2015 American Control Conference
Palmer House Hilton
July 1-3, 2015. Chicago, IL, USA

An interior point method for growing connected resistive networks

Sepideh Hassan Moghaddam and Mihailo R. Jovanović

***Abstract*— We consider the problem of growing connected networks of resistors where effective resistance is used as a performance metric. This problem can be cast as a semidefinite program by introducing an ℓ_1 regularization into the optimal control formulation. For small networks this problem can be solved via standard interior point method solvers (e.g., SeDuMi or SDPT3). In this paper, we develop a primal-dual interior point algorithm that is well-suited for large-scale problems. The search direction is obtained using the direct method based on Cholesky factorization and iterative method based on the preconditioned conjugate gradient. We illustrate that both of these significantly outperform general-purpose solvers.**

***Index Terms*— Convex optimization, interior-point method, ℓ_1 minimization, resistive networks, semidefinite programming.**

I. Introduction

Reaching consensus via distributed information exchange has emerged as an important paradigm in network science [1]. This problem is encountered in a number of applications ranging from social networks where a group of individuals is trying to agree on a certain issue [2], [3], to distributed computing networks where it is desired to evenly spread workload over a number of processors [4], [5], to cooperative control where local interactions between the vehicles are to be used in order to reach an agreement on heading direction or inter-vehicular spacing [6]–[9]. In each of these applications, it is of interest to reach an agreement by exchanging relative information between the nodes. Conventional optimal control of distributed systems relies on centralized implementation of control policies [10]. In large networks of dynamical systems, centralized information processing may impose a heavy burden on individual nodes. This motivates the development of distributed control strategies that require limited information exchange between the nodes in order to reach consensus or guarantee synchronization [11]–[15].

In this paper, we consider an optimal control problem in which it is desired to add certain number of edges to a connected resistive network (with known graph Laplacian) in order to optimally enhance performance of the closed-loop network. In general, this problem amounts to an intractable combinatorial search. Several references have examined convex relaxations or greedy algorithms in order to optimize algebraic connectivity of the network by adding edges from a given set of edges [16], [17]. To avoid combinatorial complexity, we approach this problem using recently introduced sparsity-promoting optimal control framework [18], [19]. In our formulation, the network performance is captured by the effective resistance [12] and ℓ_1 regularization is introduced in order to promote controller sparsity [14], [20]. The resulting optimal control problem is a convex optimization problem, which can be formulated as a semidefinite program (SDP). We derive the dual form of the optimal control problem which provides a lower bound on the primal objective value. We also develop a primal-dual interior point algorithm that is significantly faster than general purpose solvers. We compute the search direction using both a direct method (based on Cholesky factorization) and an iterative inexact method (based on the preconditioned conjugate gradient).

Notation: Notation is standard. We study undirected stochastically forced consensus network with n nodes. Symmetric $n\times n$ matrices L_p and L_f represent Laplacian matrices of the plant and the controller, respectively. The $n \times m$ matrix E is the incidence matrix of the controller graph. The transpose of the vector x is given by x^T and the ith component of the vector x is x_i; $D_x := \mathrm{diag}\,(x)$ is a diagonal matrix with diagonal entries determined by the vector x. The diagonal of a matrix A, $\mathrm{diag}\,(A)$, is a vector. Symmetric positive definiteness (semi-definiteness) is expressed by $A \succ 0$ ($A \succeq 0$), and elementwise inequality is denoted by $x \geq 0$. Finally, $\langle A, B\rangle$ represents the standard inner product of two matrices, i.e., $\langle A, B\rangle := \mathrm{trace}\left(A^T B\right)$.

II. Problem formulation

We consider a control problem for an undirected consensus network with n nodes

$$\begin{aligned} \dot{x} &= -L_p\, x \,+\, u \,+\, d \\ z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x \,+\, \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\ u &= -L_f\, x. \end{aligned} \tag{1}$$

Here, d and z denote disturbance input and performance output, x is the state of the network, and u is the control input. Symmetric $n \times n$ matrices L_p and L_f represent Laplacians of the plant and the controller, respectively. Upon closing the loop we obtain

$$\begin{aligned} \dot{x} &= -\left(L_p \,+\, L_f\right) x \,+\, d \\ z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2} L_f \end{bmatrix} x. \end{aligned} \tag{2}$$

Our objective is to design sparse L_f in order to minimize the steady-state variance amplification of stochastically forced

Financial support from the University of Minnesota Informatics Institute Transdisciplinary Faculty Fellowship and the National Science Foundation under award ECCS-1407958 is gratefully acknowledged.

Sepideh Hassan Moghaddam and Mihailo R. Jovanović are with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455. E-mails: hassa247@umn.edu, mihailo@umn.edu.

network (from white-in-time input d to performance output z that quantifies deviation from consensus and control effort).

