$V(\mathbf{x}(k_1)) \leq \beta \sum_{i\in\mathcal{V}} |x^* - \hat{x}_i(k_1)| \leq \beta \cdot N \cdot 2(N-1)\gamma^{-1}(\epsilon) < c$, which contradicts (20). Therefore, $c = 0$, i.e., (16) holds, implying that (3) is satisfied.

APPENDIX B
PROOF OF THEOREM 2

The proof is similar to that of Theorem 1. Let a, b, γ, and β be as defined in Appendix A. Then, due to (8), (18), (4), and Lemma 1, we have $\hat{x}_i(k) \in [a,b]\ \forall k \in \mathbb{N}\ \forall i \in \mathcal{V}$ and $x^* \in [a,b]$. From Lemma 3, $\lim_{k\to\infty} V(\mathbf{x}(k)) = c$ for some $c \geq 0$. To show that $c = 0$, assume to the contrary that $c > 0$ and let ϵ be as defined in Appendix A. Then, (20) holds for some $k_1 \in \mathbb{N}$. It follows from the proof of Lemma 3 that $f_i(\hat{x}_i(k)) - f_i(\hat{x}_i(k-1)) - f_i'(\hat{x}_i(k-1))(\hat{x}_i(k) - \hat{x}_i(k-1)) \leq V(\mathbf{x}(k-1)) - V(\mathbf{x}(k)) < \epsilon\ \forall k \geq k_1 + 1\ \forall i \in u(k)$. Thus, $|\hat{x}_i(k) - \hat{x}_i(k-1)| \leq \gamma^{-1}(\epsilon)\ \forall k \geq k_1 + 1\ \forall i \in u(k)$. This, along with (19) and the fact that $R \in \mathbb{P}$, implies $|\hat{x}_i(k) - \hat{x}_j(k)| \leq 2\gamma^{-1}(\epsilon)/(1 - (1/2^R)) \leq 4\gamma^{-1}(\epsilon)\ \forall k \geq k_1\ \forall i,j \in u(k+1)$. Then, using the same idea as in Appendix A, it can be shown that $\max_{i\in\mathcal{V}} \hat{x}_i(k_1) - \min_{i\in\mathcal{V}} \hat{x}_i(k_1) \leq 4(N-1)\gamma^{-1}(\epsilon)$. This leads to $V(\mathbf{x}(k_1)) < c$, which contradicts (20). Therefore, (16) and (3) hold.

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Augmented Lagrangian Approach to Design of Structured Optimal State Feedback Gains

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***Abstract*—We consider the design of optimal state feedback gains subject to structural constraints on the distributed controllers. These constraints are in the form of sparsity requirements for the feedback matrix, implying that each controller has access to information from only a limited number of subsystems. The minimizer of this constrained optimal control problem is sought using the augmented Lagrangian method. Notably, this approach does not require a stabilizing structured gain to initialize the optimization algorithm. Motivated by the structure of the necessary conditions for optimality of the augmented Lagrangian, we develop an alternating descent method to determine the structured optimal gain. We also utilize the sensitivity interpretation of the Lagrange multiplier to identify favorable communication architectures for structured optimal design. Examples are provided to illustrate the effectiveness of the developed method.**

***Index Terms*—Augmented Lagrangian, optimal distributed design, sparse matrices, structured feedback gains.**

I. INTRODUCTION

The design of distributed controllers for interconnected systems has received considerable attention in recent years [1]–[13]. For linear spatially invariant plants, it was shown in [1] that optimal controllers are themselves spatially invariant. Furthermore, for optimal distributed problems with quadratic performance indices the dependence of a controller on information coming from other parts of the system decays exponentially as one moves away from that controller [1]. These

Manuscript received February 09, 2011; revised May 20, 2011; accepted June 10, 2011. Date of publication June 20, 2011; date of current version December 07, 2011. This work was supported by the National Science Foundation under award CMMI-09-27720 and under CAREER Award CMMI-06-44793, and by the National Science Foundation under award CMMI-09-27509. Recommended by Associate Editor D. Arzelier.

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Color versions of one or more of the figures in this technical note are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2011.2160022

developments motivate the search for inherently *localized* controllers that communicate only to a subset of other controllers.

The main focus of this work is to search for an optimal distributed controller that is a static gain with *a priori* assigned structural constraints. The localized architectural requirements are formulated using matrix sparsity constraints. For example, for banded feedback gains, which are non-zero only on the main diagonal and a relatively small number of sub-diagonals, each controller uses information only from a limited number of neighboring subsystems. We search for structured controllers that minimize the H_2 norm and find a set of *coupled* algebraic matrix equations that characterize necessary conditions for the optimality.

The unstructured output feedback problem has been studied extensively since the original work of Levine and Athans [14]. Many computational methods have been proposed and, in general, they fall into two categories: i) the general-purpose minimization methods, which include Newton's method [15] and quasi-Newton method [16]; and ii) the special-purpose iterative methods [17], [18]. The advent of linear matrix inequality (LMI) has sparked renewed interest in fixed-order output feedback design [19]–[21]. Recently, nonsmooth optimization methods have been successfully employed for the design of the fixed-order H_∞ and H_2 controllers [22], [23]. HIFOO, a Matlab package for fixed-order controller design, provides an effective means for solving many benchmark problems [24], [25].

