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Edge addition in directed consensus networks

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Abstract— **We study the problem of performance enhancement in stochastically-forced directed consensus networks by adding edges to an existing topology. We formulate the problem as a feedback control design, and represent the links as the elements of the controller graph Laplacian matrix. The topology design of the controller network can be cast as an ℓ_1 regularized version of the $\mathcal{H}_2$ optimal control problem. The goal is to optimize the performance of the network by selecting a controller graph with low communication requirements. To deal with the structural constraints that arise from the absence of absolute measurements, we introduce a coordinate transformation to eliminate the average mode and assure convergence of all states to the average of the initial node values. By exploiting structure of the optimization problem, we develop a customized algorithm based on the alternating direction method of multipliers to design a sparse controller network that improves the performance of the closed-loop system.**

Index Terms— **Alternating direction method of multipliers, consensus, directed networks, non-convex optimization, sparsity-promoting optimal control.**

I. INTRODUCTION

Distributed computing over networks is of fundamental importance in network science [1]. Consensus problem has received significant attention because of a broad range of applications including animal group behavior [2], [3], social networks [4], [5], power systems [6]–[8], spreading processes on complex networks [9], [10], and cooperative control of vehicular formations [11]–[13]. An inherent challenge in these problems is that it is desired for all nodes to reach an agreement or to achieve synchronization by only exchanging relative information with limited number of nodes. The restriction on the absence of the absolute measurements imposes structural constraints for the analysis and design.

Reaching agreement using relative information exchange in a decentralized fashion has attracted lots of interest. In large networks, centralized implementation of control policies imposes heavy communication and computation burden on individual nodes. This motivates the development of distributed control strategies that require limited information exchange between the nodes in order to reach consensus or achieve synchronization [14]–[18].

Significant amount of research has been devoted to the study of the consensus problem in networks, where both the plant and the controller graphs are undirected. For undirected consensus network, the ℓ_1-regularized minimum variance optimal control problem is convex [19]–[23]. On the other hand, convexity is lost for directed networks.

In the absence of disturbances, a strongly connected balanced network converges to the average of the initial node values [1]. However, in the presence of additive stochastic disturbances, the network average experiences a random walk. Thus, the control objective is to minimize mean square deviation from average. To cope with structural constraints that arise from the absence of absolute information exchange, we introduce a coordinate transformation to eliminate the average mode. While it is desired to promote sparsity of controller graph in the physical domain, the $\mathcal{H}_2$ optimal control problem is solved in the transformed set of coordinates where the average mode is eliminated.

In this paper, we consider the problem of adding edges to a weakly connected directed consensus network in order to improve performance. In particular, we are interested in designing sparse communication graphs that strike a balance between the variance amplification of the closed-loop system and the number of communication links. In general, this is a combinatorial search problem and is non-convex. In undirected networks, convex relaxations or greedy algorithms have been introduced in order to optimize algebraic connectivity of the network [24], [25] or network coherence [21], [26], [27] by adding edges from a given set of edges.

We formulate the edge addition problem for directed networks with an objective of optimizing the closed-loop coherence. We consider the scenario in which the plant graph is unbalanced. Structural requirements that the closed-loop graph Laplacian is weakly connected and balanced make the optimal control problem challenging. We approach this problem using sparsity-promoting optimal control framework [19], [28], [29]. In our formulation, performance is captured by the $\mathcal{H}_2$ norm of the closed-loop network and ℓ_1 regularization is introduced as a proxy for inducing sparsity in the controller graph [21], [26]. By exploiting the structure of the problem, we develop a customized algorithm based on alternating direction method of multipliers (ADMM) [30].

The rest of the paper is structured as follows. In Section II, we provide necessary background on graph theory for directed networks, define consensus problems on graphs, and introduce an appropriate change of coordinates. In Section III, we formulate the optimal control problem. In Section IV, we develop a customized algorithm based on ADMM. In Section V, we use our algorithm for sparse edge addition to directed consensus networks. Finally, in Section VI, we

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provide concluding remarks and highlight future directions.

II. Motivation and Background

In this section, after providing the necessary background, we describe the dynamics of consensus networks, inherent structural constraints, and the challenges in the related design problems. We then introduce a change of coordinates that enables us to overcome these restrictions.

A. Directed graphs

Herein, we provide a brief overview of the background material; for additional information, see [1].