The interesting features of this problem come from structural restrictions on the matrices L_p, L_f, and Q. All of them are restricted to having an eigenvalue at zero with the corresponding eigenvector of all ones,

$$\begin{aligned} L_p\,\mathbb{1} &= 0, \quad L_p &&= L_p^T \\ L_f\,\mathbb{1} &= 0, \quad L_f &&= L_f^T \\ Q\,\mathbb{1} &= 0, \quad Q &&= Q^T. \end{aligned} \tag{3}$$

Moreover, we consider state weights that are positive definite on the orthogonal complement of the subspace spanned by the vector of all ones,

$$\bar{Q} \;:=\; Q \,+\, (1/n)\,\mathbb{1}\mathbb{1}^T \;\succ\; 0.$$

In what follows, we express L_f using incidence matrix E of the controller graph

$$L_f \;:=\; \sum_{l\,=\,1}^{m} f_l\,\mathrm{e}_l\,\mathrm{e}_l^T \;=\; E\,\mathrm{diag}\,(f)\,E^T. \tag{4}$$

Here, m is the number of edges in the controller graph L_f, and $\mathrm{diag}\,(f)$ is a diagonal matrix containing vector of edge weights $f \in \mathbb{R}^m$ of the controller graph. Vectors $\mathrm{e}_l \in \mathbb{R}^n$ determine columns of the incidence matrix E and they signify that nodes i and j are connected: the ith and jth entries of e_l are equal to 1 and -1, respectively; all other entries are equal to 0.

In order to achieve consensus, it is required that the closed-loop graph Laplacian, $L_p + L_f$, be positive definite on $\mathbb{1}^\perp$. This requirement amounts to positive definiteness of the "strengthened" graph Laplacian

$$\begin{aligned} G \;&:=\; L_p \,+\, L_f \,+\, (1/n)\,\mathbb{1}\mathbb{1}^T \\ &=\; G_p \,+\, L_f \;\succ\; 0 \end{aligned} \tag{5a}$$

where

$$G_p \;:=\; L_p \,+\, (1/n)\mathbb{1}\mathbb{1}^T. \tag{5b}$$

Structural restrictions (3) on the Laplacian matrices introduce an additional constraint on G,

$$G\,\mathbb{1} \;=\; \mathbb{1}. \tag{5c}$$

The variance amplification of the closed-loop system (2) is determined by the H_2 norm of the transfer function from d to z,

$$\|H\|_2^2 \;=\; \frac{1}{2}\,\big\langle G^{-1}, Q + L_f\,R\,L_f\big\rangle \;=:\; \frac{1}{2}\,J(G,f). \tag{6}$$

As shown in Appendix A, up to an additive constants, the objective function J can be expressed as

$$J(G,f) \;=\; \big\langle G^{-1}, Q_p\big\rangle \,+\, \mathrm{diag}\,\big(E^T R\,E\big)^T f \tag{7}$$

where

$$Q_p \;:=\; \bar{Q} \,+\, L_p\,R\,L_p.$$

The problem of designing sparse controller graph in order to minimize the H_2 norm of the closed-loop network can be formulated as

$$\begin{aligned} \underset{G,\,f}{\text{minimize}} \quad & \big\langle G^{-1}, Q_p\big\rangle \,+\, \mathrm{diag}\,\big(E^T R\,E\big)^T f \,+\, \gamma\,\|f\|_1 \\ \text{subject to} \quad & G \,-\, G_p \,-\, E\,\mathrm{diag}\,(f)\,E^T \;=\; 0 \\ & G \;\succ\; 0 \end{aligned} \tag{SP}$$

where the ℓ_1 norm of f,

$$\|f\|_1 \;:=\; \sum_{l\,=\,1}^{m} |f_l|$$

is introduced into the optimal control problem as a proxy for inducing sparsity. In (SP), the positive definite matrix $G \in \mathbb{R}^{n\times n}$ and the vector of edge weights $f \in \mathbb{R}^m$ are optimization variables; the problem data is given by the positive regularization parameter γ, the plant graph Laplacian L_p, the state and control weights Q and R, and the incidence matrix of the controller graph E.

The sparsity-promoting optimal control problem (SP) is a constrained optimization problem with convex objective function, linear equality, and positive definite inequality constraints. This implies convexity of (SP). Linear constraint comes from structural requirements on the closed-loop graph Laplacian. On the other hand, positive definiteness of the strengthened graph Laplacian, G, guarantees stability of the closed-loop network (2) on the subspace $\mathbb{1}^\perp$. The sparsity-promoting optimal control problem (SP) is a convex optimization problem which can be cast as an SDP [20], [21].

