Understanding viscoelastic flow instabilities using the Oldroyd-B model

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Abstract

The Oldroyd-B model has been used extensively to predict a host of instabilities in shearing flows of viscoelastic fluids, polymer solutions in particular. The present review, written on the occasion of the birth centenary of James Oldroyd, provides an overview of the instabilities found in major classes of shearing flows. The latter consist of the canonical rectilinear shearing flows including plane Couette, plane Poiseuille and pipe Poiseuille flows, viscometric shearing flows with curved streamlines which include the Taylor-Couette, cone-and-plate and parallel-plate geometries, non-viscometric shearing flows with an underlying extensional flow topology, and multilayer shearing flows; the underlying focus in all these cases is on results obtained using the Oldroyd-B model, and their relation to the actual instability. All the three commonly used tools of stability analysis, viz., modal linear stability, nonmodal stability, and weakly nonlinear stability analyses are discussed, with supporting evidence from experiments and numerical simulations as appropriate. Despite only accounting for a shear-rate-independent viscosity and first normal stress coefficient, the Oldroyd-B model is able to qualitatively predict the majority of instabilities in the aforementioned shearing flows. The review also highlights, where appropriate, open questions in the area of viscoelastic stability.

Keywords: Oldroyd-B fluid; purely elastic instability; elastic turbulence; elasto-inertial turbulence; nonmodal stability; nonlinear stability.

1. Introduction

Compelling differences between Newtonian and viscoelastic flow phenomena in the same geometry have been well highlighted in textbooks [1], and this contrast also applies to instabilities occurring in the same base-flow configuration. Viscoelastic flows are prone to instabilities that arise due to elasticity, in contrast to Newtonian flows where inertial effects usually play a destabilizing role [2]. Initial interest in viscoelastic flow instabilities was driven by the need to understand them, so as
to prevent their occurrence during polymer processing operations [3, 4], since the latter often placed restrictions on processing rates; fluid inertia is usually negligible in these processes, and the focus is therefore on purely elastic instabilities. For instance, during extrusion of highly viscous entangled polymer melts, the extrudate often exhibits a spiral or wavy distortion, a phenomenon referred to as ‘melt fracture’ [5], and is thought to occur via a hydrodynamic instability, although physicochemical effects such as wall slip likely play a role too [6]. A second motivation for studying viscoelastic flow instabilities arose from their discovery in the standard rheometric geometries (e.g. the Couette, cone-and-plate and parallel-plate set ups); for instance, see [7, 8]. The occurrence of purely elastic instabilities, in dilute polymer solutions, subject to simple curvilinear shearing flows in rheometric devices, hampered the latter’s use for purposes of rheological characterization. The elastic instabilities above have their origins in normal stress differences present in viscoelastic shear flows. For the rheometric devices in particular, the first normal stress difference leads to a hoop stress, on account of streamline curvature, which drives the instability.

Viscoelastic flow instabilities can also be beneficial depending on the scenario. For instance, with regard to the rheometric example above, increasing the shear rate causes the initial elastic instability to eventually saturate in a complex disorderly flow state termed ‘elastic turbulence’ (ET) [9, 10, 11]. There have also been reports of similar ET-like states in rectilinear flows of dilute polymer solutions through micro channels, especially when the flow is perturbed by finite-amplitude obstacles at the channel inlet [12, 13, 14, 15], suggesting the nonlinear nature of the initial transition. The inaccessibility of Newtonian turbulence on microfluidic scales (owing to the low Reynolds numbers typically prevalent in such flows) implies that the ET state, in either of the two cases above, can instead be exploited towards increasing mixing efficiencies, as has indeed been demonstrated in earlier efforts [10].

