle + \langle u, \mathcal{R}u\rangle)\,\mathrm{d}t, \tag{3}$$

with equation (1). If $\mathcal{Q} \geq 0$ and $\mathcal{R} > 0$ are translation invariant operators, the application of spatial Fourier transform renders equation (3) into

$$J = \frac{1}{2}\int_0^\infty \int_{\widehat{\mathbb{G}}} \left(\hat{\psi}_\kappa^*(t)\widehat{\mathcal{Q}}_\kappa \hat{\psi}_\kappa(t) + \hat{u}_\kappa^*(t)\widehat{\mathcal{R}}_\kappa \hat{u}_\kappa(t)\right)\mathrm{d}\kappa\,\mathrm{d}t \tag{4}$$

where $\mathrm{d}\kappa$ denotes the Haar measure. Thus, distributed LQR problems (1) and (3) amounts to solving the κ-parameterised family of finite dimensional LQR problems (2) and (4). If pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}^*, \mathcal{Q}^{1/2})$ are exponentially stabilisable, then the κ-parameterised family of Algebraic Riccati Equations (AREs)

$$\widehat{\mathcal{A}}_\kappa^* \widehat{\mathcal{P}}_\kappa + \widehat{\mathcal{P}}_\kappa \widehat{\mathcal{A}}_\kappa + \widehat{\mathcal{Q}}_\kappa - \widehat{\mathcal{P}}_\kappa \widehat{\mathcal{B}}_\kappa \widehat{\mathcal{R}}_\kappa^{-1} \widehat{\mathcal{B}}_\kappa^* \widehat{\mathcal{P}}_\kappa = 0, \tag{5}$$

has a unique positive definite uniformly bounded solution for every $\kappa \in \widehat{\mathbb{G}}$ (Bamieh et al., 2002). This positive definite matrix determines the optimal stabilising feedback for system (2) for every $\kappa \in \widehat{\mathbb{G}}$

$$\hat{u}_\kappa := \widehat{\mathcal{K}}_\kappa \hat{\psi}_\kappa = -\widehat{\mathcal{R}}_\kappa^{-1} \widehat{\mathcal{B}}_\kappa^* \widehat{\mathcal{P}}_\kappa \hat{\psi}_\kappa, \quad \kappa \in \widehat{\mathbb{G}}. \tag{6}$$

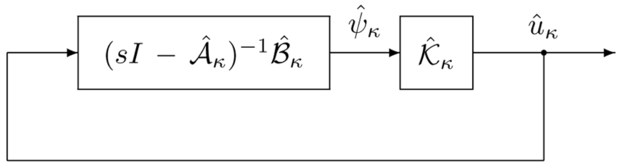
In this case, there exist an exponentially stabilising translation invariant feedback for system (1) that minimises equation (3) (Bamieh et al., 2002). This optimal stabilising feedback for equation (1) is readily obtained by taking an inverse Fourier transform of equation (6).

2.2 *Return difference equality*

System (2) with a state-feedback control law $\hat{u}_\kappa = \widehat{\mathcal{K}}_\kappa \hat{\psi}_\kappa$ can be equivalently represented by a feedback arrangement shown in Figure 1. The so-called *return difference* of system whose block diagram is shown in Figure 1 is defined by (Anderson and Moore, 1990; Kalman, 1964)

$$\widehat{\mathcal{H}}_\kappa(s) := I - \widehat{\mathcal{K}}_\kappa (sI - \widehat{\mathcal{A}}_\kappa)^{-1} \widehat{\mathcal{B}}_\kappa =: I - \widehat{\mathcal{K}}_\kappa \widehat{\mathcal{G}}_\kappa(s) \widehat{\mathcal{B}}_\kappa. \tag{7}$$

Figure 1 Block diagram of system (2) with $\hat{u}_\kappa = \widehat{\mathcal{K}}_\kappa \hat{\psi}_\kappa$



This quantity is important because its inverse determines the sensitivity function $\widehat{\mathcal{S}}_\kappa(s) := \widehat{\mathcal{H}}_\kappa^{-1}(s)$. It is readily established that $\widehat{\mathcal{H}}_\kappa(j\omega)$ for every $\omega \in \mathbb{R}$ and $\kappa \in \widehat{\mathbb{G}}$ satisfies (Anderson and Moore, 1990; Kalman, 1964)

$$\widehat{\mathcal{R}}_\kappa + \widehat{\mathcal{B}}_\kappa^* \widehat{\mathcal{G}}_\kappa^*(j\omega) \widehat{\mathcal{Q}}_\kappa \widehat{\mathcal{G}}_\kappa(j\omega) \widehat{\mathcal{B}}_\kappa = \widehat{\mathcal{H}}_\kappa^*(j\omega) \widehat{\mathcal{R}}_\kappa \widehat{\mathcal{H}}_\kappa(j\omega), \tag{8}$$

where, for example, $\widehat{\mathcal{G}}_\kappa(j\omega) := (j\omega I - \widehat{\mathcal{A}}_\kappa)^{-1}$ and $\widehat{\mathcal{G}}_\kappa^*(j\omega) := -(j\omega I + \widehat{\mathcal{A}}_\kappa^*)^{-1}$. Equation (8) is usually referred to as the *return difference equality* and it follows directly from the ARE. A straightforward consequence of this equality is

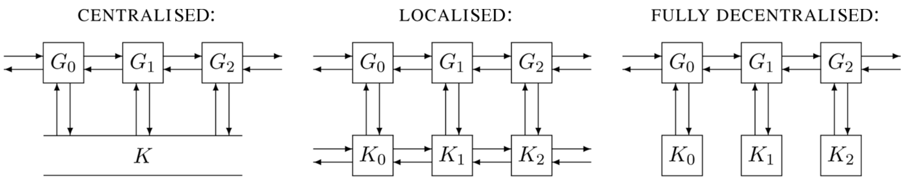
$$\widehat{\mathcal{H}}_\kappa^*(j\omega) \widehat{\mathcal{R}}_\kappa \widehat{\mathcal{H}}_\kappa(j\omega) \geq \widehat{\mathcal{R}}_\kappa. \tag{9}$$

Equations (8) and (9) are utilised in Section 3 to express a frequency domain condition for inverse optimality of distributed exponentially stabilising spatially invariant controllers.