In this paper, we restrict our attention to resistive networks [22]. This restriction implies non-negativity of the edge weights. We will also assume that the plant graph is connected which implies positive definiteness of the "strengthened" graph Laplacian of the plant, $G_p \succ 0$. Under these assumptions, the positive definite constraint in (SP) is automatically satisfied and the sparsity-promoting optimal control problem (SP) simplifies to

$$\begin{aligned} \underset{G,\,f}{\text{minimize}} \quad & \big\langle G^{-1}, Q_p\big\rangle \,+\, (\gamma\,\mathbb{1} \,+\, \mathrm{diag}\,\big(E^T R\,E\big))^T f \\ \text{subject to} \quad & G \,-\, G_p \,-\, E\,\mathrm{diag}\,(f)\,E^T \;=\; 0 \\ & -f \;\leq\; 0. \end{aligned} \tag{P}$$

A. Solving the structured H_2 problem (polishing step)

After the structure of the controller graph Laplacian has been identified, we eliminate the columns from the incidence matrix E that correspond to zero elements in the vector of the optimal edge weights $f^\star$. This yields new incidence matrix $\hat{E}$ and leads to the following optimization problem

$$\begin{aligned} \underset{G,\,f}{\text{minimize}} \quad & \big\langle G^{-1}, Q_p\big\rangle \,+\, \mathrm{diag}\,\Big(\hat{E}^T R\,\hat{E}\Big)^T f \\ \text{subject to} \quad & G \,-\, G_p \,-\, \hat{E}\,\mathrm{diag}\,(f)\,\hat{E}^T \;=\; 0 \\ & -f \;\leq\; 0 \end{aligned}$$

whose solution provides the optimal controller graph Laplacian with the desired structure. This optimization problem is obtained by setting $\gamma = 0$ in (P) and by replacing the

incidence matrix E with $\hat{E}$. The "polishing step" is used to improve performance relative to the solution of the sparsity-promoting optimal control problem (P).

III. Dual problem

In this section, we derive a Lagrange dual of the sparsity-promoting optimal control problem (P).

Proposition 1: The dual of the sparsity-promoting optimal control problem (P) is given by

$$\begin{array}{ll} \underset{Y}{\text{maximize}} & 2\ \text{trace}\left((Q_p^{1/2}\, Y\, Q_p^{1/2})^{1/2}\right)\ -\ \langle Y, G_p\rangle \\ \text{subject to} & \text{diag}\left(E^T(Y\ -\ R)\, E\right)\ -\ \gamma\, \mathbb{1}\ \leq\ 0 \\ & Y\ \succ\ 0,\ \ Y\,\mathbb{1}\ =\ \mathbb{1}. \end{array} \tag{D}$$

where $Y = Y^T \in \mathbb{R}^{n\times n}$ is the dual variable associated with the equality constraint in (P).

Proof: The Lagrangian of (P), $\mathcal{L}(G, f;\, Y, \lambda)$, is obtained by associating dual variables, Y and $\lambda\ \geq\ 0$, with equality and elementwise inequality constraints,

$$\begin{aligned} \mathcal{L}\ :=\ &\left\langle G^{-1}, Q_p\right\rangle\ +\ \left(\gamma\,\mathbb{1}\ +\ \text{diag}\left(E^T R\, E\right)\right)^T f\ + \\ &\left\langle Y, G\ -\ G_p\ -\ E\,\text{diag}\,(f)\, E^T\right\rangle\ -\ \lambda^T f. \end{aligned}$$

Using commutativity of the matrix trace, we can equivalently rewrite $\mathcal{L}$ as

$$\begin{aligned} \mathcal{L}\ =\ &\left\langle G^{-1}, Q_p\right\rangle\ +\ \langle Y, G\rangle\ -\ \langle Y, G_p\rangle\ + \\ &(\gamma\,\mathbb{1}\ -\ \text{diag}\left(E^T(Y\ -\ R)\, E\right)\ -\ \lambda)^T f. \end{aligned}$$

The dual function is obtained by minimizing the Lagrangian with respect to f and G. Minimization with respect to f yields

$$\lambda\ =\ \gamma\,\mathbb{1}\ -\ \text{diag}\left(E^T(Y\ -\ R)\, E\right)\ \geq\ 0 \tag{8}$$

where non-negativity follows from the fact that λ is the Lagrange multiplier associated with the inequality constraint in (P). On the other hand, minimization of the Lagrangian with respect to G yields

$$G^{-1}\, Q_p\, G^{-1}\ =\ Y \tag{9a}$$

or, equivalently,

$$G\ =\ Q_p^{1/2}\left(Q_p^{1/2}\, Y\, Q_p^{1/2}\right)^{-1/2} Q_p^{1/2}. \tag{9b}$$

Positive definiteness of G and Q_p implies positive definiteness of Y. Furthermore, since $Q_p\mathbb{1} = \mathbb{1}$, from (5c) and (9a) we have

$$Y\,\mathbb{1}\ =\ \mathbb{1}.$$

Using the above expressions, we can express the dual function as

$$2\ \text{trace}\left((Q_p^{1/2}\, Y\, Q_p^{1/2})^{1/2}\right)\ -\ \langle Y, G_p\rangle$$

and eliminate the slack variable λ from the dual problem. This allows us to bring the dual of (P) to (D). ∎

Any dual feasible variable Y can be used to obtain a lower bound on the optimal value of the primal problem (P). Furthermore, the difference between the objective function of the primal problem (evaluated at the primal feasible point (G, f)) and the objective function of the dual problem (evaluated at the dual feasible point Y) yields the duality gap. This positive quantity can be used to estimate distance to optimality. For the problem under study, the duality gap, η, can be expressed as

$$\eta\ =\ \lambda^T f\ =\ \mathbb{1}^T(\lambda\ \circ\ f)$$

where $\circ$ denotes elementwise (Hadamard) vector product and λ is given by (8).