While the above instances probed the low Reynolds number (Re) regime where fluid inertial effects are negligible, there have also been many reports of ‘early turbulence’ in pipe flow of polymer solutions at values of Re lower than the Newtonian threshold [16, 17, 18, 19, 20, 21, 22]; although, these studies were not systematically corroborated in the subsequent literature. The recent experiments of Hof and coworkers [23, 24], involving pipe flow of dilute polymer solutions at concentrations below or close to the overlap value, have unambiguously demonstrated that transition from the laminar state does occur at Re ~ 1000, and thence, much lower than the Newtonian threshold. The ensuing flow is neither laminar, nor does it resemble Newtonian turbulence, and was therefore christened ‘elasto-inertial turbulence’ (EIT) to emphasize the importance of both fluid elasticity and inertia, thereby also contrasting it from the ET state discussed above. This EIT state was further shown to be linked [25] to the asymptotic maximum drag reduction (MDR) regime, a universal state that arises with the progressive addition of polymers to turbulent Newtonian pipe flow [26]. This latter link is an important one. The phenomenon of turbulent drag reduction [27] is undoubtedly one of the most spectacular manifestations of viscoelasticity, and while there exists a large body literature in this regard [28, 29, 30], the prevailing viewpoint regards the aforementioned MDR regime as a drag-reduced state accessible only from the Newtonian turbulent state; even relatively recent dynamical-systems-based interpretations have attempted to understand MDR in terms of the existence of essentially dynamical-systems-based instabilities, modified by elasticity [31, 32]. As a consequence, advances in viscoelastic stability and drag reduction have occurred largely independently with very little cross-pollination of theoretical viewpoints. However, as we discuss later in this review, the aforementioned experiments, together with recent theoretical work [33], show that the MDR regime, at least for moderate Re, can be viewed as a ‘drag-enhanced’ state arising from an elastoinertial instability of the laminar state.

The present review article, written on the occasion of the birth centenary of James Oldroyd who proposed the now-eponymous constitutive equation [34], attempts to provide a state-of-the-art summary of the understanding of flow instabilities using the Oldroyd-B model. While there have been many earlier review articles on the subject of viscoelastic flow instabilities [3, 4, 7, 8], the present review focuses on the developments over the last two decades. Further, in contrast to some of the review articles above which have focused almost exclusively on the hoop-stress-driven elastic instabilities that arise in the curvilinear rheometric geometries, this review covers instabilities in both the canonical rectilinear and curvilinear shearing flows, with the latter including viscometric and non-viscometric flow configurations. In fact, the experimental observations in the preceding paragraph, pertaining to the moderate-to-high Re regimes, have spawned renewed interest in viscoelastic instabilities that occur in the canonical rectilinear shearing flows, including pressure-driven flow through a pipe or a channel, and the present review lays a greater emphasis on these more recent findings. The review by Renardy and Thomases [35] of this special issue presents a different perspective, by focusing on open mathematical challenges related to the Oldroyd-B model. Another recent multi-author review article [36], based on the virtual workshop on viscoelastic flow instabilities and elastic turbulence organized by the Princeton Center for Theoretical Sciences, also provides a state-of-the-art summary of the various challenges in this broad area.

It is worth recalling that Oldroyd, in his seminal 1950 paper [34], proposed a constitutive model for viscoelastic flows purely from a continuum viewpoint, by requiring the model to satisfy the principle of material frame indifference. This stipulates that a constitutive relation should not depend on translation, rotation or acceleration of the reference frame; to this end, Oldroyd introduced the ‘upper-convected derivative’. While the usual material derivative in fluid mechanics denotes the instantaneous rate of change of a fluid property (e.g. velocity, temperature) in a reference frame that translates with a given fluid particle, the upper convected derivative denotes the rate of change in a reference frame that, in addition, deforms (affinely) with the fluid motion.