2.3 Distributed controller architectures

Figure 2 illustrates different control strategies that can be used for control of spatially distributed systems: centralised, localised, and fully decentralised. Centralised controllers require information from all plant units for achieving the desired control objective. On the other hand, in fully decentralised strategies control unit K_n uses only information from the nth plant unit G_n on which it acts. An example of a localised distributed control architecture with nearest neighbour interactions is shown in Figure 2.

Figure 2 Distributed architectures for centralised, localised (with nearest neighbour interactions), and fully decentralised control strategies



2.4 Examples of optimal distributed design

We next provide two examples of spatially invariant systems with fully distributed measurements and controls:

- *diffusion equation over an infinite domain* ($\mathbb{G} := \mathbb{R}$)
- *mass-spring system on an infinite line* ($\mathbb{G} := \mathbb{Z}$).

We demonstrate that the LQR design with *fully decentralised performance indices* yields *centralised optimal controllers* for these systems.

2.4.1 Diffusion equation

We consider a one-dimensional diffusion equation

$$\psi_t(t,\xi) = \psi_{\xi\xi}(t,\xi) + c\psi(t,\xi) + u(t,\xi), \quad \xi \in \mathbb{R}. \tag{10}$$

The application of the standard spatial Fourier transform yields

$$\dot{\hat{\psi}}_\kappa(t) = (c-\kappa^2)\hat{\psi}_\kappa(t) + \hat{u}_\kappa(t) =: \widehat{\mathcal{A}}_\kappa \hat{\psi}_\kappa(t) + \widehat{\mathcal{B}}_\kappa \hat{u}_\kappa(t), \quad \kappa \in \mathbb{R},$$

which implies that equation (10) is not (open-loop) exponentially stable if $c \geq 0$. Choosing, for example, $\mathcal{Q} := qI$ and $\mathcal{R} := rI$ in equation (3), with ($q = \text{const.} > 0$, $r = \text{const.} > 0$), yields the following positive definite solution to the κ-parameterised ARE (equation (5)):

$$\widehat{\mathcal{P}}_\kappa = r(c-\kappa^2) + \sqrt{r^2(c-\kappa^2)^2 + rq},$$

which gives the optimal control of the form (6) with

$$\widehat{\mathcal{K}}_\kappa = -\big((c-\kappa^2) + \sqrt{(c-\kappa^2)^2 + q/r}\big).$$

Since $\widehat{\mathcal{K}}_\kappa$ is irrational function of κ, it cannot be implemented by a PDE (in t and ξ). Rather, the optimal control in the physical space assumes the form

$$u(t,\xi) = \int_{\mathbb{R}} \mathcal{K}(\xi-\zeta)\psi(t,\zeta)\,\mathrm{d}\zeta. \tag{11}$$

In Bamieh et al. (2002), it was established that $\mathcal{K}$ decays exponentially fast as a function of its argument which is a desirable property for implementation. Despite this nice feature, equation (11) represents a centralised controller.

2.4.2 Mass-spring system

A system consisting of an infinite number of identical masses and springs on a line is shown in Figure 3. If restoring forces are considered as linear functions of displacements, the dynamics of the mass indexed by $\xi \in \mathbb{Z}$ are given by

$$\ddot{x}(t,\xi) = x(t,\xi-1) - 2x(t,\xi) + x(t,\xi+1) + u(t,\xi),$$

where $x(t,\xi)$ represents the displacement from a reference position of the mass ξ, and $u(t,\xi)$ is the control applied on the mass ξ. A state-space representation of this system is given by

$$\begin{aligned}\dot{\psi}(t,\xi) &= \begin{bmatrix} 0 & 1 \\ T_{-1} - 2 + T_1 & 0 \end{bmatrix} \psi(t,\xi) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t,\xi), \\ \psi(t,\xi) &:= [x(t,\xi) \quad \dot{x}(t,\xi)]^*, \quad \xi \in \mathbb{Z},\end{aligned} \tag{12}$$

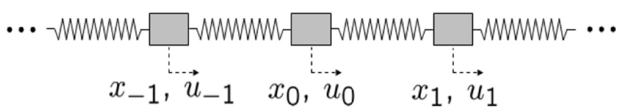
where T_{-1} and T_1 respectively denote the operators of translation by -1 and 1 (in the mass' index). We utilise the fact that system (12) has spatially invariant dynamics over discrete spatial lattice $\mathbb{Z}$ and apply the appropriate Fourier transform (spatial $\mathcal{Z}$ transform evaluated on a unit circle) to obtain

$$\begin{aligned}\dot{\hat{\psi}}_\kappa(t) &= \begin{bmatrix} 0 & 1 \\ a_\kappa & 0 \end{bmatrix} \hat{\psi}_\kappa(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}_\kappa(t), =: \widehat{\mathcal{A}}_\kappa \hat{\psi}_\kappa(t) + \widehat{\mathcal{B}}_\kappa \hat{u}_\kappa(t), \\ a_\kappa &:= 2(\cos\kappa - 1), \quad \kappa \in [0, 2\pi),\end{aligned} \tag{13}$$

where, for example,

$$\hat{u}(t,\kappa) := \sum_{\xi\in\mathbb{Z}} u(t,\xi)\mathrm{e}^{-j\kappa\xi}.$$