In this note we employ the augmented Lagrangian method to design structured optimal state feedback gains. This approach does not require knowledge of a stabilizing structured gain to initialize the algorithm. A sequence of unstructured problems is instead minimized and the resulting minimizers converge to the optimal structured gain. We note that the augmented Lagrangian method was previously used to design decentralized dynamic controllers [26] and fixed-order H_∞ controllers [27], [28]. In contrast to these papers, we utilize structure of the necessary conditions for optimality of the augmented Lagrangian to develop an alternating descent method to determine the structured optimal gain. Furthermore, we use sensitivity interpretation of the Lagrange multiplier to identify favorable architectures for performance improvement.

Our presentation is organized as follows. In Section II, we formulate the structured optimal state feedback problem, introduce the augmented Lagrangian approach, and demonstrate how sensitivity interpretation of Lagrange multiplier can be utilized to identify favorable sparsity patterns for performance improvement. In Section III, we develop an alternating descent method for the minimization of the augmented Lagrangian. In Section IV, we illustrate the effectiveness of the proposed approach via two examples. We summarize our developments and comment on future directions in Section V.

II. Problem Formulation and Augmented Lagrangian Method

Let a linear time-invariant system be given by its state-space representation

$$\dot{x} = A\,x + B_1\,d + B_2\,u,\qquad z = \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix} \tag{1}$$

where x is the state vector, d is the disturbance, u is the control input, and z is the performance output. All matrices are of appropriate dimensions, and $Q^{1/2}$ and $R^{1/2}$ denote the square-roots of the state and control performance weights. We consider the *structured* state feedback design problem

$$u = -Fx$$

where matrix F is subject to structural constraints that dictate the zero entries of F. For a mass-spring system in Fig. 1, if the controller acting on the ith mass has access to displacement and velocity of the ith mass and displacements of the two neighboring masses, then the feedback gain can be partitioned into $F = [\,F_p \;\; F_v\,]$ where F_p is a tridiagonal matrix and F_v is a diagonal matrix.

Fig. 1. Mass-spring system on a line.

For systems defined on general graphs the feedback matrix sparsity patterns can be more complex. Let the subspace $\mathcal{S}$ encapsulate these structural constraints and let us assume that there exists a non-empty set of stabilizing F that belongs to $\mathcal{S}$. Upon closing the loop, we have

$$\dot{x} = (A - B_2F)\,x + B_1\,d,\qquad z = \begin{bmatrix} Q^{1/2} \\ -R^{1/2}F \end{bmatrix} x,\quad F \in \mathcal{S}.$$

Our objective is to find $F \in \mathcal{S}$ that minimizes the H_2 norm of the transfer function from d to z. This structured optimal control problem can be formulated as

$$\begin{array}{ll} \text{minimize} & J(F) = \operatorname{trace}\left(B_1^T \displaystyle\int_0^\infty e^{(A-B_2F)^T t} \times (Q + F^TRF)\, e^{(A-B_2F)t}\,dt\,B_1\right) \\ \text{subject to} & F \in \mathcal{S}. \end{array} \tag{2}$$

For stabilizing F, the integral in (2) is bounded and it can be evaluated by solving the Lyapunov equation

$$(A - B_2F)^T P + P(A - B_2F) = -\left(Q + F^TRF\right) \tag{3}$$

thereby yielding $J(F) = \operatorname{trace}(B_1^T P(F) B_1)$.

The closed-loop H_2 norm of a stabilizable and detectable system increases to infinity as the least stable eigenvalue of $A_{\rm cl} := A - B_2F$ goes towards the imaginary axis. For marginally stable and unstable systems, we define the H_2 norm to be infinity, which is consistent with the integral in the definition of the H_2 norm (2). Furthermore, $J(F)$ is a smooth function of F, since it is a product of the exponential and polynomial functions of the feedback gain. Therefore, the closed-loop H_2 norm is a smooth function that increases to infinity as one approaches the boundary of the set of stabilizing feedback gains. However, in general, the H_2 norm of the closed-loop system is not convex in the feedback gain [29], that is, $J(F)$ is not a convex function of F. Moreover, the set of all stabilizing feedback gains is not a convex set. On the other hand, $\mathcal{S}$ defines a linear subspace and thus $F \in \mathcal{S}$ is a linear constraint on the feedback gain.

If a stabilizing $F \in \mathcal{S}$ is known, descent algorithms can be employed to determine a local minimum of (2). However, finding a structured stabilizing gain is in general a challenging problem. To alleviate this difficulty, we employ the augmented Lagrangian method in Section II-A. We then provide the sensitivity interpretation of the Lagrange multiplier in Section II-B and introduce an alternating method for the minimization of the augmented Lagrangian in Section III.

A. Augmented Lagrangian Method

The augmented Lagrangian method minimizes a sequence of *unstructured* problems such that the minimizers of the unstructured problems converge to the minimizer of (2). Therefore, the augmented Lagrangian method does not require a stabilizing *structured* feedback gain to initialize the optimization algorithm.