Weighted digraph: a weighted directed graph is denoted by $\mathcal{D} = (V, E, w)$ where V and E are the sets of nodes and edges and w is the vector of edge weights. The edges are directed and a value w_{ij} is the edge weight between nodes v_i and v_j. If the ordered pair $(v_i, v_j) \in E$ then v_i is the tail of the edge and v_j is its head.

Strongly connected digraph: a digraph is strongly connected if there is a directed path between every pair of nodes.

Weakly connected digraph: a digraph is weakly connected if it is a connected graph when the directions are omitted.

Adjacency matrix: the ijth element is given by

$$A_{ij} \;=\; \begin{cases} w_{ij}, & (v_j, v_i) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Degree matrix is a diagonal matrix with $D_{ii} = d_{\text{in}}(v_i)$ where $d_{\text{in}}(v_i)$ is the weighted in-degree of node v_i,

$$d_{\text{in}}(v_i) \;=\; \sum_j w_{ij}.$$

We sum over j's for $(v_j, v_i) \in E$.

Laplacian matrix is a matrix $L = D - A$, where $L\,\mathbb{1} = 0$ by definition.

Balanced digraph: a digraph is balanced if the weighted in-degree and the weighted out-degree for each node are equal.

B. Feedback design in consensus networks

We consider a consensus dynamics

$$\dot{x} \;=\; -L_p\, x \,+\, u \,+\, d$$

where d and u are the disturbance and control inputs, x is the vector of the states, and $L_p \in \mathbb{R}^{n\times n}$ is the graph Laplacian of the plant network. We assume that the plant network is weakly connected and formulate the edge addition problem as a feedback design problem with

$$u \;=\; -L_k\, x$$

where $L_k \in \mathbb{R}^{n\times n}$ is the weighted directed Laplacian matrix of the controller. This matrix represents the locations, directions, and edge weights. Note that each nonzero element in L_k can either indicate addition of an edge or re-tuning of an existing edge gain.

The closed-loop system is given by

$$\dot{x} \;=\; -\,(L_p \,+\, L_k)\, x \,+\, d. \tag{1a}$$

Our goal is to optimally design the feedback gain matrix L_k in order to achieve the desired tradeoff between the controller sparsity and network performance. The performance is quantified by the steady-state variance amplification of the stochastically-forced network, from the white-in-time input d to the performance output z that quantifies deviation from consensus and control effort,

$$z \;=\; \begin{bmatrix} Q^{1/2} \\ -R^{1/2} L_k \end{bmatrix} x. \tag{1b}$$

Here, the matrices $Q = Q^T \succeq 0$ and $R \succ 0$ are the state and control weights, respectively, and $\succ$ ($\succeq$) signifies positive definiteness (semi-definiteness) of a matrix.

In consensus networks, each node updates its state using the relative information exchange with its neighbors. In the presence of white noise, the average mode $\bar{x} = (1/n)x^T\mathbb{1}$ experiences a random walk and variance increased linearly with time. A key property of the Laplacian matrix of the controller is $L_k\mathbb{1} = 0$. Since our primary control objective is to achieve consensus, only differences between node values are penalized in the performance output. Therefore, the state weight matrix Q has a zero eigenvalue with the corresponding eigenvector of all ones, i.e., $Q\,\mathbb{1} = 0$. Furthermore, we choose Q to be positive definite on the orthogonal complement of the subspace $\text{span}\,(\mathbb{1})$,

$$Q \,+\, (1/n)\,\mathbb{1}\mathbb{1}^T \,\succ\, 0.$$

The following lemma summarizes the well-known conditions for achieving consensus in the absence of disturbances in directed networks [1].

Lemma 1: The agreement protocol over a digraph reaches the average consensus, $(1/n)\mathbb{1}\mathbb{1}^T x(0)$, for every initial condition if and only if it is weakly connected and balanced.

Thus, for a weekly connected L_p which is not necessarily balanced, it is required that the closed-loop graph Laplacian, $L_p + L_k$ be balanced, which amounts to

$$\mathbb{1}^T\,(L_p \,+\, L_k) \;=\; 0. \tag{2}$$

The problem of designing a controller graph that provides a desired tradeoff between performance index J of the network and the sparsity of the controller L_k can be formulated as

$$\begin{array}{ll} \underset{L_k}{\text{minimize}} & J(L_k) \,+\, \gamma\,\|W \circ L_k\|_1 \\ \text{subject to} & \mathbb{1}^T\,(L_k \,+\, L_p) \;=\; 0 \\ & L_k\,\mathbb{1} \;=\; 0. \end{array} \tag{3}$$

The positive scalar γ is the sparsity-promoting parameter that characterizes a trade-off between network performance and sparsity of the controller and $\circ$ denotes elementwise multiplication. The first condition guarantees asymptotic consensus to the initial network average value in the absence of disturbances and the second condition secures the row stochastic property of the controller graph Laplacian. We gurantee that the closed-loop graph Laplacian remains weakly connected via proper step-size selection in the algorithm.