Figure 3 Mass-spring system



Selecting, for example, fully decentralised weights

$$\mathcal{Q} := \begin{bmatrix} q_1 I & 0 \\ 0 & q_2 I \end{bmatrix}, \quad \mathcal{R} := rI,$$

with ($q_1 = \text{const.} > 0$, $q_2 = \text{const.} \geq 0$, $r = \text{const.} > 0$), renders equation (3) into

$$J := \frac{1}{2} \int_0^\infty \left(\sum_{\xi \in \mathbb{Z}} q_1 x^2(t,\xi) + q_2 \dot{x}^2(t,\xi) + r u^2(t,\xi) \right) \mathrm{d}t,$$

and yields the following optimal control

$$\hat{u}_\kappa(t) = \widehat{\mathcal{K}}_\kappa \hat{\psi}_\kappa(t) := [\widehat{\mathcal{K}}_{1\kappa} \quad \widehat{\mathcal{K}}_{2\kappa}] \hat{\psi}_\kappa(t),$$
$$\widehat{\mathcal{K}}_{1\kappa} = 2(1 - \cos\kappa) - \sqrt{4(\cos\kappa - 1)^2 + q_1/r}, \quad \widehat{\mathcal{K}}_{2\kappa} = -\sqrt{-2\widehat{\mathcal{K}}_{1\kappa} + q_2/r}.$$

Again, since $\widehat{\mathcal{K}}_\kappa$ is irrational function of κ, it cannot be implemented by a localised distributed controller. Rather, the optimal control in the physical space is a centralised controller of the form

$$u(t,\xi) = \sum_{\zeta \in \mathbb{Z}} \mathcal{K}(\xi - \zeta) \psi(t,\zeta), \quad \xi \in \mathbb{Z}. \tag{14}$$

In Section 4, we illustrate that both spatially localised and fully decentralised exponentially stabilising controllers for diffusion equation (10) and mass-spring system (12) can be inversely optimal with respect to physically appealing cost functionals. In particular, for a diffusion equation, these cost functionals incorporate penalties on spatial derivatives of ψ (in addition to penalties on ψ), which implies that they are no longer fully decentralised.

3 The inverse problem of optimal distributed control

In this section, we consider the inverse problem of optimal exponential stabilisation of LSI system (1). This problem is inverse because we assume that an exponentially stabilising state-feedback control law for equation (1) is available and search for performance indices of the form (3) for which this control law is optimal. In other words, operators $\mathcal{Q}$ and $\mathcal{R}$ in equation (3) are not *a priori* assigned; rather, they are determined *a posteriori* by the exponentially stabilising state-feedback. We state a frequency domain condition that separates distributed controllers that are never optimal (in the LQR sense) from distributed controllers that are optimal (in the LQR sense). This condition represents an extension of a well-known result for finite dimensional LTI systems (Kalman, 1964) to a class of systems considered in this paper. In particular, for single input systems, the inverse optimality of an exponentially stabilising control law $\mathcal{K}$ is guaranteed if and only if the absolute value of the return difference:

$$\widehat{\mathcal{H}}_\kappa(j\omega) := I - \widehat{\mathcal{K}}_\kappa (j\omega I - \widehat{\mathcal{A}}_\kappa)^{-1} \widehat{\mathcal{B}}_\kappa =: I - \widehat{\mathcal{K}}_\kappa \widehat{\mathcal{G}}_\kappa(j\omega) \widehat{\mathcal{B}}_\kappa,$$

is not less than one at any spatio-temporal frequency pair (κ, ω) (see Theorem 1 for precise formulation).

Theorem 1 and Corollary 2 are readily established by recognising that the application of the appropriate spatial Fourier transform renders LSI systems into a κ-parameterised family of finite dimensional LTI systems. We refer the reader to Appendix A for proof of Theorem 1 and to Anderson and Moore (1990) and Kalman (1964) for finite dimensional LTI results.

Theorem 1: *Let a triple* $\{\mathcal{A}, \mathcal{B}, \mathcal{K}\}$ *for LSI system* (1) *satisfy:*

(a) $\mathcal{A}$ *is a generator of a* C_o *semigroup*

(b) $(\mathcal{A}, \mathcal{B})$ *is exponentially controllable*

(c) $\mathcal{K}$ *is an exponentially stabilising translation invariant state-feedback operator.*

Then, a necessary and sufficient condition for $\mathcal{K}$ *to be an optimal control law with respect to a performance index given by equation* (3), *with* $\mathcal{R} > 0$ *and* $(\mathcal{Q}^{1/2}, \mathcal{A})$ *exponentially observable, is that the return difference equality* (8) *holds for all* $\omega \in \mathbb{R}$ *and* $\kappa \in \widehat{\mathbb{G}}$.

Remark 1: From Theorem 1 it is straightforward to see that an exponentially stabilising translation invariant state-feedback controller $\mathcal{K}$ is inversely optimal if the return difference inequality

$$\sigma_{\min}\left\{\widehat{\mathcal{R}}_\kappa^{1/2}\widehat{\mathcal{H}}_\kappa(j\omega)\widehat{\mathcal{R}}_\kappa^{-1/2}\right\} \geq 1, \tag{15}$$

holds for all $\omega \in \mathbb{R}$ and $\kappa \in \widehat{\mathbb{G}}$. For a single input LSI system (1), condition (15) simplifies to

$$|\widehat{\mathcal{H}}_\kappa(j\omega)| \geq 1, \quad \forall \omega \in \mathbb{R}, \quad \forall \kappa \in \widehat{\mathbb{G}}. \tag{16}$$

Corollary 2: *Let a quadruple* $\{\mathcal{A}, \mathcal{B}, \mathcal{K}, \mathcal{R}\}$ *for LSI system* (1) *satisfy:*

(a) $\mathcal{A}$ *is a generator of a* C_o *semigroup*

(b) $(\mathcal{A}, \mathcal{B})$ *is exponentially stabilisable*

(c) $\mathcal{K}$ *is an exponentially stabilising translation invariant state-feedback operator*

(d) $\mathcal{R} > 0$

(e) *return difference inequality* (15) *holds for all* $\omega \in \mathbb{R}$ *and* $\kappa \in \widehat{\mathbb{G}}$.