We begin by providing an algebraic characterization of the structural constraint $F \in \mathcal{S}$. Let $I_{\mathcal{S}}$ be the *structural identity* of the subspace $\mathcal{S}$ with its ijth entry defined as

$$[I_{\mathcal{S}}]_{ij} = \begin{cases} 1, & \text{if } F_{ij} \text{ is a free variable;} \\ 0, & \text{if } F_{ij} = 0 \text{ is required.} \end{cases}$$

If $I_{\mathcal{S}}^c := \mathbf{1} - I_{\mathcal{S}}$ denotes the structural identity of the *complementary* subspace $\mathcal{S}^c$, where $\mathbf{1}$ is the matrix with all its entries equal to one, then

$$F \in \mathcal{S} \Leftrightarrow F \circ I_{\mathcal{S}} = F \Leftrightarrow F \circ I_{\mathcal{S}}^c = 0$$

where $\circ$ denotes the entry-wise multiplication of matrices. Therefore, the structured H_2 optimal control problem (2) can be rewritten as

$$\begin{array}{ll} \text{minimize} & J(F) = \text{trace}\left(B_1^T P(F) B_1\right) \\ \text{subject to} & F \circ I_{\mathcal{S}}^c = 0 \end{array} \tag{SH2}$$

where $P(F)$ is the solution of (3).

The Lagrangian function for (SH2) is given by

$$\mathcal{L}(F,V) = J(F) + \text{trace}\left(V^T (F \circ I_{\mathcal{S}}^c)\right).$$

From Lagrange duality theory [30]–[32], it follows that there exists a unique Lagrange multiplier $V^\star \in \mathcal{S}^c$ such that the minimizer of $\mathcal{L}(F, V^\star)$ with respect to F is a local minimum of (SH2). The Lagrange dual approach minimizes $\mathcal{L}(F,V)$ with respect to *unstructured* F for fixed V (the estimate of $V^\star$), and then updates V such that it converges to the Lagrange multiplier $V^\star$. Consequently, as V converges to $V^\star$, the minimizer of $\mathcal{L}(F,V)$ with respect to F converges to the minimizer of (SH2). This Lagrange dual approach is most powerful for convex problems [32]; for nonconvex problems, it relies on local convexity assumptions [31] that may not be satisfied in problem (SH2).

In what follows, a quadratic term is introduced to locally convexify the Lagrangian [30], [31] yielding the augmented Lagrangian for (SH2)

$$\mathcal{L}_c(F,V) = J(F) + \text{trace}\left(V^T (F \circ I_{\mathcal{S}}^c)\right) + \left(\frac{c}{2}\right) \|F \circ I_{\mathcal{S}}^c\|^2$$

where the penalty weight c is a positive scalar and $\|\cdot\|$ is the Frobenius norm. Starting with an initial estimate of the Lagrange multiplier, e.g., $V^0 = 0$, the augmented Lagrangian method iterates between minimizing $\mathcal{L}_c(F, V^i)$ with respect to unstructured F (for fixed V^i) and updating

$$V^{i+1} = V^i + c\,(F^i \circ I_{\mathcal{S}}^c)$$

where F^i is the minimizer of $\mathcal{L}_c(F, V^i)$. Note that, by construction, V^i belongs to the complementary subspace $\mathcal{S}^c$, that is

$$V^i \circ I_{\mathcal{S}}^c = V^i.$$

It can be shown [30], [31] that the sequence $\{V^i\}$ converges to the Lagrange multiplier $V^\star$, and consequently, the sequence of the minimizers $\{F^i\}$ converges to the structured optimal feedback gain $F^\star$.

Augmented Lagrangian method for (SH2):

Let $V^0 = 0$ and $c^0 > 0$, **for** $i = 0, 1, \ldots,$ **do**
(1) for fixed V^i, minimize $\mathcal{L}_c(F, V^i)$ with respect to unstructured F (see Section III);
(2) update $V^{i+1} = V^i + c^i\,(F^i \circ I_{\mathcal{S}}^c)$;
(3) update $c^{i+1} = \gamma\, c^i$ with $\gamma > 1$;
until: the stopping criterion $\|F^i \circ I_{\mathcal{S}}^c\| < \epsilon$ is reached.

The convergence rate of the augmented Lagrangian method depends heavily on the penalty weight c. In general, large c results in fast convergence rate. However, large values of c may introduce computational difficulty in minimizing the augmented Lagrangian. This is because the condition number of the Hessian matrix $\nabla^2 \mathcal{L}_c(F,V)$ becomes larger as c increases. It is thus recommended [30] to increase the penalty weight gradually until it reaches a certain threshold value τ. Our numerical experiments suggest that $c^0 \in [1, 5]$, $\gamma \in [3, 10]$, and $\tau \in [10^4, 10^6]$ work well in practice. Additional guidelines for choosing these parameters can be found in [30, Section 4.2].

B. Sensitivity Interpretation

It is a standard fact that the Lagrange multiplier provides useful information about the sensitivity of the optimal value with respect to the perturbations of the constraints [30]–[32]. In particular, for the structured design problem, the Lagrange multiplier indicates how sensitive the optimal H_2 norm is with respect to the change of the structural constraints. We use this sensitivity interpretation to identify favorable sparsity patterns for improving H_2 performance.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of matrices $\langle M_1, M_2 \rangle = \text{trace}\,(M_1^T M_2)$. It is readily verified that $\|F \circ I_{\mathcal{S}}^c\|^2 = \langle F \circ I_{\mathcal{S}}^c, F \circ I_{\mathcal{S}}^c \rangle = \langle F \circ I_{\mathcal{S}}^c, F \rangle$ and $\langle V, F \circ I_{\mathcal{S}}^c \rangle = \langle V \circ I_{\mathcal{S}}^c, F \rangle = \langle V, F \rangle$ where we used the fact that $V \circ I_{\mathcal{S}}^c = V$. Thus the augmented Lagrangian can be rewritten as

$$\mathcal{L}_c(F,V) = J(F) + \langle V, F \rangle + \left(\frac{c}{2}\right) \langle F \circ I_{\mathcal{S}}^c, F \rangle$$

and its gradient with respect to F is given by

$$\nabla \mathcal{L}_c(F,V) = \nabla J(F) + V + c\,(F \circ I_{\mathcal{S}}^c).$$