Interestingly, the Oldroyd-B constitutive equation can also be derived from a coarse-grained, mesoscopic model wherein
In the limit of zero solvent viscosity, \( \eta_s = 0 \), the Oldroyd-B model reduces to the UCM model. For steady simple shear flow, the Oldroyd-B model predicts a shear-rate-independent viscosity and first normal stress coefficient, and yields a zero second-normal stress difference. Consequently, the Oldroyd-B model is not applicable for strongly shear-thinning systems such as polymer melts or water-based dilute polymer solutions; nor can it describe phenomena ascribable exclusively to an \( N_2 \) which, in principle, include spanwise instabilities in a rectilinear shearing flow. While the model correctly predicts an extension-thickening behavior, it also predicts an unbounded growth of the extensional viscosity beyond a threshold extension rate. The latter is due to the infinite extensibility of the dumbbells in the microscopic picture underlying the Oldroyd-B model; upon inclusion of a nonlinear spring force, the extensional viscosity saturates to a large but finite value in accord with experimental observations [37].

Even for the simplest shear flows driven by the motion of rigid boundaries, and that are characterized by a single length (\( H \)) and velocity (\( V \)) scale, the stability of an Oldroyd-B fluid is governed by three dimensionless parameters: the Reynolds number \( Re = HVp/\eta_s \), the Weissenberg number \( Wi = AV/\eta_s \) which is the product of the polymer relaxation time and a typical shear rate, and the ratio of solvent to solution viscosity \( \beta = \eta_p/\eta_s \); here, \( \rho \) and \( \eta_s \) refer to, respectively, the density and total viscosity of the polymer solution. We note in passing that the capillary number, denoting the ratio of viscous to surface tension forces, will become relevant for viscoelastic flows with a free surface. Alternatively, the ‘elastocapillary’ number, which is the ratio of Weissenberg and capillary numbers, and measures the relative importance of elastic and capillary forces, is also used [38]. For internal flows, in lieu of the Weissenberg number, the flow-independent elasticity number \( E = Wi/Re = \lambda\eta_s/\rho H^2 \) is used below, when describing elastoinertial instabilities, and represents the ratio of the polymer relaxation time to the momentum diffusion timescale. In some applications, the Deborah number (\( De \)), which is the ratio of the relaxation time to a characteristic flow timescale \( T \), is also used; here, \( T \) can be either the residence time of a fluid element or the characteristic timescale for a flow transient such as the time period of an oscillatory shear flow. The Deborah number and Weissenberg numbers, often used interchangeably, are sometimes related to each other by a dimensionless geometric factor such as the aspect ratio of a particular flow configuration [39]; the latter will be seen to be the case for the viscometric curvilinear shearing flows examined in section 3.

In light of the above, the subject of transition in viscoelastic shearing flows, unlike their Newtonian counterparts, is not a ‘single problem’. Instead, one may have different asymptotic regimes including weakly-elastic inertial-dominated flows (\( Wi \ll 1, Re \gg 1 \)), strongly-elastic inertialess flows (\( Wi \sim O(1), Re \ll 1 \)), and elasto-inertial (\( Wi, Re > 1, E \sim O(1) \)) flows where both inertia and elasticity are equally important. In addition, the solvent viscosity ratio \( \beta \), which is a proxy for polymer concentration, allows one to span the regimes from ultra-dilute polymer solutions (\( \beta \to 1 \)) to polymer melts (\( \beta \to 0 \)). Importantly, the higher-dimensional nature of the parameter space governing the stability of an Oldroyd-B fluid implies that transition from the steady laminar state, to states with nontrivial spatiotemporal dynamics, can occur via multiple pathways in the \( Re-Wi-\beta \) space; we return to the aspect of multiple transition scenarios in Section 2.3.

The first step in analyzing the stability of a laminar flow is to consider its response to infinitesimal disturbances, which allows for the linearization of the governing equations about the laminar base state. Within this linear stability framework, there are two different approaches. The classical approach is modal stability, and involves expressing the perturbation fields in the normal mode form with an exponential dependence in time, in turn leading to an eigenvalue problem for the growth rate as a function of the wavenumber and other relevant dimensionless parameters. According to the convention usually followed, a change in sign of the imaginary part of the eigenvalue (the growth rate), from negative to positive, corresponds to the onset of instability. For sheared base states in particular, the non-normality of the differential operator governing linear stability implies that the aforementioned modal stability analysis only pertains to the asymptotic behavior for long times when the evolution is dominated by exponentially growing unstable modes [2]. Even in the absence of unstable modes, however, small amplitude disturbances can grow algebraically for shorter times. The machinery for a detailed analysis of this so-called transient growth is now well developed, and has been extensively applied.
in the Newtonian context [40].