At optimality, the duality gap η for the primal problem (P) and the dual problem (D) is zero. Strong duality follows from convexity of the primal problem (P) and strict feasibility of the constraints in (P); Slater's condition is satisfied with, e.g., $f = \mathbb{1}$. Furthermore, if $(G^\star, f^\star)$ are optimal points of the primal problem (P), then $Y^\star = (G^\star)^{-1} Q_p (G^\star)^{-1}$ is the optimal point of the dual problem (D). Similarly, if $Y^\star$ is the optimal point of (D), $G^\star = Q_p^{1/2}\left(Q_p^{1/2} Y^\star Q_p^{1/2}\right)^{-1/2} Q_p^{1/2}$ is the optimal point of (P). The optimal vector of edge weights $f^\star$ is determined by the non-zero off-diagonal elements of the controller graph Laplacian, $L_f^\star = G^\star - G_p$.

IV. A primal-dual interior point method

We develop a customized algorithm based on primal-dual interior point method (see, e.g., [23]–[26]) for growing connected resistive networks. By exploiting structure of the optimality conditions, we achieve significant speedup relative to standard interior point method solvers (e.g., SeDuMi or SDPT3).

We find it convenient to write optimality conditions as

$$\left(G_p + E\, D_f\, E^T\right)^{-1} Q_p \left(G_p + E\, D_f\, E^T\right)^{-1}\ =\ Y \tag{10a}$$

$$\gamma\,\mathbb{1}\ -\ \text{diag}\left(E^T(Y\ -\ R)\, E\right)\ -\ \lambda\ =\ 0 \tag{10b}$$

$$f\ \geq\ 0,\ \ \lambda\ \geq\ 0,\ \ \lambda\ \circ\ f\ =\ 0 \tag{10c}$$

where $D_f := \text{diag}\,(f)$. Several comments are in order

- Conditions (10a) and (10b) result from minimization of the Lagrangian with respect to G and f, respectively.
- Condition (10a) establishes relation between the vector of edge weights f and the dual variable Y. It is obtained by substitution of the equality constraint in (P) to (9a).
- Condition (10b) follows from (8) and it relates dual variables Y and λ.
- Conditions (10c) follow from non-negativity constraints on f and λ and complementary slackness (i.e., vanishing duality gap) requirement.
- If Y and f satisfy (10a) with $f \geq 0$, then $Y \succ 0$ and $Y\mathbb{1} = \mathbb{1}$. This is because the plant graph is connected (i.e., $G_p \succ 0$), $G_p\mathbb{1} = \mathbb{1}$, and $Q_p\mathbb{1} = \mathbb{1}$.

A. Central path equations and search direction

Condition (10a) allows us to express Y in terms of f, $Y = Y(f)$. This facilitates the use of infeasible primal-dual

interior point method to solve central path equations

$$\gamma\,\mathbb{1} \;-\; \mathrm{diag}\left(E^T(Y \;-\; R)\,E\right) \;-\; \lambda \;=\; 0 \tag{11a}$$

$$\lambda \circ f \;=\; \sigma\,\mu\,\mathbb{1}. \tag{11b}$$

Equation (11b) is obtained by relaxing complementary slackness condition in (10c), where μ and σ are positive parameters that provide continuous deformation of the optimality conditions.

For $\bar{f} > 0$ and $\bar{\lambda} > 0$ that are infeasible (i.e., do not satisfy (11a) with $Y(f)$ given by (10a)), the dual residual is determined by

$$r_d(\bar{f}, \bar{\lambda}) \;:=\; \gamma\,\mathbb{1} \;-\; \mathrm{diag}\left(E^T(Y(\bar{f}) \;-\; R)\,E\right) \;-\; \bar{\lambda}.$$

The search direction $(\tilde{f}, \tilde{\lambda})$ is obtained by solving the linearized system of central path equations,

$$M\,\tilde{f} \;-\; \tilde{\lambda} \;=\; -\,r_d(\bar{f}, \bar{\lambda}) \tag{12a}$$

$$D_{\bar{\lambda}}\,\tilde{f} \;+\; D_{\bar{f}}\,\tilde{\lambda} \;=\; \sigma\,\mu\,\mathbb{1} \;-\; \bar{\lambda} \circ \bar{f}. \tag{12b}$$