In what follows, we quantify the network performance using the $\mathcal{H}_2$ norm. The weighted ℓ_1 norm [31] of L_k is a convex approximation of the cardinality function. In each iteration, each element of the weight matrix $W(i,j)$ is chosen to be inversely proportional to the magnitude of $L_k(i,j)$ in the previous iteration and can be determined as

$$W(i,j) \;=\; 1/(|L_k(i,j)| \,+\, \varepsilon). \tag{4}$$

This puts larger emphasis on smaller optimization variables, where a small positive parameter ε ensures that $W(i,j)$ is well-defined.

C. Elimination of the average mode

Since both Q and L_k have a zero eigenvalue associated with the vector of all ones, the average mode $\bar{x}$ is not observable from the output z. In order to eliminate the average mode, we introduce the following change of coordinates

$$\begin{bmatrix} \psi \\ \bar{x} \end{bmatrix} \;=\; \begin{bmatrix} U^T \\ \mathbb{1}^T/n \end{bmatrix} x$$

where $U \in \mathbb{R}^{n\times(n-1)}$ is a full-rank matrix and its columns span the orthogonal complement of $\mathbb{1}$. Using the properties of the matrix U,

$$U^T U \;=\; I, \quad U U^T \;=\; I \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T, \;\; U^T \mathbb{1} \;=\; 0,$$

we have

$$\begin{bmatrix} \dot{\psi} \\ \dot{\bar{x}} \end{bmatrix} \;=\; \begin{bmatrix} -U^T(L_p \,+\, L_k)\,U & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \bar{x} \end{bmatrix} + \begin{bmatrix} U^T \\ (1/n)\,\mathbb{1}^T \end{bmatrix} d.$$

In the new coordinates, the performance output is given by

$$z \;=\; \begin{bmatrix} Q^{1/2}\,U & 0 \\ -\,R^{1/2} L_k\,U & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \bar{x} \end{bmatrix}.$$

The network average $\bar{x} = (1/n)\,\mathbb{1}^T d$ experiences a random walk and the vector ψ quantifies the deviation from average of the states. Since our control objective is to minimize the deviation from average in the nodes, the average mode is not of interest. Therefore, the minimal representation of the system containing only the state ψ is given by

$$\begin{aligned} \dot{\psi} \;&=\; -\,U^T(L_p \,+\, L_k)\,U\,\psi \,+\, U^T d \\ z \;&=\; \begin{bmatrix} Q^{1/2}\,U \\ -\,R^{1/2} L_k\,U \end{bmatrix} \psi. \end{aligned} \tag{5}$$

In order to guarantee that the closed-loop graph Laplacian $(L_p + L_k)$ is balanced, we introduce the following change of variables

$$L_k \;=\; U F U^T \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p \;\;\Leftrightarrow\;\; F \;=\; U^T L_k\,U. \tag{6}$$

Note that the main contribution that differentiates this work from previous results [26] is that we are considering a weakly-connected directed unbalanced network, and we require the closed-loop network to be balanced. The problem of adding edges to a directed network has not been addressed in [26]. Moreover, the presented framework cannot be used to solve the current constrained problem due to an additional bias term. In our framework, a new change of variables is introduced to accommodate structural constraints.

Using the properties of the matrix U, the constraints (2) and $L_k\,\mathbb{1} = 0$ are automatically satisfied. The equation (6) demonstrates how we can move back and forth between two variables F and L_k.

Substituting L_k given by (6) into (5), we can write the state-space representation of the closed-loop system as

$$\begin{aligned} \dot{\psi} \;&=\; -\,(\,U^T L_p\,U \,+\, F\,)\,\psi \,+\, U^T d \\ z \;&=\; \begin{bmatrix} Q^{1/2}\,U \\ -\,R^{1/2}(U\,F \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p\,U) \end{bmatrix} \psi. \end{aligned} \tag{7}$$

Next, we formulate the optimal control problem and propose a framework to design sparse controller L_k.

III. Topology design for directed networks

In this section, we approach the problem of topology design as a regularized optimal control problem.