Going beyond linear stability, finite amplitude disturbances are often considered within the framework of an amplitude expansion, an approach that originated in the efforts of Stuart [41] and Watson [42]. In fact, the nonmodal and nonlinear stability approaches have acquired prominence owing to the failure of the classical modal approach to explain transitions in any of the canonical Newtonian shearing flows (plane Couette, plane Poiseuille and pipe flows). All the three approaches above are covered in this review within the context of the Oldroyd-B model. It is important to note that, in recent times, direct numerical simulations of viscoelastic flows complement the aforementioned approaches, providing detailed structural information in the nonlinear regime, although the computational expense implies that the parameter space explored by such simulations is often restricted.

The rest of this review article is organized as follows: We first begin with instabilities in simple rectilinear flows, and discuss the nature of the viscoelastic spectrum in the inertialless limit in Sec. 2.2, and point out that rectilinear flows are generally linearly stable in the $Wi$-space, with the exception of plane Poiseuille flow which becomes unstable for $Wi \gg 1$ and $\beta \to 1$. In Sec. 2.3, the finite-$Re$ elasto-inertial spectrum for the canonical rectilinear shear flows is discussed, and it is shown that while plane Couette flow is always stable in the $Re$-$Wi$-space, plane- and pipe-Poiseuille flows are unstable in significant domains of this space. The nature of instabilities in these flows is discussed briefly, and we further provide an overview of various possible transition scenarios in viscoelastic flows in the $Re$-$Wi$-space. In Sec. 2.5, we discuss instabilities in two-layer flows of viscoelastic fluids wherein a jump in the first normal stress difference leads to a novel instability absent for Newtonian two-layer flows. This section also includes a brief summary of instabilities in shear-banded flows which are closely related to interfacial instabilities in two-layer flows.

Purely elastic instabilities in curvilinear viscoelastic flows are surveyed in Sec. 3. Here, we first discuss (Sec. 3.1) the role of elasticity on the Newtonian (centrifugal) instability in the Taylor-Couette geometry, before moving on to a discussion of the purely elastic instability in the same geometry (Sec. 3.2). In fact, the nonmodal and nonlinear stability approaches have acquired prominence owing to the failure of the classical modal approach to explain transitions in any of the canonical Newtonian shearing flows (plane Couette, plane Poiseuille and pipe flows). All the three approaches above are covered in this review within the context of the Oldroyd-B model. It is important to note that, in recent times, direct numerical simulations of viscoelastic flows complement the aforementioned approaches, providing detailed structural information in the nonlinear regime, although the computational expense implies that the parameter space explored by such simulations is often restricted.

In this section, we examine the stability of canonical rectilinear shear flows, comprising plane Couette, plane Poiseuille and pipe Poiseuille flows, from the modal perspective. Further results from the non-modal viewpoint are presented later in Sec. 5. A prerequisite to understanding the non-trivial structure of the full elastoinertial spectrum, and associated instabilities, is an understanding of the spectra arising from inertial and elastic forces acting separately. Thus, we begin with a discussion of the inertialless elastic spectrum associated with an Oldroyd-B fluid. The latter subsection highlights the recent, and unexpected, discovery of a purely elastic instability in plane Poiseuille flow.

### 2. Rectilinear shearing flows: Results from modal analyses in the $Re$-$Wi$-$\beta$ space

In this section, we examine the stability of canonical rectilinear shear flows, comprising plane Couette, plane Poiseuille and pipe Poiseuille flows, from the modal perspective. Further results from the non-modal viewpoint are presented later in Sec. 5. A prerequisite to understanding the non-trivial structure of the full elastoinertial spectrum, and associated instabilities, is an understanding of the spectra arising from inertial and elastic forces acting separately. Thus, we begin with a discussion of the inertialless elastic spectrum associated with an Oldroyd-B fluid. The latter subsection highlights the recent, and unexpected, discovery of a purely elastic instability in plane Poiseuille flow.