Since the minimizer $F^\star$ of $\mathcal{L}_c(F, V^\star)$ satisfies $\nabla \mathcal{L}_c(F^\star, V^\star) = 0$ and $F^\star \circ I_{\mathcal{S}}^c = 0$, we have

$$\nabla J(F^\star) + V^\star = 0.$$

Let the structural constraints $\{F_{ij} = 0,\, (i,j) \in \mathcal{S}^c\}$ be relaxed to $\{|F_{ij}| \le w,\, (i,j) \in \mathcal{S}^c\}$ with $w > 0$, and let $\hat{F}$ be the minimizer of

$$\begin{array}{ll} \text{minimize} & J(F) \\ \text{subject to} & \{|F_{ij}| \le w,\ (i,j) \in \mathcal{S}^c\}. \end{array} \tag{RH2}$$

Since the constraint set in (RH2) contains the constraint set in (SH2), $J(\hat{F})$ is smaller than or equal to $J(F^\star)$

$$J(\hat{F}) := J(F^\star + \tilde{F}^\star) \le J(F^\star) \tag{4}$$

where $\tilde{F}^\star$ denotes the difference between $\hat{F}$ and $F^\star$. Now, the Taylor series expansion of $J(F^\star + \tilde{F}^\star)$ around $F^\star$ in conjunction with (4) yields

$$\begin{aligned} J(F^\star) - J(F^\star + \tilde{F}^\star) &= -\langle \nabla J(F^\star), \tilde{F}^\star \rangle + O(\|\tilde{F}^\star\|^2) \\ &= \langle V^\star, \tilde{F}^\star \rangle + O(\|\tilde{F}^\star\|^2) \ge 0. \end{aligned}$$

Furthermore

$$\begin{aligned} \langle V^\star, \tilde{F}^\star \rangle &\le \sum_{i,j} |V_{ij}^\star||\tilde{F}_{ij}^\star| \\ &= \sum_{(i,j)\in\mathcal{S}} |V_{ij}^\star||\tilde{F}_{ij}^\star| + \sum_{(i,j)\in\mathcal{S}^c} |V_{ij}^\star||\tilde{F}_{ij}^\star| \\ &\le w \sum_{(i,j)\in\mathcal{S}^c} |V_{ij}^\star| \end{aligned}$$

where we have used the fact that $V_{ij}^\star = 0$ for $(i,j) \in \mathcal{S}$ and $|\tilde{F}_{ij}^\star| \leq w$ for $(i,j) \in \mathcal{S}^c$. Thus, up to the first order in $\tilde{F}^\star$, we have

$$J(F^\star) - J(F^\star + \tilde{F}^\star) \leq w \sum_{(i,j)\in\mathcal{S}^c} |V_{ij}^\star|.$$

Note that larger $|V_{ij}^\star|$ indicates larger decrease in the H_2 norm if the corresponding constraint $F_{ij} = 0$ is relaxed. This sensitivity interpretation can be utilized to identify favorable controller architectures; see Section IV-B for an illustrative example.

III. Alternating Method for Minimization of Augmented Lagrangian

In this section, we develop an alternating iterative method for minimization of the augmented Lagrangian. This method is motivated by the structure encountered in the necessary conditions for optimality (NC-L), (NC-P), and (NC-F) given below. We note that Newton's method, which is well-suited for dealing with ill-conditioning in $\mathcal{L}_c$ for large values of c [30], can also be employed to minimize the augmented Lagrangian.

Using standard techniques [14], [16], we obtain the expression for the gradient of $\mathcal{L}_c(F)$[1]

$$\begin{aligned}\nabla\mathcal{L}_c(F) &= \nabla J(F) + V + c\,(F \circ I_\mathcal{S}^c)\\ &= 2(RF - B_2^T P)L + V + c\,(F \circ I_\mathcal{S}^c).\end{aligned}$$

Here, L and P are the controllability and observability Gramians of the closed-loop system

$$(A - B_2F)L + L(A - B_2F)^T = -B_1B_1^T, \qquad \text{(NC-L)}$$

$$(A - B_2F)^T P + P(A - B_2F) = -(Q + F^T RF), \qquad \text{(NC-P)}$$

and the necessary condition for optimality of $\mathcal{L}_c(F)$ is given by

$$2(RF - B_2^T P)L + V + c\,(F \circ I_\mathcal{S}^c) = 0. \qquad \text{(NC-F)}$$

Solving the system of equations (NC-L), (NC-P), and (NC-F) is a non-trivial task. In the absence of structural constraints, setting $\nabla J(F) = 2(RF - B_2^T P)L = 0$ yields the optimal unstructured feedback gain

$$F_c = R^{-1}B_2^T P$$

where the pair $(A - B_2F, B_1)$ is assumed to be controllable and therefore L is invertible. Here, P is the positive definite solution of the algebraic Riccati equation obtained by substituting F_c in (NC-P)

$$A^T P + PA + Q - PB_2R^{-1}B_2^T P = 0.$$

Starting with $F = F_c$, we can solve Lyapunov equations (NC-L) and (NC-P), and then solve (NC-F) to obtain a new feedback gain $\bar{F}$. We can thus alternate between solving (NC-L), (NC-P) and solving (NC-F).