A. Sparsity-promoting optimal control problem

The $\mathcal{H}_2$ norm of the transfer function from d to z,

$$\|H\|_2^2 \;=\; J(F) \;=\; \mathrm{trace}\,(P(F))$$

quantifies the variance amplification of the closed-loop network (7). Here, $P(F)$ is the closed-loop observability Gramian which is the solution of the Lyapunov equation,

$$(\,U^T L_p\,U \,+\, F)^T\,P \,+\, P\,(\,U^T L_p\,U \,+\, F) \;=\; C^T C \tag{8}$$

where

$$C \;=\; \begin{bmatrix} Q^{1/2}\,U \\ -\,R^{1/2}(U\,F \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p\,U) \end{bmatrix}.$$

The control design problem (3) takes the following form

$$\begin{aligned} &\underset{F,\,L_k}{\text{minimize}} \quad J(F) \,+\, \gamma\,\|W \circ L_k\|_1 \\ &\text{subject to} \quad U\,F\,U^T \,-\, L_k \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p \;=\; 0. \end{aligned} \tag{9}$$

In (9), the Laplacian matrix L_k and the matrix F are the optimization variables; the problem data is given by plant graph Laplacian L_p, state and control weights Q and R, and positive regularization parameter γ. The matrix W is the weight matrix that imposes a penalty on the magnitude of the elements in L_k. With the problem formulation (9), we are able to minimize the $\mathcal{H}_2$ norm of the network $J(F)$ in the transformed coordinates where the average mode is eliminated, while promoting sparsity of L_k in the physical domain. For $\gamma = 0$, the solution to (9) is typically given by a matrix L_k with all non-zero components. As the regularization parameter increases, the number of non-zero elements in the controller graph decreases.

B. Structured optimal control problem

After the structure of the controller graph Laplacian L_k has been designed, we fix the sparsity pattern $\mathcal{S}$ and then solve the following problem

$$\begin{array}{ll} \underset{F}{\text{minimize}} & J(F) \\ \text{subject to} & U\,F\,U^T \,-\, (1/n)\,\mathbb{1}\,\mathbb{1}^T L_p \in \mathcal{S} \end{array} \tag{10}$$

whose solution provides the optimal controller graph Laplacian with the desired structure. This optimization problem is obtained by setting $\gamma = 0$ in (9) and adding the sparsity pattern $\mathcal{S}$. This "polishing" or "debiasing" step is used to improve the performance relative to the solution of the sparsity-promoting optimal control problem (9).

IV. An ADMM-based algorithm

We next exploit the structure of the constrained optimization problem (9) and develop a customized algorithm based on ADMM. The augmented Lagrangian associated with (9) is given by

$$\begin{aligned} \mathcal{L}_\rho(F,\,L_k;\,\Lambda) \;=\; & J(F) \,+\, \gamma\,\|W \,\circ\, L_k\|_1 \,+ \\ & \left\langle \Lambda, U\,F\,U^T \,-\, L_k \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p \right\rangle \,+ \\ & \frac{\rho}{2}\,\|U\,F\,U^T \,-\, L_k \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p\|_F^2 \end{aligned} \tag{11}$$

where the matrix Λ is the Lagrange multiplier, ρ is a positive scalar, and $\|\cdot\|_F$ is the Frobenius norm. The ADMM algorithm consists of the following steps at each iteration

$$\begin{aligned} F^{k+1} \;&=\; \underset{F}{\text{argmin}}\; \mathcal{L}_\rho(F,\,L_k^k;\,\Lambda^k) \\ L_k^{k+1} \;&=\; \underset{L_k}{\text{argmin}}\; \mathcal{L}_\rho(F^{k+1},\,L_k;\,\Lambda^k) \\ \Lambda^{k+1} \;&=\; \Lambda^k \,+\, \rho\,\left(U\,F^{k+1}\,U^T \,-\, L_k^{k+1} \,-\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p\right). \end{aligned}$$

The algorithm terminates when $\|L_k^{k+1} - L_k^k\|_F \,\leq\, \epsilon_1$ and $\|U\,F^{k+1}\,U^T - L_k^{k+1} - (1/n)\,\mathbb{1}\mathbb{1}^T L_p)\| \leq \epsilon_2$, where ϵ_1 and ϵ_2 are desired tolerances.

Note that, the smooth part $J(F)$ and the non-smooth part $\|W \circ L_k\|_1$ are now operating in different coordinates; therefore, descent algorithms can be utilized in the F-minimization step. Moreover, the ℓ_1 norm is a separable function with respect to each element of L_k. Thus, we can determine the solution to the L_k-minimization step analytically. In the Lagrange multiplier update step, we use the step-size equal to ρ in order to guarantee the dual feasibility [30].