The system of equations (12) represents linearization of (11) around $(\bar{f}, \bar{\lambda})$. The matrix M in (12b) is determined by the gradient of the function $\mathrm{diag}\,(E^T Y(\bar{f})E)$ at $\bar{f} > 0$,

$$\mathrm{diag}\,(E^T Y(\bar{f} + \tilde{f})E) \;=\; \mathrm{diag}\,(M_1) \;-\; M\,\tilde{f} \;+\; O(\|\tilde{f}\|^2)$$

where

$$\begin{aligned} M &= 2\,(M_1 \circ M_2) \\ M_1 &= E^T\,Y(\bar{f})\,E \\ M_2 &= E^T\left(G_p \;+\; E\,D_{\bar{f}}\,E^T\right)^{-1} E. \end{aligned}$$

B. Algorithm

Starting with points $\bar{f} > 0$ and $\bar{\lambda} > 0$, the primal-dual interior point algorithm is outlined next (for details, please see [27]):

1) Compute the dual residual $r_d(\bar{f}, \bar{\lambda})$ and the duality gap η, and evaluate the stopping criteria $\|r_d\| \le \epsilon_1$ and $\eta \le \epsilon_2$. Terminate if these are satisfied.
2) Compute affine scaling direction $(\tilde{f}_a, \tilde{\lambda}_a)$ by solving the linear system of equations

$$\begin{aligned} M\,\tilde{f}_a \;-\; \tilde{\lambda}_a \;&=\; -r_d(\bar{f}, \bar{\lambda}) \\ D_{\bar{\lambda}}\,\tilde{f}_a \;+\; D_{\bar{f}}\,\tilde{\lambda}_a \;&=\; -\,\bar{\lambda} \circ \bar{f}. \end{aligned}$$

3) Select barrier parameters

$$\begin{aligned} \alpha_f &= \underset{\alpha}{\mathrm{argmax}}\,\{\alpha \in [0,\,1];\; \bar{f} \;+\; \alpha\,\tilde{f}_a \;\ge\; 0\} \\ \alpha_\lambda &= \underset{\alpha}{\mathrm{argmax}}\,\{\alpha \in [0,\,1];\; \bar{\lambda} \;+\; \alpha\,\tilde{\lambda}_a \;\ge\; 0\} \end{aligned}$$

and set

$$\mu \;=\; \frac{\bar{f}^T\bar{\lambda}}{m}, \quad \sigma \;=\; \left(\frac{(\bar{f} \;+\; \alpha_f\,\tilde{f}_a)^T(\bar{\lambda} \;+\; \alpha_\lambda\,\tilde{\lambda}_a)}{\bar{f}^T\bar{\lambda}}\right)^3.$$

4) Compute search direction $(\tilde{f}, \tilde{\lambda})$ by solving the linear system of equations

$$\begin{aligned} M\,\tilde{f} \;-\; \tilde{\lambda} \;&=\; -r_d(\bar{f}, \bar{\lambda}) \\ D_{\bar{\lambda}}\,\tilde{f} \;+\; D_{\bar{f}}\,\tilde{\lambda} \;&=\; \sigma\,\mu\,\mathbb{1} \;-\; \bar{\lambda} \circ \bar{f} \;-\; \tilde{\lambda}_a \circ \tilde{f}_a \end{aligned}$$

where Mehrotra correction $\tilde{\lambda}_a \circ \tilde{f}_a$ has been added to the linearized system of central path equations (12).
5) Determine maximum steps to the boundary

$$\begin{aligned} \alpha_f &= \underset{\alpha}{\mathrm{argmax}}\,\{\alpha \in [0,\,1];\; \bar{f} \;+\; \alpha\,\tilde{f} \;\ge\; 0\} \\ \alpha_\lambda &= \underset{\alpha}{\mathrm{argmax}}\,\{\alpha \in [0,\,1];\; \bar{\lambda} \;+\; \alpha\,\tilde{\lambda} \;\ge\; 0\} \end{aligned}$$

and update $\bar{f}$ and $\bar{\lambda}$,

$$\begin{aligned} \bar{f} &= \bar{f} \;+\; \min\{1,\,0.99\,\alpha_f\}\,\tilde{f} \\ \bar{\lambda} &= \bar{\lambda} \;+\; \min\{1,\,0.99\,\alpha_\lambda\}\,\tilde{\lambda}. \end{aligned}$$

Return to 1).

The challenging aspect of the primal-dual interior point algorithm is the computation of the search directions $(\tilde{f}_a, \tilde{\lambda}_a)$ and $(\tilde{f}, \tilde{\lambda})$. In order to obtain these, we need to solve linear systems of equations. For example, the affine search direction can be computed by expressing $\tilde{\lambda}_a$ in terms of $\tilde{f}_a$,

$$\tilde{\lambda}_a \;=\; -D_{\bar{f}}^{-1}D_{\bar{\lambda}}\,\tilde{f}_a \;-\; \bar{\lambda} \tag{13a}$$

which yields the following equation for $\tilde{f}_a$,

$$A\,\tilde{f}_a \;=\; b \tag{13b}$$

where

$$\begin{aligned} A \;&:=\; M \;+\; D_{\bar{f}}^{-1}D_{\bar{\lambda}} \\ b \;&:=\; -\left(\bar{\lambda} \;+\; r_d(\bar{f}, \bar{\lambda})\right). \end{aligned} \tag{13c}$$

The same matrix A also appears in the equations for the search direction $\tilde{f}$. Positive definiteness of $D_{\bar{f}}^{-1}D_{\bar{\lambda}}$ and M (elementwise product of two positive definite matrices is positive definite) implies positive definiteness of A. Thus, for moderately sized problems, Cholesky factorization of A followed by back solve operations can be used to to determine search directions. These respectively take $O(m^3)$ and $O(m^2)$ operations.