In Proposition 1, we show that the difference between two consecutive steps $\bar{F} - F$ is a descent direction of $\mathcal{L}_c(F)$. Therefore, we can employ the Armijo rule to choose the step-size s in $F + s(\bar{F} - F)$ such that the alternating method converges to a stationary point of $\mathcal{L}_c(F)$. By virtue of the fact that the augmented Lagrangian $\mathcal{L}_c(F)$ is locally convex [30], [31], the stationary point indeed provides a local minimum of $\mathcal{L}_c(F)$. We then update V and c in the augmented Lagrangian (see Section II-A for details), and use the minimizer of $\mathcal{L}_c(F)$ to initialize another round of the alternating descent iterations. As V converges to $V^\star$, the minimizer of $\mathcal{L}_c(F)$ converges to $F^\star$. Therefore, the augmented Lagrangian method traces a solution path (parameterized by V and c) between the unstructured optimal gain F_c and the structured optimal gain $F^\star$. Here, we assume that F_c is contained in a connected set of stabilizing feedback gains that has a non-empty intersection with the subspace $\mathcal{S}$.

We summarize this approach in the following algorithm.

Alternating method to minimize augmented Lagrangian $\mathcal{L}_c(F, V^i)$

For $V^0 = 0$, start with the optimal unstructured feedback gain F_c;
For V^i with $i \geq 1$, start with the minimizer of $\mathcal{L}_c(F, V^{i-1})$;
for $k = 0, 1, \ldots,$ **do**
(1) solve Lyapunov equations (NC-L) and (NC-P) with $F = F_k$ to obtain L_k and P_k;
(2) solve linear equation (NC-F) with $L = L_k$ and $P = P_k$ to obtain $\bar{F}_k$;
(3) update $F_{k+1} = F_k + s_k(\bar{F}_k - F_k)$ where s_k is determined by Armijo rule;
until: The stopping criterion $\|\nabla\mathcal{L}_c(F_k)\| < \epsilon$ is reached.
Armijo rule [30, Section 1.2] for step-size s_k:
Let $s_k = 1$, repeat $s_k = \beta s_k$
until

$$\mathcal{L}_c(F_k + s_k(\bar{F}_k - F_k)) < \mathcal{L}_c(F_k) + \alpha\, s_k \,\langle \nabla\mathcal{L}_c(F_k), \bar{F}_k - F_k\rangle$$

where $\alpha, \beta \in (0, 1)$, e.g., $\alpha = 0.3$ and $\beta = 0.5$.

The descent property of $\bar{F}_k - F_k$ established in Proposition 1 , continuity of $\bar{F}_k$ with respect to F_k, and the step-size selection using the Armijo rule guarantee the convergence of the alternating method [30]. Furthermore, for F_k sufficiently close to the local minimum we have established the linear convergence rate of the alternating method; due to page limitations these convergence rate results will be reported elsewhere.

We conclude this section by establishing the descent property of the difference between two consecutive steps in the alternating method, $\bar{F} - F$.

Proposition 1: The difference between two consecutive steps, $\tilde{F} := \bar{F} - F$, is a descent direction of the augmented Lagrangian, $\langle \nabla\mathcal{L}_c(F), \tilde{F}\rangle < 0$. Moreover, $\langle \nabla\mathcal{L}_c(F), \tilde{F}\rangle = 0$ if and only if F is a stationary point of $\mathcal{L}_c(F)$, that is, $\nabla\mathcal{L}_c(F) = 0$.

Proof: Substituting $\bar{F} = F + \tilde{F}$ in (NC-F) yields

$$2R\tilde{F}L + c\,(\tilde{F} \circ I_\mathcal{S}^c) + \nabla\mathcal{L}_c(F) = 0. \tag{5}$$

Since R and L are positive definite matrices, we have

$$\langle \nabla\mathcal{L}_c(F), \tilde{F}\rangle = -2\,\langle R\tilde{F}L, \tilde{F}\rangle - c\,\langle \tilde{F} \circ I_\mathcal{S}^c, \tilde{F} \circ I_\mathcal{S}^c\rangle \leq 0. \tag{6}$$

We next show that the equality is achieved if and only if F is a stationary point, that is

$$\langle \nabla\mathcal{L}_c(F), \tilde{F}\rangle = 0 \Leftrightarrow \nabla\mathcal{L}_c(F) = 0.$$

The necessity is immediate and the sufficiency follows from two facts: (i) equality in (6) implies $\tilde{F} = 0$, that is

$$-2\,\langle R\tilde{F}L, \tilde{F}\rangle - c\,\langle \tilde{F} \circ I_\mathcal{S}^c, \tilde{F} \circ I_\mathcal{S}^c\rangle = 0 \Rightarrow \tilde{F} = 0$$

[1]Since V is fixed in minimizing $\mathcal{L}_c(F, V)$, we will use $\mathcal{L}_c(F)$ to denote the augmented Lagrangian.