A. F-minimization step

We bring the F-minimization step to the following form by using the properties of the matrices L_p and U,

$$F^{k+1} \;=\; \underset{F}{\text{argmin}}\; J(F) \,+\, \frac{\rho}{2}\,\|F \,-\, S^k\|_F^2$$

where

$$\begin{aligned} S^k \;&=\; U^T\,\left(L_k^k \,+\, (1/n)\,\mathbb{1}\mathbb{1}^T L_p \,-\, (1/\rho)\,\Lambda^k\right)\,U \\ &=\; U^T\,\left(L_k^k \,-\, (1/\rho)\,\Lambda^k\right)\,U. \end{aligned}$$

We employ the Anderson-Moore method to solve this problem [32]. This algorithm converges faster compared to the gradient method and its implementation is simpler compared to Newton method [28].

We next summarize the first- and second-order derivatives of the objective function J. The second order approximation of the smooth part of objective function J around $\bar{F}$ is given by

$$J(\bar{F}+\tilde{F}) \approx J(\bar{F}) + \left\langle \nabla_F J(\bar{F}), \tilde{F} \right\rangle + \frac{1}{2}\left\langle \nabla_F^2 J(\bar{F},\,\tilde{F}), \tilde{F} \right\rangle.$$

For related developments we refer the reader to [32].

Proposition 2: The gradient and the Hessian of J at $\bar{F}$ are determined by

$$\begin{aligned} \nabla J(\bar{F}) &= 2\,(\bar{R}\,F \,-\, (1/n)\,U^T R\,\mathbb{1}\mathbb{1}^T\,L_p\,U \,-\, P)\,L \\ \nabla^2 J(\bar{F},\,\tilde{F}) &= 2\,\Big((\bar{R}\,\tilde{F} \,-\, \tilde{P})\,L \\ &\qquad + (\bar{R}\,F - (1/n)\,U^T R\,\mathbb{1}\mathbb{1}^T\,L_p\,U - P)\,\tilde{L}\Big) \end{aligned} \tag{12}$$

where $\bar{R} = U^T R\,U$ and the matrix P is given by (8) and is the observability Gramian. The matrix L is the controllability Gramian and is determined by

$$(U^T\,L_p\,U \,+\, F)\,L \,+\, L\,(U^T\,L_p\,U \,+\, F)^T \;=\; U^T\,U \tag{13}$$

where $U^T U = I$ is identity. The matrices $\tilde{P}$ and $\tilde{L}$ are the solutions to the following Lyapunov equations

$$\begin{aligned} &(U^T\,L_p\,U \,+\, F)\,\tilde{L} + \tilde{L}\,(U^T\,L_p\,U \,+\, F) = -\,\tilde{F}\,L \,-\, L\,\tilde{F} \\ &(U^T\,L_p\,U \,+\, F)^T\tilde{P} \,+\, \tilde{P}\,(U^T\,L_p\,U \,+\, F) \;= \\ &\qquad \tilde{F}^T(\bar{R}\,F - \,P \,-\, (1/n)\,U^T R\,\mathbb{1}\,\mathbb{1}^T Lp\,U) + \\ &\qquad (F^T\,\bar{R} \,-\, P \,-\, (1/n)\,U^T L_p^T\mathbb{1}\,\mathbb{1}^T R\,U)\,\tilde{F}. \end{aligned}$$

By setting $\nabla_F \mathcal{L}_\rho = \nabla_F J + \rho\,(F - S^k) = 0$, we obtain

$$2\left(\bar{R}\,F - (1/n)\,U^T R\,\mathbb{1}\mathbb{1}^T\,L_p\,U - P\right)\,L + \rho\,(F - S^k) \;=\; 0. \tag{14}$$

The necessary conditions for the optimality of $\mathcal{L}_\rho(F,\,L_k;\,\Lambda)$ are given by (8), (13) and (14).

The Anderson-Moore method solves the F-minimization step iteratively. In each iteration, the algorithm starts with a stabilizing F and solves two Lyapunov equations and one Sylvester equation. It first solves the Lyapunov equations (8), (13) with a fixed F to obtain controllability and observability Gramians L and P, respectively. Then it solves the Sylvester equation (14) for F with fixed L and P. Then we use Newton's method to find the descent direction between two consecutive steps by utilizing (12). We next employ a line search strategy to determine an appropriate step-size in order to guarantee convergence to a stationary point and the closed-loop stability.