#### 2.1. The Newtonian Spectrum

It is useful to first recall features of the $Re$-dependent Newtonian eigenspectrum for the canonical rectilinear shearing flows [40]. By way of illustration, consider plane Poiseuille flow in the $x$-direction with velocity profile $U(y) \propto (1 - (y/H)^2)$;
here, $2H$ is the separation between the walls with the spanwise base-state vorticity pointing along the $z$-direction. In the linear stability analysis, and within the modal framework, the perturbed velocity field is assumed to be of the form $[U(y) + v'_{\chi}(x, y, z, t), v'_{\chi}(x, y, z, t), v'_{\chi}(x, y, z, t)]$, where the primes denote the perturbation components which are taken to be Fourier modes $v'_{\chi}(x, y, z, t) = \hat{v}_{\chi}(y) \exp(ikx + ilz - i\omega t)$; here, $k$ and $l$ are the wavenumbers in the $x$ and $z$-directions, $\hat{v}_{\chi}$ are the shapes (eigenfunctions) of the perturbations in the $y$ direction, and the complex wavespeed $c = c_r + ic_i$ is the (unknown) eigenvalue. If $c_i > 0$, the flow is temporally unstable, and if $c_i < 0$, the flow is asymptotically stable in that perturbations decay away exponentially for sufficiently long times. On account of Squire’s theorem, which remains valid for both Newtonian [2] and Oldroyd-B [43] fluids, it is sufficient to restrict attention to two-dimensional perturbations ($l = 0$). Note, however, that the theorem is applicable only within the normal-mode ansatz, and it is possible to have nonmodal growth of three-dimensional perturbations, as will indeed be seen in Sec. 5.

Figure 1 shows the Newtonian spectrum at $Re = 10^4$ and $k = 1$, which has a characteristic ‘Y-shaped’ structure in the $c_r - c_i$ plane. It consists of (i) the ‘A branch’, corresponding to wall modes with phase speeds approaching zero with decreasing decay rates, (ii) the ‘P branch’ corresponding to center modes with phase speeds approaching the base-state maximum, again with decreasing decay rates, and (iii) the ‘S branch’ with modes having a common phase speed intermediate between the wall and the base-state maximum, and with decay rates asymptoting to infinity. The aforementioned Y-shaped structure only emerges above a threshold $Re$. Below this threshold, whose value is dependent on the particular base-state profile, the Newtonian spectrum comprises only of the S-modes. After its emergence, however, the Y-locus itself remains invariant, with the density of modes along each of the three branches increasing with increasing $Re$. As evident from Figure 1, the first mode belonging to the A branch is unstable for the chosen parameters, and corresponds to the well known Tollmien-Schlichting (TS) instability. On account of an exact antisymmetry about the centerline, plane Couette flow does not possess a P branch; instead, both arms of the Y correspond to wall modes.

2.2. The purely elastic spectrum and the elastic centermode instability ($Re = 0, Wi \neq 0$)

In the absence of inertia, the governing equations in the Newtonian case reduce to the Stokes equations. For Stokes flows driven by the motion of rigid boundaries, the quasi-steady nature of the governing equations and boundary conditions implies there can be no associated spectrum. With reference to the preceding subsection, the S-modes in the finite-$Re$ Newtonian spectrum recede down to negative infinity in the Stokes limit. In contrast, the stress relaxation term in the Oldroyd-B equation provides for an intrinsic time scale, and as discussed below, gives rise to a non-trivial spectrum even in the absence of inertia. We discuss below the nature of this inertialess spectrum whose structure is a function of $Wi$ and $\beta$. Unstable modes in this spectrum correspond to purely elastic instabilities.