TABLE I

MASS-SPRING SYSTEM WITH $n = 100$ MASSES: p DETERMINES THE SPATIAL SPREAD OF THE DISTRIBUTED CONTROLLER, ALT# IS THE NUMBER OF ALTERNATING STEPS

p	ALT#	$J(F^\star)$	$J(F_c \circ I_{\mathcal{S}})$
0	92	499.9	546.5
1	83	491.2	497.2
2	71	488.0	489.6
3	70	486.8	487.6

and (ii) setting $\tilde{F} = 0$ in (5) yields $\nabla \mathcal{L}_c(F) = 0$. This completes the proof.

Remark: If R is a diagonal matrix, we can write the jth row of (5) as

$$\tilde{F}_j \left(2R_{jj} L + c\, \mathrm{diag}\,\{I^c_{\mathcal{S}\,j}\} \right) + \nabla \mathcal{L}_c(F)_j = 0$$

where $(\cdot)_j$ denotes the jth row of a matrix and $\mathrm{diag}\,\{I^c_{\mathcal{S}\,j}\}$ is a diagonal matrix with $I^c_{\mathcal{S}\,j}$ on its main diagonal. Therefore each row of $\tilde{F}$ can be computed independently.

IV. EXAMPLES

We next demonstrate the utility of the augmented Lagrangian approach in the design of optimal structured controllers. The mass-spring system in Section IV-A illustrates the efficiency of the augmented Lagrangian method, and the vehicle formation example in Section IV-B illustrates the effectiveness of the Lagrange multiplier in identifying favorable controller architectures for improving H_2 performance.

A. Mass-Spring System

We consider a mass-spring system with unit masses and unit spring constants shown in Fig. 1. If restoring forces are considered as linear functions of displacements, the state-space representation of this system is given by (1) with

$$A = \begin{bmatrix} O & I \\ T & O \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} O \\ I \end{bmatrix}$$

where I and O are $n \times n$ identity and zero matrices, and T is an $n \times n$ symmetric Toeplitz matrix with the first row given by $[-2\ 1\ 0\ \cdots\ 0] \in \mathbb{R}^n$. The state and control weights are assigned to be $Q = I$ and $R = 10I$.

We consider a situation in which the control applied to the ith mass has access to displacement and velocity of the ith mass, and displacements of p neighboring masses on the left and p neighboring masses on the right. Thus, $I_{\mathcal{S}} = [\,S_p \quad I\,]$ where S_p is a banded matrix with ones on p upper and p lower sub-diagonals. For $n = 100$ masses with $p = 0, 1, 2, 3$, the results are summarized in Table I. Here, the stopping criterion for the augmented Lagrangian method is $\|F \circ I^c_{\mathcal{S}}\| < 10^{-6}$, and the stopping criterion for the alternating method is $\|\nabla \mathcal{L}_c(F)\| < 10^{-3}$.

We note that as the spatial spread p of the distributed controller increases (i) the improvement of $J(F^\star)$ becomes less significant; and (ii) $J(F_c \circ I_{\mathcal{S}}) \approx J(F^\star)$, i.e., near optimal performance can be achieved by the truncated optimal unstructured controller $F_c \circ I_{\mathcal{S}}$. These observations are consistent with the spatially decaying property of the optimal unstructured controller on the information from neighboring subsystems [1], [10].

B. Formation of Vehicles

We consider a formation of nine vehicles in a plane. The control objective is to keep constant distances between neighboring vehicles.

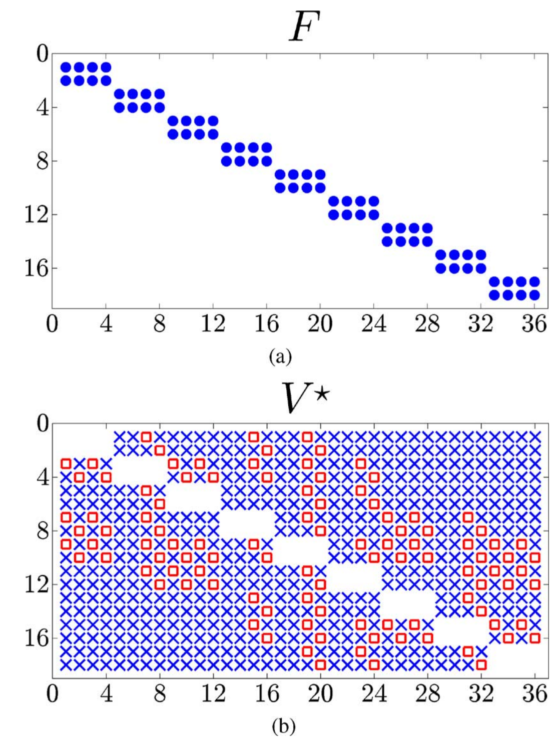

Fig. 2. (a) Block diagonal feedback gain F where each block signifies that the two control inputs acting on each vehicle only have access to the four states of that vehicle; (b) Lagrange multiplier $V^\star$ with entries separated into groups small ($\times$) and large ($\Box$) according to (7).