B. L_k-minimization step

Using the expression for the augmented Lagrangian (11), we can write the L_k-minimization step as

$$L_k^{k+1} \;=\; \underset{L_k}{\text{argmin}}\; \gamma\,\|L_k\|_1 \,+\, \frac{\rho}{2}\,\|L_k \,-\, T^k\|_F^2$$

where

$$T^k \;=\; U\,F^{k+1}\,U^T - (1/n)\,\mathbb{1}\mathbb{1}^T L_p \,+\, (1/\rho)\,\Lambda^k.$$

The solution is given by soft-thresholding

$$L_k(i,j)^{k+1} = \begin{cases} (1 - \dfrac{\upsilon}{|T^k(i,j)|})\, T^k(i,j), & |T^k(i,j)| > \upsilon \\ 0, & \text{otherwise} \end{cases}$$

where $\upsilon = (\gamma/\rho)\, W(i,j)$.

Convergence analysis of ADMM for convex problems can be found in [30]. For non-convex problems, the quadratic term in the augmented Lagrangian locally convexify the problem for a large ρ. A recent result on convergence analysis of ADMM for a family of non-convex problems can be found in [33]. Computational results also show that ADMM works well when ρ is sufficiently large [34], [35].

V. Computational experiments

A. Synthetic example

In this section, we employ our customized algorithm based on ADMM to add certain number of edges to a given directed network. The plant network is a randomly generated graph with n nodes and with edge weight $l_i \in (0,1)$ for the ith edge which is obtained randomly.

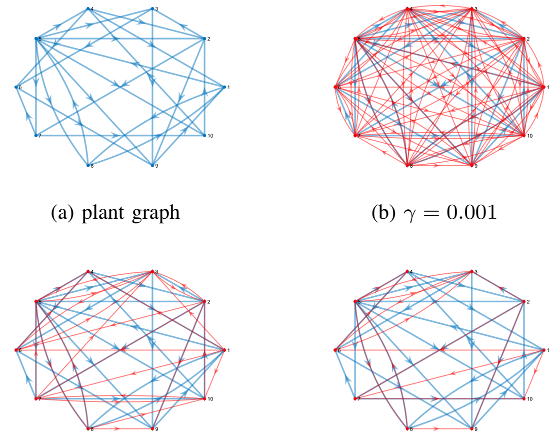
We set $R = I$ and choose the state weight that penalizes the mean-square deviation from the network average,

$$Q = I - (1/n)\, \mathbb{1}\mathbb{1}^T.$$

We solve the sparsity-promoting optimal control problem (9) for controller graph for 100 logarithmically-spaced values of $\gamma \in [0.001,\, 3]$ using the path-following iterative reweighted algorithm as a proxy for inducing sparsity [31]. We set the weights to be inversely proportional to the magnitude of the solution L_k to (9) at the previous value of γ. We choose $\varepsilon = 10^{-3}$ in (4) and initialize weights for $\gamma = 0.01$ using the solution to (9) with LQR state-feedback matrix. Topology design is followed by a polishing step that computes the optimal weights of identified edges; see Section I-A.

B. Performance improvement

The randomly generated plant graph in Fig. 1a is a directed graph with $n = 20$ nodes. The plant graph is weakly connected but unbalanced. Thus, it will not converge to the initial nodes average value. In order to reach consensus and improve the performance, adding edges to the plant network is required. Figure 1 illustrates that by increasing γ, the controller graph becomes sparser. The number of added edges to the network is equal to the number of nonzero off-diagonal elements in the controller. Specifically, for $\gamma = 0.001$, the number of nonzero elements in the controller graph is 84, among which 74 edges are added to the original network, and the other 10 nonzero elements represent the diagonal elements. By increasing γ to 0.0788, there are 33 nonzero elements are in the controller graph, and only 25 edges are added to the plant network. It is noteworthy that the Laplacian matrix of the controller graph can not be a zero matrix, because the plant network is an unbalance graph and a nonzero Laplacian matrix of the controller is needed to make the closed-loop graph Laplacian balanced.



(a) plant graph (b) $\gamma = 0.001$

(c) $\gamma = 0.0788$ (d) $\gamma = 3$

Fig. 1: Topologies of the plant (blue lines) and controller graphs (red lines) for a randomly generated weighted network.