C. Search direction via the PCG algorithm

Since the direct method based on Cholesky factorization is not well-suited for large problems, we next provide an efficient inexact method for computing search directions $(\tilde{f}_a, \tilde{\lambda}_a)$ and $(\tilde{f}, \tilde{\lambda})$. Our approach utilizes the preconditioned conjugate gradient (PCG) algorithm, an indirect iterative method for solving a linear system of equations [28]. In theory, conjugate gradient converges in m iterations. Conjugate gradient algorithm consists of inner and outer iterations. Each inner iteration requires a few inner products and one matrix-vector multiplication. If the matrix A is dense, then the matrix-vector multiplication costs $O(m^2)$. So, the total cost is $O(m^3)$ which is the same as the direct method based on Cholesky factorization. However, an advantage can be gained if the matrix-vector multiplication is cheaper than $O(m^2)$. Moreover, in many problems, an acceptable solution can be reached in less than m outer iterations. On the other

hand, conjugate gradient method can perform poorly in the presence of round-off errors. In many problems, introduction of suitably selected preconditioners ensures fast convergence or even convergence of the conjugate gradient method. In our implementation, we use the diagonal preconditioner $P := I \circ A$.

V. Computational experiments

In this section, we provide several examples to illustrate utility of our customized algorithms. Moreover, we compare the performance of exact and inexact primal-dual interior point algorithms with CVX [29]. The exact algorithm uses Cholesky factorization to compute the search direction, and the inexact algorithm uses preconditioned conjugate gradient (PCG) method. We have implemented all algorithms in MATLAB, and all tests were executed on a two quad-core 2.8 GHz machine with Intel Xeon X5560 "Nehalem EP"-class processors.

In all examples we set $L_p = E_p E_p^T$, where E_p is the incidence matrix of the plant graph. The incidence matrix of the controller graph is selected to satisfy the following requirements: (i) in the absence of the sparsity-promoting term, the closed-loop network is given by a complete graph; and (ii) there are no joint edges between the plant and the controller graphs. In all plots that illustrate the graph structure, we use black dots to denote nodes, blue color to identify edges in the plant graph, and red color to identify edges in the controller graph. Finally, we set $R = I$ and choose the state weight that penalizes the mean-square deviation from the network average, $Q = I - (1/n)\,\mathbb{1}\mathbb{1}^T$.

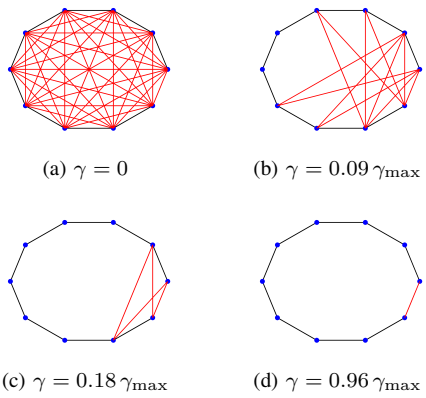
Figure 1 shows the results obtained by applying our customized algorithm to the problem of growing a resistive path graph with 10 nodes. For $\gamma = 0$, we obtain a centralized controller that requires information exchange between all nodes. With increase in γ, the number of added edges gradually decreases. For

$$\gamma \;>\; \gamma_{\max} \;:=\; \|E^T\, G_p^{-1}\, Q\, G_p^{-1}\, E\|_\infty$$

all edge weights in the controller graph are equal to zero. As shown in Fig. 1d, for $\gamma = 0.96\gamma_{\max}$ the single edge is added and this edge generates the longest cycle. This is in agreement with [30] where it was shown that the longest cycle is most beneficial for improving the $\mathcal{H}_2$ performance of tree networks.

Table I shows the performance comparison between the exact and inexact interior point methods and CVX. We employ our customized algorithms to solve the (P) for Toeplitz graphs with different number of nodes. Absolute value of the residual, r_d, and the duality gap, η, can be used as stopping criteria. For customized algorithms, we set the tolerance for both $\|r_d\|$ and η to 10^{-6}. We set $\gamma = \gamma_{\max}/2$. The performance (in terms of speed and the number of iterations) is compared.