Modeling these independently actuated vehicles as double-integrators, in both horizontal and vertical directions, yields the state-space representation (1) with

$$A = \mathrm{diag}\{A_i\}, \quad B_1 = \mathrm{diag}\{B_{1i}\}, \quad B_2 = \mathrm{diag}\{B_{2i}\},$$
$$A_i = \begin{bmatrix} O_2 & I_2 \\ O_2 & O_2 \end{bmatrix}, \ B_{1i} = B_{2i} = \begin{bmatrix} O_2 \\ I_2 \end{bmatrix}, \ i = 1, \ldots, 9$$

where I_2 and O_2 are 2×2 identity and zero matrices. The control weight R is set to identity, and the state weight Q is obtained by penalizing both the absolute and the relative position errors

$$x^T Q\, x = \sum_{i=1}^{9} \left(p_{1i}^2 + p_{2i}^2 + 10 \sum_{j \in \mathcal{N}_i} \left((p_{1i} - p_{1j})^2 + (p_{2i} - p_{2j})^2 \right) \right)$$

where p_{1i} and p_{2i} are the absolute position errors of the ith vehicle in the horizontal and vertical directions, respectively, and set $\mathcal{N}_i$ determines neighbors of the ith vehicle.

The decentralized control architecture with no communication between vehicles specifies the block diagonal structure $\mathcal{S}_d$; see Fig. 2(a). We solve (SH2) for $F \in \mathcal{S}_d$ and obtain the Lagrange multiplier $V^\star \in \mathcal{S}_d^c$; see Fig. 2(b). Let

$$V_{ij}^\star \ \text{ be in group } \begin{cases} \text{small}, & \text{if } 0 < |V_{ij}^\star| \leq 0.5\, V_M, \\ \text{large}, & \text{if } |V_{ij}^\star| > 0.5\, V_M \end{cases} \tag{7}$$

where V_M is the maximum absolute value of the entries of $V^\star$. We solve (SH2) for $F \in \mathcal{S}_s$ or $F \in \mathcal{S}_l$, where $\mathcal{S}_s$ and $\mathcal{S}_l$ are the subspaces obtained from removing the constraints $\{F_{ij} = 0\}$ corresponding to $\{V_{ij}^\star\}$ in groups small and large, respectively. We also consider the performance of the optimal controller in the unstructured subspace $\mathcal{S}_u$ with no constraints on F.

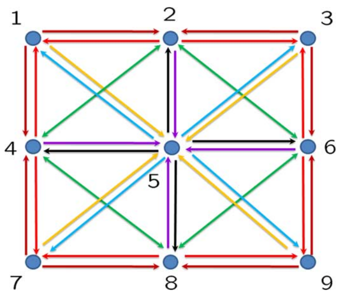

Fig. 3. Localized controller architecture in which each vehicle communicates only with its neighbors. The arrow directed from node i to node j indicates that node i is sending information to node j. Priority order of communication channels is determined by the absolute values of $V_{ij}^{\star}$, ranging from the highest to the lowest: brown, red, orange, green, blue, purple, and black.

TABLE II
PERFORMANCE IMPROVEMENT, $\kappa = (J_d^{\star} - J^{\star})/J_d^{\star}$, RELATIVE TO THE OPTIMAL H_2 NORM $J_d^{\star} = 65.4154$ WITH DECENTRALIZED STRUCTURE $\mathcal{S}_d$. HERE, ρ IS THE NUMBER OF EXTRA VARIABLES IN $\mathcal{S}_s$, $\mathcal{S}_l$, AND $\mathcal{S}_u$ COMPARED WITH $\mathcal{S}_d$, AND κ/ρ IS THE PERFORMANCE IMPROVEMENT PER VARIABLE

	$J^{\star}$	κ	ρ	κ/ρ
$\mathcal{S}_s$	64.1408	1.95%	472	0.0041%
$\mathcal{S}_l$	64.2112	1.84%	104	0.0177%
$\mathcal{S}_u$	62.1183	5.04%	576	0.0088%

Table II shows the influence of the number of optimization variables on the performance improvement. Note that $\mathcal{S}_l$ has the largest improvement *per variable* among all three structures $\mathcal{S}_s$, $\mathcal{S}_l$, and $\mathcal{S}_u$. As illustrated in Fig. 3, $\mathcal{S}_l$ determines a *localized* communication architecture in which each vehicle communicates only with its neighbors. Therefore, the Lagrange multiplier $V^{\star}$ identifies distributed controller with nearest neighbor interactions as the favorable controller architecture. This is in agreement with [10] where it was shown that optimal unstructured controllers for systems on general graphs possess spatially decaying property; similar result was proved earlier for spatially invariant systems [1].

V. Conclusion

In this note, we consider the design of structured optimal state feedback gains for interconnected systems. We employ the augmented Lagrangian method and utilize the sensitivity interpretation of the Lagrange multiplier to identify favorable communication architectures for structured optimal design. The necessary conditions for optimality of the augmented Lagrangian are given by coupled matrix equations. Motivated by the structure of these equations, we develop an alternating descent method for obtaining the optimal feedback gain. The proposed approach does not require a stabilizing structured controller to initialize the iterative procedure and its utility is illustrated by two examples.

Although we focus on structural equality constraints, we note that it is also possible to incorporate inequality constraints, e.g., $|F_{ij}| \leq w_{ij}$, in the augmented Lagrangian method [30], [31]. This extension is expected to be useful in applications where controller saturations or limited communication budgets are incorporated in the design.