Figure 2 shows that the closed-loop performance deteriorates and the number of nonzero elements in the controller graph Laplacian L_k decreases as γ increases. As shown in Fig. 2b, relative to the optimal LQR controller, L_c, the $\mathcal{H}_2$ performance loss decreases as the sparsity of the controller graph Laplacian matrix L_k increases. In particular, for $\gamma = 3$, there are only 24 nonzero elements in the controller graph. This is equivalent to have only 16 added edges. The identified sparse controller in this case uses only 0.057% of the edges, relative to the optimal LQR controller, i.e., $\mathbf{card}(L_k)/\mathbf{card}(L_c) = 29.629\%$ and achieves a performance loss of 15.118%, i.e., $(J - J_c)/J_c = 15.118\%$.

VI. Concluding remarks

We consider the ℓ_1 regularized version of optimal control problem for adding edges to directed consensus networks in order to reach consensus and optimally enhance performance. Although the given plant network is not necessarily balanced, in order to reach agreement, we restrict the closed-loop graph Laplacian to be balanced. The performance is measured by the $\mathcal{H}_2$ norm from the disturbance to the output of the closed-loop network. In general, this problem is a combinatorial search problem. We use sparsity promoting optimal control framework and introduce weighted ℓ_1 regularization as a proxy for promoting sparsity of the controller. By exploiting structure of the problem, we develop an algorithm based on ADMM. An example is provided to demonstrate the utility of the developed algorithm. In the future work, we would like to extend the existing framework to the case where the directed network is arbitrarily unbalanced, and the corresponding performance index operates in the subspace that spans the null space of the left eigenvector of the closed-loop Laplacian matrix.

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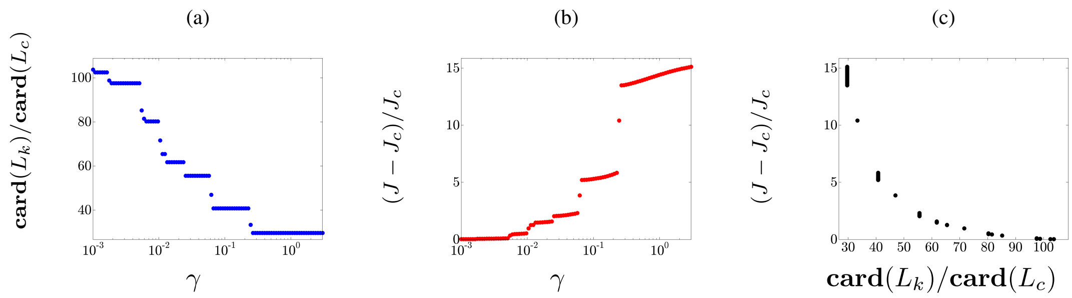

Fig. 2: (a) Sparsity level; (b) performance degradation; and (c) the optimal tradeoff between the performance degradation and the sparsity level of the optimal sparse L_k compared to the optimal centralized controller L_c. The results are obtained for the weighted random plant network with topology shown in Fig. 1a.

References

[1] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[2] C. W. Reynolds, "Flocks, herds and schools: A distributed behavioral model," *ACM SIGGRAPH Computer Graphics*, vol. 21, no. 4, pp. 25–34, 1987.

[3] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, p. 1226, 1995.

[4] M. H. DeGroot, "Reaching a consensus," *J. Amer. Statist. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974.

[5] B. Golub and M. Jackson, "Naive learning social networks and the wisdom of crowds," *Am. Econ. J.*, vol. 2, no. 1, pp. 112–149, 2010.

[6] F. Dörfler, M. R. Jovanovic, M. Chertkov, and F. Bullo, "Sparsity-promoting optimal wide-area control of power networks," *IEEE Trans. Power Syst.*, vol. 29, no. 5, pp. 2281–2291, September 2014.

[7] X. Wu, F. Dörfler, and M. R. Jovanovic, "Decentralized optimal control of inter-area oscillations in bulk power systems," in *Proceedings of the 54th IEEE Conference on Decision and Control*, 2015, pp. 5532–5537.

[8] X. Wu, F. Dörfler, and M. R. Jovanovic, "Input-output analysis and decentralized optimal control of inter-area oscillations in power systems," *IEEE Trans. Power Syst.*, vol. 31, no. 3, pp. 2434–2444, 2016.

[9] V. Preciado, M. Zargham, C. Enyioha, A. Jadbabaie, and G. Pappas, "Optimal vaccine allocation to control epidemic outbreaks in arbitrary networks," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, 2013, pp. 7486–7491.

[10] E. Ramírez-Llanos and S. Martínez, "A distributed nonlinear dynamics for virus spread control," *Automatica*, vol. 76, pp. 41–48, 2017.