Then, there exist a translation invariant operator $\mathcal{Q} = \mathcal{D}\mathcal{D}^*$ *with* $(\mathcal{A}^*, \mathcal{D})$ *exponentially stabilisable such that the optimal state-feedback operator* $\bar{\mathcal{K}}$ *associated with the LQR problem* (1), (3) *satisfies* $\bar{\mathcal{K}}\mathcal{G}(j\omega)\mathcal{B} = \mathcal{K}\mathcal{G}(j\omega)\mathcal{B}$. *If a pair* $(\mathcal{A}, \mathcal{B})$ *is exponentially controllable, then* $\mathcal{K} = \bar{\mathcal{K}}$.

If Corollary 2 is satisfied, then a translation invariant operator $\mathcal{D}$ ($\mathcal{Q} = \mathcal{D}\mathcal{D}^*$) can be determined from return difference equality (8) using polynomial matrix fraction description of $\widehat{\mathcal{H}}_\kappa(j\omega)$. This operator can always be selected to guarantee exponential stabilisability of pair $(\mathcal{A}^*, \mathcal{D})$ (we refer the reader to Anderson and Moore (1990) and Kalman (1964) for finite dimensional LTI version). In general, for a given $\mathcal{R}$ there are many different $\mathcal{D}$'s that satisfy (8) and yield $\mathcal{K}$ as a solution to the corresponding LQR problem. For single input systems, $\widehat{\mathcal{D}}_\kappa$ can be determined from Anderson and Moore (1990) and Kalman (1964)

$$\|\widehat{\mathcal{D}}_\kappa^* \widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa\|^2 = \widehat{\mathcal{R}}_\kappa(|\widehat{\mathcal{H}}_\kappa(j\omega)|^2 - 1). \tag{17}$$

4 Examples of inversely optimal distributed design

In this section, we investigate inverse optimality of localised distributed exponentially stabilising controllers for diffusion equation (10) and mass-spring system (12). We utilise results of Section 3 to distinguish between controllers that are never optimal and controllers that are optimal. In the latter case, we show that inverse optimality is guaranteed with respect to physically appealing performance criteria. For a diffusion equation, we demonstrate that localised distributed and even fully decentralised optimal controllers can be obtained by incorporating spatial derivatives of ψ (in addition to ψ) in the performance index. Similarly, for a mass-spring system, we establish that spatially localised cost functionals can produce controllers with favourable architectures. These observations should be compared to the results of Section 2.4, where it was shown that LQR design with fully decentralised performance criteria results into centralised optimal controllers for both these systems. Finally, in Section 4.3, we consider a vehicular platoon that is not spatially invariant and demonstrate how ideas of this paper can be used to identify conditions for inverse optimality in spatially varying distributed control problems.

4.1 Diffusion equation

It is readily established that the following spatially invariant localised distributed controller

$$\begin{gathered} u(t,\xi) = -(\beta\psi_{\xi\xi}(t,\xi) + (c+\alpha)\psi(t,\xi)), \\ \Updownarrow \\ \hat{u}_\kappa(t) = \widehat{\mathcal{K}}_\kappa \hat{\psi}_\kappa(t) = -(c + \alpha - \beta\kappa^2)\hat{\psi}_\kappa(t), \end{gathered} \tag{18}$$

provides exponential stability of equation (10) so long as κ-independent real design parameters α and β respectively satisfy $\alpha > 0$ and $\beta \in (-\infty, 1]$. Based on Theorem 1, it follows that controller (18) is inversely optimal if and only if

$$(\alpha - c + (2-\beta)\kappa^2)(\alpha + c - \beta\kappa^2) \geq 0,$$

holds for all $\kappa \in \mathbb{R}$. This condition is satisfied for all $\kappa \in \mathbb{R}$ if and only if $\alpha \geq c$ and $\beta \leq 0$. Thus, *if either* $\alpha < c$ *or* $\beta \in (0, 1]$ *then controller given by equation* (18) *is never optimal in the LQR sense*. In other words, for this choice of design parameters α and β it is not possible to select a pair $(\mathcal{Q} \geq 0, \mathcal{R} > 0)$ for which equation (18) is obtained as a solution to the corresponding LQR problem (1), (3). This implies that this exponentially stabilising control law does not have any stability margins: with a slightly perturbed feedback closed-loop system becomes unstable. On the other hand, for $\alpha \geq c$ and $\beta \leq 0$ there always exist $(\mathcal{Q} \geq 0, \mathcal{R} > 0)$ in equation (3) with respect to which controller given by equation (18) is inversely optimal. Choosing, for example, $\mathcal{R} := rI$ in equation (3), with $r = \text{const.} > 0$, yields the following state penalty:

$$\widehat{\mathcal{Q}}_\kappa = r((\alpha^2 - c^2) + 2(c + \alpha(1-\beta))\kappa^2 + \beta(\beta - 2)\kappa^4),$$
$$\Updownarrow$$
$$\mathcal{Q} = r((\alpha^2 - c^2)I - 2(c + \alpha(1-\beta))\partial_{\xi\xi} + \beta(\beta - 2)\partial_{\xi\xi\xi\xi}).$$

Since for any $\kappa \in \mathbb{R}$, κ-parameterised diffusion equation represents a scalar system, this state penalty is obtained as a unique solution to equation (17) for any $r > 0$.