Our results illustrate that both of our customized interior point algorithms significantly outperform CVX. While



(a) $\gamma = 0$ (b) $\gamma = 0.09\,\gamma_{\max}$ (c) $\gamma = 0.18\,\gamma_{\max}$ (d) $\gamma = 0.96\,\gamma_{\max}$

Fig. 1: Path graph

TABLE I: Comparison between algorithms on a Toeplitz tridiagonal network with unit edge weights.

Algorithm	$n = 100$ $m = 4851$	$n = 200$ $m = 19701$	$n = 300$ $m = 44551$
Exact	69.143 sec/10 iter	3.750 e3/11	5.881 e4/13
Inexact	60.009 sec/7iter	1.569 e3/8	2.008 e4/9
CVX	1561.687 sec	–	–

the performance of the algorithm that uses Cholesky factorization is robust, our inexact interior point algorithm implementation can be further improved by more efficient implementation and a better choice of preconditioner.

VI. Concluding Remarks

We have examined the problem of growing connected networks of resistors. Our approach formulates a distributed control problem aimed at balancing performance of stochastically forced network with the number of edges that are added to the plant network. We derive a dual of this convex optimization problem and provide an efficient implementation of an inexact primal-dual interior point method. Our algorithm is significantly faster than general purpose solvers. Currently, we are working on a more efficient implementation of an iterative method based on the preconditioned conjugate gradient. We are also developing customized first-order algorithms for the design of consensus and synchronization networks.

Appendix

A. Network performance

The $\mathcal{H}_2$ norm of the closed-loop system is given by

$$\|H\|_2^2 \;=\; \mathrm{trace}\,(P)$$

where the symmetric matrix P is the solution of the following Lyapunov equation

$$(L_p + L_f)\, P + P\, (L_p + L_f) \;=\; Q + L_f\, R\, L_f.$$

Here, $L_f = E\,\mathrm{diag}\,(f)\,E^T$. Using the formula for the solution P and walking through some algebraic steps, $\mathcal{H}_2$ norm can be written as (6). Here, we describe the transition from (6) to (7). As shown in (6), the $\mathcal{H}_2$ norm is given by

$$2\,\|H\|_2^2 \;=\; \left\langle G^{-1}, Q\right\rangle + \left\langle G^{-1}, L_f\,R\,L_f\right\rangle.$$

The second part $\left\langle G^{-1}, L_f\,R\,L_f\right\rangle$ can be written as

$$\begin{aligned}
&\left\langle (G_p + L_f)^{-1}, L_f\,R\,L_f\right\rangle = \\
&\left\langle (G_p + L_f)^{-1}, (G_p + L_f - G_p)\,R\,L_f\right\rangle = \\
&\langle R, L_f\rangle - (G_p + L_f)^{-1}\,G_p\,R\,(L_f + G_p - G_p) = \\
&\langle R, L_f - G_p\rangle + \left\langle (G_p + L_f)^{-1}, G_p\,R\,G_p\right\rangle.
\end{aligned}$$

We can write $\left\langle (G_p + L_f)^{-1}, G_p\,R\,G_p\right\rangle$ as

$$\begin{aligned}
&\left\langle (G_p + L_f)^{-1}, G_p\,R\,(L_p + (1/n)\,\mathbb{1}\mathbb{1}^T)\right\rangle = \\
&\left\langle G^{-1}, G_p\,R\,L_p\right\rangle + \left\langle G^{-1}\,G_p\,R, (1/n)\,\mathbb{1}\mathbb{1}^T\right\rangle = \\
&\left\langle G^{-1}, L_p\,R\,L_p\right\rangle + (1/n)\,\mathbb{1}^T R\,\mathbb{1}.
\end{aligned}$$

Thus, $J = 2\,\|H\|_2^2$ is given by

$$\begin{aligned}
J \;=\; &\left\langle G^{-1}, (Q + L_p\,R\,L_p)\right\rangle + \langle R, L_f\rangle - \langle R, G_p\rangle + \\
&(1/n)\,\mathbb{1}^T R\,\mathbb{1}.
\end{aligned}$$

Considering structural restrictions given in (3), we can write $\left\langle G^{-1}, (Q + L_p R L_p)\right\rangle$ as

$$\begin{aligned}
&\left\langle G^{-1}, (Q + (1/n)\,\mathbb{1}\mathbb{1}^T + L_p\,R\,L_p - (1/n)\,\mathbb{1}\mathbb{1}^T)\right\rangle = \\
&\left\langle G^{-1}, (Q + (1/n)\,\mathbb{1}\mathbb{1}^T + L_p\,R\,L_p)\right\rangle - 1
\end{aligned}$$

therefore,

$$\begin{aligned}
J \;=\; &\left\langle G^{-1}, (Q + L_p\,R\,L_p + (1/n)\,\mathbb{1}\mathbb{1}^T)\right\rangle + \langle R, L_f\rangle - \\
&\langle R, G_p\rangle + (1/n)\,\mathbb{1}^T R\,\mathbb{1} - 1.
\end{aligned}$$

Replacing Q_p and ignoring constant terms, J is given by

$$J(G, f) \;=\; \left\langle G^{-1}, Q_p\right\rangle + \mathrm{diag}\left(E^T R\,E\right)^T f.$$

REFERENCES

[1] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[2] M. H. DeGroot, "Reaching a consensus," *J. Amer. Statist. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974.