[11] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[12] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automat. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[13] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Automat. Control*, vol. 50, no. 2, pp. 169–182, 2005.

[14] L. Xiao, S. Boyd, and S.-J. Kim, "Distributed average consensus with least-mean-square deviation," *J. Parallel Distrib. Comput.*, vol. 67, no. 1, pp. 33–46, 2007.

[15] A. Ghosh, S. Boyd, and A. Saberi, "Minimizing effective resistance of a graph," *SIAM Review*, vol. 50, no. 1, pp. 37–66, 2008.

[16] D. Zelazo and M. Mesbahi, "Edge agreement: Graph-theoretic performance bounds and passivity analysis," *IEEE Trans. Automat. Control*, vol. 56, no. 3, pp. 544–555, 2011.

[17] D. Zelazo, S. Schuler, and F. Allgöwer, "Performance and design of cycles in consensus networks," *Syst. Control Lett.*, vol. 62, no. 1, pp. 85–96, 2013.

[18] I. Poulakakis, G. F. Young, L. Scardovi, and N. E. Leonard, "Information centrality and ordering of nodes for accuracy in noisy decision-making networks," *IEEE Trans Automat Contr*, vol. 61, no. 4, pp. 1040–1045, 2016.

[19] M. Fardad, F. Lin, and M. R. Jovanovic, "Sparsity-promoting optimal control for a class of distributed systems," in *Proceedings of the 2011 American Control Conference*, 2011, pp. 2050–2055.

[20] S. Hassan-Moghaddam and M. R. Jovanovic, "Customized algorithms for growing connected resistive networks," in *Proceedings of the 10th IFAC Symposium on Nonlinear Control Systems*, 2016, pp. 986–991.

[21] F. Lin, M. Fardad, and M. R. Jovanovic, "Identification of sparse communication graphs in consensus networks," in *Proceedings of the 50th Annual Allerton Conference on Communication, Control, and Computing*, 2012, pp. 85–89.

[22] S. Hassan-Moghaddam and M. R. Jovanovic, "Topology design for stochastically-forced consensus networks," *IEEE Trans. Control Netw. Syst.*, 2017, arXiv:1506.03437v3.

[23] S. Hassan-Moghaddam and M. R. Jovanovic, "An interior point method for growing connected resistive networks," in *Proceedings of the 2015 American Control Conference*, 2015, pp. 1223–1228.

[24] A. Ghosh and S. Boyd, "Growing well-connected graphs," in *Proceedings of the 45th IEEE Conference on Decision and Control*, 2006, pp. 6605–6611.

[25] R. Dai and M. Mesbahi, "Optimal topology design for dynamic networks," in *50th IEEE Conference on Decision and Control*, 2011, pp. 1280–1285.

[26] X. Wu and M. R. Jovanovic, "Sparsity-promoting optimal control of consensus and synchronization networks," in *Proceedings of the 2014 American Control Conference*, 2014, pp. 2948–2953.

[27] T. Summers, I. Shames, J. Lygeros, and F. Dörfler, "Topology Design for Optimal Network Coherence," in *Proceedings of the 14th European Control Conference*, July 2015.

[28] F. Lin, M. Fardad, and M. R. Jovanovic, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Automat. Control*, vol. 58, no. 9, pp. 2426–2431, 2013.

[29] M. R. Jovanovic and N. K. Dhingra, "Controller architectures: tradeoffs between performance and structure," *Eur. J. Control*, vol. 30, pp. 76–91, July 2016.

[30] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundation and Trends in Machine Learning*, vol. 3, no. 1, pp. 1–124, 2011.

[31] E. J. Candès, M. B. Wakin, and S. P. Boyd, "Enhancing sparsity by reweighted ℓ_1 minimization," *J. Fourier Anal. Appl*, vol. 14, pp. 877–905, 2008.

[32] T. Rautert and E. W. Sachs, "Computational design of optimal output feedback controllers," *SIAM J. Optim.*, vol. 7, no. 3, pp. 837–852, 1997.

[33] M. Hong, Z. Q. Luo, and M. Razaviyayn, "Convergence analysis of alternating direction method of multipliers for a family of nonconvex problems," *SIAM J. Optim.*, vol. 26, no. 1, pp. 337–364, 2016.

[34] M. Fortin and R. Glowinski, *Augmented Lagrangian method: Application to the numerical solution of boundary-value problems*. North-Holland, 1983.

[35] R. Glowinski and P. Le Tallec, *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics*. SIAM, 1989.