The simplest flow is, of course, plane Couette flow. The elastic plane Couette eigenspectrum was first examined in the UCM limit ($\beta = 0$) by Gorodtsov & Leonov [44], who showed, analytically, that there is a continuous spectrum (abbreviated as ‘CS’ henceforth) along with two discrete modes, all of which are stable. We refer to the two stable discrete modes as the zero-Reynolds number Gorodtsov-Leonov (‘ZRGL’). The elastic continuous spectrum is a generic presence, and owes its origin to the spatially local evolution of the polymeric stress (in accordance with the simple fluid paradigm); the CS eigenfunctions decay exponentially on the scale of the polymer relaxation time. The above picture was generalised to the Oldroyd-B fluid by Wilson, Renardy & Renardy in 1999 [45]. While the flow continues to remain stable, the spectrum becomes considerably more complicated, pointing to the singular nature of the UCM limit. The continuous spectrum associated with the UCM fluid is qualitatively unchanged, as are the two ZRGL modes. But, there is an additional stable continuous spectrum which moves in from $c_i = -\infty$ as $\beta$ increases from zero. Further, unlike the UCM-continuous spectrum, this so-called viscous-continuous spectrum is associated with a branch cut, and discrete eigenvalues can emerge from, or disappear into, the viscous continuous spectrum with varying $\beta$. The number of discrete modes increases with decreasing $\beta$, with there existing an infinite sequence of discrete modes in the limit $\beta \to 0$; for moderate $\beta$, all of these discrete modes are all more stable than the viscous continuous modes.

In the aforementioned effort, the authors also analyzed the spectrum of plane Poiseuille flow. In the UCM limit, the authors showed that the equivalent of the Gorodtsov–Leonov spectrum has six discrete modes (instead of the two found for plane Couette flow above); numerical computations showed that the discrete modes continued to remain stable. The addition of a solvent viscosity, leading to the Oldroyd-B model, again resulted in a spectrum similar to plane Couette flow; thus, a second stable continuous spectrum arose for any non-zero $\beta$, along with a large family of stable discrete modes which disappear into this viscous-continuous spectrum as $\beta$ is increased.

The preceding two paragraphs had, until very recently, represented our understanding of the elastic stability characteristics of rectilinear shearing flows. Thus, although never proven, such shearing flow configurations have nonetheless been thought to be linearly stable (this is the scenario even for Newtonian pipe flow; although in this case observations clearly point to nonlinear mechanisms). As a consequence, purely elastic linear instabilities are synonymous with curvilinear flow configurations [7], with the analog of such instabilities in rectilinear flows thought to have a nonlinear character (see section 6.1); in either case, streamline curvature is regarded as a necessary prerequisite for instability [47]. However, recent work by Khalid, Shankar, and Subramanian [48] has demonstrated that inertialess plane Poiseuille flow of an Oldroyd-B fluid is, in fact, linearly unstable at sufficiently high $Wi$(of $O(1000)$), and for $\beta > 0.99$. Figure 2 shows the structure of the elastic spectrum at such high $Wi's$, and Fig. 3 shows neutral curves, which are in the form of the unstable tongues in the $Wi - k$ plane; the instability appears to arise due to a critical-layer mechanism,
in contrast to the hoop-stress-based mechanism that is operative in curvilinear shearing flows. The work of Buza, Page, and Kerswell [49], using the FENE-P model has confirmed that the aforementioned instability continues to exist with the incorporation of finite extensibility. The said authors have also carried out a weakly nonlinear stability analysis to show that the instability in the creeping-flow limit is subcritical, pointing to a potentially larger unstable region in the $Wi - \beta$ plane. At present, it is not yet clear whether this instability is directly relevant to recent experimental observations from the Paulo Arratia [13, 14] and Victor Steinberg groups [15] which clearly indicate an EL-like state for $Re \ll 1$ even in rectilinear shearing flows, albeit at $\beta \sim 0.5-0.7$.