Thus, we have established optimality of spatially invariant localised distributed controller given by equation (18) with respect to the following performance index

$$J = \frac{r}{2}\int_0^\infty ((\alpha^2 - c^2)\langle \psi, \psi\rangle + 2(c + \alpha(1-\beta))\langle \psi_\xi, \psi_\xi\rangle + \beta(\beta-2)\langle \psi_{\xi\xi}, \psi_{\xi\xi}\rangle + \langle u, u\rangle)\mathrm{d}t, \quad r > 0, \quad \alpha \geq c, \quad \beta \leq 0. \tag{19}$$

In particular, for $\beta = 0$ controller given by equation (18) is fully decentralised and equation (19) simplifies to

$$J = \frac{r}{2}\int_0^\infty ((\alpha^2 - c^2)\langle \psi, \psi\rangle + 2(\alpha + c)\langle \psi_\xi, \psi_\xi\rangle + \langle u, u\rangle)\mathrm{d}t, \quad r > 0, \quad \alpha \geq c. \tag{20}$$

To recap:

- *a fully decentralised controller,* $u(t,\xi) = -(c+\alpha)\psi(t,\xi)$, *with* $\alpha \geq c$ *represents exponentially stabilising solution to the* LQR problem (10), (20)
- *a localised distributed controller,* $u(t,\xi) = -(\beta\psi_{\xi\xi}(t,\xi) + (c+\alpha)\psi(t,\xi))$, *with* $\alpha \geq c$, $\beta < 0$ *represents exponentially stabilising solution to the* LQR problem (10), (19).

Remark 2: Our analysis indicates that a choice of the state-space on which optimal control problems are formulated can significantly influence localisation properties of resulting distributed optimal controllers. Example of Section 2.4.1 illustrates that the spatially invariant LQR problem (for a diffusion equation) formulated on the space of square integrable functions $L_2(-\infty,\infty)$ yields centralised controllers. On the other hand, the LQR design performed on the Sobolev spaces $H_1(-\infty,\infty)$ or $H_2(-\infty,\infty)$ can result into localised distributed and fully decentralised controllers provided that the penalties on $\langle \psi, \psi\rangle$, $\langle \psi_\xi, \psi_\xi\rangle$, and $\langle \psi_{\xi\xi}, \psi_{\xi\xi}\rangle$ are appropriately selected.

4.2 Mass-spring system

It is easily shown that the exponential stability of equation (12) is guaranteed with the spatially invariant localised distributed controller of the form

$$\begin{aligned} u(t,\xi) &= -\left[\ \alpha+\beta(T_{-1}-2+T_1) \quad \gamma\ \right]\psi(t,\xi) \\ &= -(\alpha-2\beta)x(t,\xi)-\gamma\dot{x}(t,\xi)-\beta(x(t,\xi-1)+x(t,\xi+1)), \quad \xi\in\mathbb{Z}, \\ &\Updownarrow \\ \hat{u}_\kappa(t) &= \widehat{\mathcal{K}}_\kappa\hat{\psi}_\kappa(t) = -\left[\ \alpha+\beta a_\kappa \quad \gamma\ \right]\hat{\psi}_\kappa(t), \quad \kappa\in[0,2\pi), \end{aligned} \tag{21}$$

so long as κ-independent real design parameters α, β, and γ satisfy

$$\gamma>0 \quad \text{and} \quad \begin{cases} \alpha>0, & \beta\in(-\infty,1], \\ \alpha>4(\beta-1), & \beta>1. \end{cases}$$

Based on Theorem 1, it follows that controller (21) is inversely optimal if and only if

$$(\gamma^2-2(\alpha+\beta a_\kappa))\omega^2+(\alpha+\beta a_\kappa)(\alpha+(\beta-2)a_\kappa)\geq 0,$$

holds for all $\omega\in\mathbb{R}$, $\kappa\in[0,2\pi)$. This criterion for inverse optimality is satisfied for all $\omega\in\mathbb{R}$, $\kappa\in[0,2\pi)$ if and only if either $\{\alpha>0,\ \beta\leq 0,\ \gamma\geq\sqrt{2(\alpha-4\beta)}\}$ or $\{\alpha\geq 4\beta, \beta>0, \gamma\geq\sqrt{2\alpha}\}$. Hence, *controller* (21) *is never optimal in the* LQR *sense if parameters* α*,* β*, and* γ *fail to satisfy either of these two conditions*. On the other hand, if α, β, and γ satisfy either of these two conditions than there always exist $(\mathcal{Q}\geq 0, \mathcal{R}>0)$ in equation (3) with respect to which controller (21) is optimal. Selecting, for example,

$$\left.\begin{aligned} &\mathcal{R}:=rI \\ &r=\text{const.}>0 \end{aligned}\right\}, \quad \mathcal{Q}:=\begin{bmatrix}\mathcal{Q}_{11} & 0 \\ 0 & \mathcal{Q}_{22}\end{bmatrix}, \tag{22}$$

in equation (3) yields the following state penalty:

$$\begin{aligned} &\widehat{\mathcal{Q}}_{11\kappa}=r(\alpha^2+2\alpha(\beta-1)a_\kappa+\beta(\beta-2)a_\kappa^2), \quad \widehat{\mathcal{Q}}_{22\kappa}=r(\gamma^2-2(\alpha+\beta a_\kappa)), \\ &\Updownarrow \\ &\mathcal{Q}_{11}=r(\alpha^2+2\alpha(\beta-1)(T_{-1}-2+T_1)+\beta(\beta-2)(T_{-2}-4T_{-1}+6-4T_1+T_2)), \\ &\mathcal{Q}_{22}=r(\gamma^2-2(\alpha+\beta(T_{-1}-2+T_1))). \end{aligned}$$