[3] B. Golub and M. Jackson, "Naive learning social networks and the wisdom of crowds," *American Economic Journal: Microeconomics*, vol. 2, no. 1, pp. 112–149, 2010.

[4] G. Cybenko, "Dynamic load balancing for distributed memory multiprocessors," *J. Parallel Distrib. Comput.*, vol. 7, no. 2, pp. 279–301, 1989.

[5] J. E. Boillat, "Load balancing and Poisson equation in a graph," *Concurrency: Practice and Experience*, vol. 2, no. 4, pp. 289–313, 1990.

[6] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[7] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automat. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[8] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Automat. Control*, vol. 50, no. 2, pp. 169–182, 2005.

[9] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri, "Communication constraints in the average consensus problem," *Automatica*, vol. 44, no. 3, pp. 671–684, 2007.

[10] G. E. Dullerud and F. Paganini, *A course in robust control theory*. Springer, 2000.

[11] L. Xiao, S. Boyd, and S.-J. Kim, "Distributed average consensus with least-mean-square deviation," *J. Parallel Distrib. Comput.*, vol. 67, no. 1, pp. 33–46, 2007.

[12] A. Ghosh, S. Boyd, and A. Saberi, "Minimizing effective resistance of a graph," *SIAM Review*, vol. 50, no. 1, pp. 37–66, 2008.

[13] D. Zelazo and M. Mesbahi, "Edge agreement: Graph-theoretic performance bounds and passivity analysis," *IEEE Trans. Automat. Control*, vol. 56, no. 3, pp. 544–555, 2011.

[14] D. Zelazo, S. Schuler, and F. Allgöwer, "Performance and design of cycles in consensus networks," *Syst. Control Lett.*, vol. 62, no. 1, pp. 85–96, 2013.

[15] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Systems & Control Letters*, vol. 53, no. 1, pp. 65–78, 2004.

[16] A. Ghosh and S. Boyd, "Growing well-connected graphs," in *Proceedings of the 45th IEEE Conference on Decision and Control*, 2006, pp. 6605–6611.

[17] R. Dai and M. Mesbahi, "Optimal topology design for dynamic networks," in *50th IEEE Conference on Decision and Control*, 2011, pp. 1280–1285.

[18] M. Fardad, F. Lin, and M. R. Jovanović, "Sparsity-promoting optimal control for a class of distributed systems," in *Proceedings of the 2011 American Control Conference*, San Francisco, CA, 2011, pp. 2050–2055.

[19] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Automat. Control*, vol. 58, no. 9, pp. 2426–2431, 2013.

[20] F. Lin, M. Fardad, and M. R. Jovanović, "Identification of sparse communication graphs in consensus networks," in *Proceedings of the 50th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, 2012, pp. 85–89.

[21] X. Wu and M. R. Jovanović, "Sparsity-promoting optimal control of consensus and synchronization networks," in *Proceedings of the 2014 American Control Conference*, Portland, OR, 2014, pp. 2948–2953.

[22] A. Ghosh, S. Boyd, and A. Saberi, "Minimizing effective resistance of a graph," *SIAM review*, vol. 50, no. 1, pp. 37–66, 2008.

[23] J. F. Sturm, "Implementation of interior point methods for mixed semidefinite and second order cone optimization problems," *Optimization Methods and Software*, vol. 17, no. 6, pp. 1105–1154, 2002.

[24] F. E. Curtis, O. Schenk, and A. Wächter, "An interior-point algorithm for large-scale nonlinear optimization with inexact step computations," *SIAM Journal on Scientific Computing*, vol. 32, no. 6, pp. 3447–3475, 2010.

[25] T. Roh and L. Vandenberghe, "Discrete transforms, semidefinite programming, and sum-of-squares representations of nonnegative polynomials," *SIAM Journal on Optimization*, vol. 16, no. 4, pp. 939–964, 2006.

[26] Z. Lu, R. D. Monteiro, and J. W. O'Neal, "An iterative solver-based infeasible primal-dual path-following algorithm for convex quadratic programming," *SIAM Journal on optimization*, vol. 17, no. 1, pp. 287–310, 2006.

[27] J. Nocedal and S. J. Wright, *Numerical Optimization*. Springer, 2006.

[28] J. Nocedal and S. J. Wright, *Conjugate gradient methods*. Springer, 2006.

[29] M. Grant and S. P. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.0 beta," http://cvxr.com/cvx, September 2013.

[30] D. Zelazo, S. Schuler, and F. Allgwer, "Performance and design of cycles in consensus networks," *Systems & Control Letters*, vol. 62, no. 1, pp. 85 – 96, 2013.