In our ongoing efforts, we are applying the tools developed here to the control of vehicular formations [33], [34], and to the design of consensus-type algorithms for optimal performance over general connected networks. These problems have received considerable attention in recent years but a systematic procedure for the design of optimal localized controllers is yet to be developed. The algorithms developed here will also be useful in analyzing the scaling of different performance measures with respect to the network size [34]. Such analysis will provide insight into the fundamental limitations of the performance achievable using localized control strategies with relative information exchange in systems with arbitrary communication topologies.

Acknowledgment

The authors would like to thank anonymous reviewers and the associate editor for their valuable comments.

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Switched Affine Systems Using Sampled-Data Controllers: Robust and Guaranteed Stabilization

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***Abstract*—The problem of robust and guaranteed stabilization is addressed for switched *affine* systems using sampled state feedback controllers. Based on the existence of a control Lyapunov function for a relaxed system, we propose three sampled-data controls. Global attracting sets, computed by solving a sequence of optimization problems, guarantee practical and global asymptotic stabilization for the whole system trajectories. In addition, robust margins with respect to parameters uncertainties and non uniform sampling are provided using input-to-state stability. Finally, a buck-boost converter is considered to illustrate the effectiveness of the proposed approaches.**

***Index Terms*—Input-to-state stability, robust control, stabilization of hybrid systems, switched systems.**

I. Introduction

Most of the results related to the stabilization of switched systems deal with subsystems sharing zero as common equilibrium. In this technical note, we treat the case of *affine* switched systems for which generally no common equilibrium can be defined. In this context, the referred targets, named operating points, are defined as the equilibria of a relaxed system – obtained by relaxing the control domain to its convex hull. The control goal is then to steer, in average value, the state variables to these targets. Advanced control methods based on Lyapunov functions [1], sliding modes [2], optimal [3], [4] and predictive controls [5], [6] have been extensively proposed. In [7], in the case of pulsewidth modulated systems with no common equilibrium, a systematic method for stability analysis is provided. Sufficient conditions for global and local exponential stability are stated in terms of matrix inequalities. Nevertheless, there are few results concerning the estimation of the attracting set. In [8] and into a discrete time framework, a positively invariant set [9] formed by the union of bounded ellipsoids is determined and used in a predictive control algorithm to steer the state inside. However, the method uses a LMI formulation to compute these ellipsoids which introduces some conservatism in the result. Indeed, LMIs imply that the switched system possesses a switching sequence S of a prescribed length for which a property of uniform stability w.r.t. the initial condition is satisfied. So, the computed invariants are not particularly tight around the target.

In this technical note, assuming that a continuous time Control Lyapunov Function (CLF) is known for a relaxed system, robust stability for different sampled switched strategies is investigated. Precisely, we prove that tight positive invariant sets around the targets can be obtained by solving optimization problems. The global and practical asymptotic stabilization is thus guaranteed. The robustness aspects of the proposed sampled switched strategies in case of non uniform sampling and parameter uncertainties are also studied and discussed.

The technical note is organized as follows. The system description is given and the operating points are defined in Section II. In Section III, we propose three sampled-data controls for the switched system. In Section IV, a set of optimization problems is also formulated and we prove that the solutions allow to define global attracting sets (see [9], for definition) for the sampled switched affine system. An extension of those results in the case of parameter uncertainties and non-uniform sampling is given in Section V. The computational aspects are addressed in Section VI. A buck-boost converter is used in Section VII as illustration.

Notation

$\mathbb{N}^*$ denotes the set of strictly positive natural numbers and $\mathbb{N}_{\leq a}$, the set $\{k \in \mathbb{N} \mid k \leq a\}$.

II. System Description

In this technical note, the class of *affine* systems is considered

$$\dot{x}(t) = A_0 x(t) + B_0 + \sum_{i=1}^{m} u^i(t)(A_i x(t) + B_i) \tag{1}$$

where $u^i(t), i = 1, \ldots, m$ are component values of the control u and $x(t) \in \mathbb{R}^n$ represents the state value at time t. A_i and B_i are real matrices of appropriate dimensions. In the sequel, from (1), two systems are distinguished by their control set: Switched System (SS) when $u(t) \in U = \{0,1\}^m$ and Relaxed System (RS) when $u(t) \in \mathrm{co}(U) = [0,1]^m$ where $\mathrm{co}(U)$ stands for the convex hull.

(SS) belongs to the class of nonsmooth systems for which the notion of solution can be properly defined and generalized in the sense given by Fillipov [10]. The link between the solutions of (SS) and (RS) is established by a density theorem in infinite time in ([11], Theorem 1). This theorem guarantees that switching laws $u \in L^\infty([0,+\infty), U)$ (where L^∞ denotes the Banach space of all essentially bounded measurable functions) exist such that the trajectory of (RS) can be approached as close as desired by the one of (SS). For this reason, the

Manuscript received February 14, 2011; revised March 17, 2011; accepted June 16, 2011. Date of publication June 23, 2011; date of current version December 07, 2011. This work was supported in part by ANR project ArHyCo, Programme "Systèmes Embarqués et Grandes Infrastructures" under Contract ARPEGE ANR-2008 SEGI 004 01-30011459, by the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement n° 257462: HYCON2 Network of Excellence "Highly-Complex and Networked Control Systems," and from Contrat de Projet État-Région 2007–2013, Région Lorraine. Recommended by Associate Editor J. H. Braslavsky.

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Digital Object Identifier 10.1109/TAC.2011.2160598