Thus, we have established optimality of spatially invariant localised distributed controller (21) with respect to the following performance index

$$J=\frac{1}{2}\int_0^\infty\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(\psi^*(t,n)Q_{n-m}\psi_m(t,m)+u^*(t,n)R_{n-m}u(t,m))\mathrm{d}t, \tag{23}$$

where

$$\{R_0 = r = \text{const.} > 0;\ R_n = 0, \forall n \in \mathbb{Z} \setminus \{0\}\}, \quad \{Q_n = 0, \forall n \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}\},$$
$$Q_0 = r \begin{bmatrix} \alpha^2 - 4\alpha(\beta - 1) + 6\beta(\beta - 2) & 0 \\ 0 & \gamma^2 - 2(\alpha - 2\beta) \end{bmatrix}, \tag{24}$$
$$Q_{\pm 1} = r \begin{bmatrix} 2\alpha(\beta - 1) - 4\beta(\beta - 2) & 0 \\ 0 & -2\beta \end{bmatrix}, \quad Q_{\pm 2} = r \begin{bmatrix} \beta(\beta - 2) & 0 \\ 0 & 0 \end{bmatrix}.$$

In particular, for $\beta = 0$ controller (21) is fully decentralised and equation (24) simplifies to

$$\{R_0 = r = \text{const.} > 0;\ R_n = 0, \forall n \in \mathbb{Z} \setminus \{0\}\}, \quad \{Q_n = 0, \forall n \in \mathbb{Z} \setminus \{0, \pm 1\}\},$$
$$Q_0 = r \begin{bmatrix} \alpha^2 + 4\alpha & 0 \\ 0 & \gamma^2 - 2\alpha \end{bmatrix}, \quad Q_{\pm 1} = r \begin{bmatrix} -2\alpha & 0 \\ 0 & 0 \end{bmatrix}. \tag{25}$$

To recap:

- *a fully decentralised controller,* $u(t, \xi) = -(\alpha x(t, \xi) + \gamma \dot{x}(t, \xi))$, *with* $\{\alpha > 0, \gamma > \sqrt{2\alpha}\}$ *represents exponentially stabilising solution to the LQR* problem (12), (23), (25)
- *a nearest neighbour interaction controller*, $u(t, \xi) = -(\alpha - 2\beta)x(t, \xi) - \gamma \dot{x}(t, \xi) - \beta(x(t, \xi - 1) + x(t, \xi + 1))$, with either $\{\alpha > 0, \beta < 0, \gamma \geq \sqrt{2(\alpha - 4\beta)}\}$ or $\{\alpha \geq 4\beta, \beta > 0, \gamma \geq \sqrt{2\alpha}\}$ represents exponentially stabilising solution to the LQR problem (12), (23), (24).

Remark 3: The above penalties on $\{x(t, \xi)\}_{\xi \in \mathbb{Z}}$ and $\{\dot{x}(t, \xi)\}_{\xi \in \mathbb{Z}}$ represent unique solutions to equation (17) provided that equation (22) is satisfied (that is, $\mathcal{Q}_{12} \equiv 0$). However, for given $\mathcal{R} := rI > 0$, there are many other operators $\mathcal{Q} = \mathcal{Q}^* \geq 0$ with non-zero off-diagonal elements (that is, $\mathcal{Q}_{12} \neq 0$) that satisfy equation (17) and give controller (21) as a solution to the corresponding LQR problem.

Remark 4: Example of Section 2.4.2 illustrates that the spatially invariant LQR design (for a mass-spring system) with fully decentralised performance indices yields centralised controllers. On the other hand, spatially localised performance indices (with penalties on positions and velocities of several neighbouring masses) can yield localised distributed and fully decentralised controllers, provided that these penalties are appropriately assigned. However, it is very difficult to choose these cost functionals *a priori*. Rather, they have been determined *a posteriori* by the exponentially stabilising control law using the return difference equality.

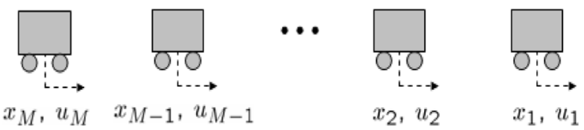
4.3 An example of a spatially varying problem

We next present an example of a problem that is not spatially invariant where approach of this paper can be utilised to identify conditions for inverse optimality (for additional detail, see Jovanović et al. (2008)). We consider a system of M identical vehicles, shown in Figure 4. Each vehicle is modelled as a double-integrator

$$\ddot{x}_n = u_n, \quad n \in \{1, \ldots, M\}, \tag{26}$$

where x_n represents the position of the nth vehicle, and u_n is the control applied on the nth vehicle.

Figure 4 Platoon of M vehicles



A design goal is to provide a desired constant cruising velocity v_d and to keep the inter-vehicular distance at a constant level δ. For each vehicle, we introduce the position and velocity error variables with respect to the (absolute) desired trajectories

$$\xi_n(t) := x_n(t) - v_d t + n\delta, \quad \zeta_n(t) := \dot{x}_n(t) - v_d,$$

and rewrite equation (26) as

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u =: A\psi + Bu, \tag{27}$$

where $\xi := \text{col}\{\xi_n\}$, $\zeta := \text{col}\{\zeta_n\}$, and $u := \text{col}\{u_n\}$. The relative position errors between neighbouring vehicles are determined by

$$\eta_n(t) := x_n(t) - x_{n-1}(t) + \delta = \xi_n(t) - \xi_{n-1}(t), \quad n \in \{2, \ldots, M\}.$$

We propose the following static distributed controller

$$u = K\psi = -[aI + bL \quad cI]\psi, \tag{28}$$

where a, b, and c denote positive design parameters, and $L \in \mathbb{R}^{M \times M}$ is a matrix describing information exchange between different vehicles. If L is a diagonal matrix then there is no information exchange between the vehicles and control strategy (28) is *fully decentralised*. This approach ignores the fact that a vehicle is a part of the platoon and as such is not safe for implementation. If L is a full matrix then there is communication between all the vehicles and controller (28) is *centralised*. In this case, every vehicle utilises information from all other vehicles for achieving the desired control objective which usually results in best performance, but it requires excessive communication. If L is a banded matrix then there is a communication between few neighbouring vehicles, and equation (28) represents a *localised distributed* controller. For example, if L is given by

$$L := \begin{bmatrix} 1 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 1 \end{bmatrix}, \qquad (29)$$

then the controller for vehicle n utilises information about absolute position and velocity of vehicle n, and information about the distances between vehicle n and neighbouring vehicles $n-1$ and $n+1$. The architecture of this localised controller with nearest neighbour interactions is shown in the middle plot of Figure 2.

A spectral decomposition of L

$$L = V\Lambda V^*, \quad VV^* = V^*V = I, \quad \Lambda = \text{diag}\{\lambda_1(L), \ldots, \lambda_M(L)\},$$

can be used to establish stability of equations (27)–(29) for any choice of positive design parameters a, b, and c. This follows directly from the fact that the eigenvalues of L are determined by (see, for example, Grenander and Szegö (1984)):

$$\lambda_n(L) = \begin{cases} 2\left(1 - \cos\dfrac{n\pi}{M}\right) & n \in \{1, \ldots, M-1\}, \\ 0 & n = M. \end{cases}$$

Next, we address the question of whether it is possible to select the weights in the LQR problem

$$J = \frac{1}{2}\int_0^\infty (\xi^* Q_\xi \xi + \zeta^* Q_\zeta \zeta + u^* R u)\mathrm{d}t, \qquad (30)$$

with $Q_\xi = Q_\xi^* \geq 0$, $Q_\zeta = Q_\zeta^* \geq 0$, and $R = R^* > 0$, to obtain a *localised distributed controller* for equation (27). Any quadratic cost functional for system (27) that does not penalise products between positions and velocities, can be represented by equation (30). By selecting $R = rI$ in equation (30) with $r > 0$, a spectral decomposition of L can be used to show that controller (28) and (29) represents an inversely optimal controller for the LQR problem (27), (30) if and only if

$$c^2 \geq 2(a + b\lambda_1(L)).$$

If this condition is not satisfied than controller given by equations (28) and (29) fails to be optimal in the LQR sense. If this condition is satisfied, the state penalty

$$Q := \begin{bmatrix} Q_\xi & 0 \\ 0 & Q_\zeta, \end{bmatrix},$$

can be determined from return difference equality, and it is given by $Q_\xi = r(aI + bL)^2$, $Q_\zeta = r((c^2 - 2a)I - 2bL)$. These penalties on ξ and ζ represent unique solutions to return difference equality provided that the products between positions and velocities are not penalised in J. However, for given $R := rI > 0$, there are many other matrices

$$Q = \begin{bmatrix} Q_\xi & Q_{\xi\zeta} \\ Q_{\xi\zeta}^* & Q_\zeta \end{bmatrix} \geq 0,$$

with non-zero off-diagonal elements (that is, $Q_{\xi\zeta} \neq 0$) that satisfy return difference equality and give controller (28), (29) as a solution to the corresponding LQR problem.

To recap:

- *A localised distributed controller,* $u = -((aI + bL)\xi + c\zeta)$, $\{a > 0,\ b > 0,\ c \geq \sqrt{2(a + b\lambda_1(L))}\}$, *represents a stabilising solution to the LQR problem* (27), (30) *with L given by equation* (29) *and* $\{Q_\xi = r(aI + bL)^2,\ Q_\zeta = r((c^2 - 2a)I - 2bL),\ r > 0\}$.

5 Concluding remarks

This paper deals with the inverse problem of optimal distributed stabilisation of LSI systems. We establish a frequency domain criterion that separates controllers that are never optimal (in the LQR sense) from controllers that are optimal (in the LQR sense). This criterion is expressed in terms of return difference and, for systems with a single input field, the return difference is required to be at least equal to one at all spatial and temporal frequencies. We provide examples of localised distributed controllers that are inversely optimal with respect to physically appealing performance indices. A distinctive feature of these indices is the absence of fully decentralised state penalties that always seem to yield centralised optimal controllers. Our results indicate that a judicious selection of spatial weights in distributed optimal control problems can lead to controllers with favourable architectural properties.

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Appendix

A. Proof of Theorem 1

Necessity: Suppose there exists an exponentially stabilising control law $\mathcal{K}$ for system (1) that is optimal with respect to a performance index (3). Then, for each $\kappa \in \widehat{\mathbb{G}}$, there is a unique positive definite $\widehat{\mathcal{P}}_\kappa$ satisfying equation (5) which yields stabilising feedback gain for LQR problem (2), (4), $\widehat{\mathcal{K}}_\kappa = -\widehat{\mathcal{R}}_\kappa^{-1}\widehat{\mathcal{B}}_\kappa^*\widehat{\mathcal{P}}_\kappa$. Adding and subtracting $j\omega\widehat{\mathcal{P}}_\kappa$ to the ARE (5) results into

$$\widehat{\mathcal{P}}_\kappa(j\omega I - \widehat{\mathcal{A}}_\kappa) + (-j\omega I - \widehat{\mathcal{A}}_\kappa^*)\widehat{\mathcal{P}}_\kappa = \widehat{\mathcal{Q}}_\kappa - \widehat{\mathcal{P}}_\kappa\widehat{\mathcal{B}}_\kappa\widehat{\mathcal{R}}_\kappa^{-1}\widehat{\mathcal{B}}_\kappa^*\widehat{\mathcal{P}}_\kappa.$$

If we multiply the left-hand-size and the right-hand-side of the last equation by $\widehat{\mathcal{B}}_\kappa^*\widehat{\mathcal{G}}_\kappa^*(j\omega)$ and $\widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa$, respectively, we obtain

$$\widehat{\mathcal{B}}_\kappa^*\widehat{\mathcal{G}}_\kappa^*(j\omega)\widehat{\mathcal{P}}_\kappa\widehat{\mathcal{B}}_\kappa + \widehat{\mathcal{B}}_\kappa^*\widehat{\mathcal{P}}_\kappa\widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa = \widehat{\mathcal{B}}_\kappa^*\widehat{\mathcal{G}}_\kappa^*(j\omega)(\widehat{\mathcal{Q}}_\kappa - \widehat{\mathcal{P}}_\kappa\widehat{\mathcal{B}}_\kappa\widehat{\mathcal{R}}_\kappa^{-1}\widehat{\mathcal{B}}_\kappa^*\widehat{\mathcal{P}}_\kappa)\widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa.$$

From this expression, and definitions of optimal feedback gain $\widehat{\mathcal{K}}_\kappa$ and matrix $\widehat{\mathcal{H}}_\kappa(j\omega)$ (see equation (7)) it is easy to obtain return difference equality (8).

Sufficiency: Let $\mathcal{K}$ be a given exponentially stabilising control law and let return difference equality (8) hold. If $(\mathcal{A}, \mathcal{B})$ is exponentially controllable and $(\mathcal{Q}^{1/2}, \mathcal{A})$ is exponentially observable then there is a unique positive definite solution $\overline{\mathcal{P}}$ to corresponding ARE yielding an optimal control law $\overline{\mathcal{K}} = -\mathcal{R}^{-1}\mathcal{B}^*\overline{\mathcal{P}}$. Using equation (8), we will show that $\mathcal{K} = \overline{\mathcal{K}}$, which will establish sufficiency of return difference equality for optimality.

If we substitute expression for $\widehat{\mathcal{B}}_\kappa^* \widehat{\mathcal{G}}_\kappa^*(j\omega)\widehat{\mathcal{Q}}_\kappa \widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa$ from equation (8) to

$$\begin{aligned}
&\widehat{\mathcal{B}}_\kappa^* \widehat{\mathcal{G}}_\kappa^*(j\omega)\widehat{\widehat{\mathcal{P}}}_\kappa \widehat{\mathcal{B}}_\kappa + \widehat{\mathcal{B}}_\kappa^* \widehat{\widehat{\mathcal{P}}}_\kappa \widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa \\
&\quad = \widehat{\mathcal{B}}_\kappa^* \widehat{\mathcal{G}}_\kappa^*(j\omega)(\widehat{\mathcal{Q}}_\kappa - \widehat{\widehat{\mathcal{P}}}_\kappa \widehat{\mathcal{B}}_\kappa \widehat{\mathcal{R}}_\kappa^{-1} \widehat{\mathcal{B}}_\kappa^* \widehat{\widehat{\mathcal{P}}}_\kappa)\widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa,
\end{aligned} \tag{31}$$

we obtain

$$\widehat{\mathcal{H}}_\kappa^* \widehat{\mathcal{R}}_\kappa \widehat{\mathcal{H}}_\kappa = \widehat{\widehat{\mathcal{H}}}_\kappa^* \widehat{\mathcal{R}}_\kappa \widehat{\widehat{\mathcal{H}}}_\kappa,$$

where $\widehat{\widehat{\mathcal{H}}}_\kappa := I - \widehat{\widehat{\mathcal{K}}}_\kappa \widehat{\mathcal{G}}_\kappa(j\omega)\widehat{\mathcal{B}}_\kappa$. We note that equation (31) represents a direct consequence of corresponding ARE leading to optimal controller $\widehat{\widehat{\mathcal{K}}}_\kappa$. Now, using a polynomial matrix fraction description of $\widehat{\mathcal{H}}_\kappa$ and $\widehat{\widehat{\mathcal{H}}}_\kappa$, after some manipulations the last equation can be transformed to (Anderson and Moore, 1990)

$$\begin{aligned}
&\widehat{\mathcal{R}}_\kappa (I + (\widehat{\widehat{\mathcal{K}}}_\kappa - \widehat{\mathcal{K}}_\kappa)(j\omega I - \widehat{\mathcal{A}}_\kappa - \widehat{\mathcal{B}}_\kappa \widehat{\widehat{\mathcal{K}}}_\kappa)^{-1} \widehat{\mathcal{B}}_\kappa) \\
&\quad = (I - \widehat{\mathcal{B}}_\kappa^* (-j\omega I - \widehat{\mathcal{A}}_\kappa^* - \widehat{\mathcal{K}}_\kappa^* \widehat{\mathcal{B}}_\kappa^*)^{-1} (\widehat{\widehat{\mathcal{K}}}_\kappa^* - \widehat{\mathcal{K}}_\kappa^*))\widehat{\mathcal{R}}_\kappa.
\end{aligned}$$

At each κ, the left-hand-side in the last equation is a transfer function matrix with the poles in the open left-half of the complex plane, and the right-hand-side is a transfer function matrix with the poles in the open right-half of the complex plane. Thus, both the left and the right-hand sides are constant and this constant can be obtained by setting $\omega = \infty$, which results into:

$$\widehat{\mathcal{R}}_\kappa + (\widehat{\widehat{\mathcal{K}}}_\kappa - \widehat{\mathcal{K}}_\kappa)(j\omega I - \widehat{\mathcal{A}}_\kappa - \widehat{\mathcal{B}}_\kappa \widehat{\widehat{\mathcal{K}}}_\kappa)^{-1} \widehat{\mathcal{B}}_\kappa = \widehat{\mathcal{R}}_\kappa.$$

Owing to exponential controllability of $(\widehat{\mathcal{A}}_\kappa, \widehat{\mathcal{B}}_\kappa)$ we conclude $\widehat{\widehat{\mathcal{K}}}_\kappa = \widehat{\mathcal{K}}_\kappa$.