2.3. The elasto-inertial spectrum ($Re, Wi \neq 0$): the center- and wall-mode instabilities at finite $Re$

The work of Gorodtsov and Leonov [44], referred to in the section above in the context of the inertialless elastic spectrum, also analyzed plane Couette flow of a UCM fluid for small but finite $Re$. In addition to the aforementioned pair of stable ZRGL modes, the authors found a new class of modes, corresponding to damped shear waves in a viscoelastic fluid with phase speeds of $O(\sqrt{G/\rho})$, $G \sim \eta/\lambda$ being the shear modulus. We refer to this family of modes as the high-frequency-Gorodtsov-Leonov (‘HFGL’) modes, since in the base-state velocity scale, the phase speed (frequency) of the HFGL modes is $O(DeRe)^{-\frac{1}{2}}$, and therefore these modes recede to infinity (parallel to the $c_t$-axis) in the inertialless limit. Although Gorodtsov and Leonov [44] predicted an instability due to the HFGL modes in the limit $k Wi \gg 1$, this was later shown to be incorrect [50]; the HFGL modes remain damped for any finite $Wi$, with $c_t \rightarrow -1/2 Wi$ for $Re \rightarrow 0$. Note that the original elastic continuous spectrum continues to be present at finite $Re$, with the CS-modes having phase speeds in the base-state range of velocities, with decay rates of $c_t = -1/4 Wi$ (this corresponds to the dimensional decay rate equaling the inverse relaxation time, as mentioned in section 2.2). Thus, the elastoinertial spectrum of plane Couette flow of a UCM fluid has been shown [51, 50] to consist of the finite-$Re$ continuation of the ZRGL modes, the elastic continuous spectrum, and the HFGL modes. Although a rigorous proof does not exist, plane Couette flow does appear to be stable in the $Re - Wi$ plane for $\beta = 0$; the conclusion remains unchanged on consideration of an Oldroyd-B fluid. [52, 50, 53, 54]. The stability of viscoelastic plane Couette flow therefore mirrors that of its Newtonian counterpart [2], although there exists a rigorous proof in the latter case (see Romanov (1973)). In summary, there appears no evidence of a linear instability in plane Couette flow in the $Re - Wi - \beta$ space.

However, and in contrast, plane Poiseuille flow of a Newtonian fluid ($Wi = 0$) becomes susceptible to the TS instability [2] at $Re_c \approx 5772$. As already shown in section 2.1, the unstable TS eigenvalue belongs to the A-branch, and is therefore a wall mode. A continuation of this instability is expected for small $Wi$ regardless of $\beta$, including for the case of a UCM fluid. The key question is whether there are new unstable modes in plane Poiseuille flow of a UCM fluid that have an essentially elastic origin, and are therefore absent in the Newtonian limit. This question was first addressed by Porteus and Denn [55], who found three unstable modes for sufficiently high $Re (> 2000)$, only one of which was a continuation of the TS mode; the other two unstable modes are absent in the Newtonian limit. The authors showed that increase in elasticity in the range $0 < E < 10^{-2}$ resulted in a decrease in $Re_c$ from its Newtonian value to $Re_c \sim 2000$. Elasticity was also shown to have a destabilizing effect on one of the other two unstable modes, albeit with limited data. On the other hand, plane Poiseuille flow of a UCM fluid was found to be stable at low $Re$ [56, 52].
Sureshkar and Beris [57] found two different unstable families, of which one was a continuation of the Newtonian TS mode. The critical Reynolds number $Re_c$ showed a nonmonotonic behavior, showing an initial decrease for very small $E_s$ and an eventual decrease at higher $E$. Both the modes analyzed by Sureshkar and Beris, and the initial decrease in $Re_c$ with $E$, are consistent with the earlier results of Porteus and Denn [55].

The recent study of Chaudhary et al. [54] presented a more comprehensive picture of the elasto-inertial spectrum of a UCM fluid, emphasizing features common to both plane Couette and Poiseuille flows. As shown in Fig. 4, for $Re \sim 1000$ and higher, the elastoinertial spectrum for both flows contains: (i) a ballooned manifestation of the horizontal line ($c_i = -1/k Wi$; $c_r \in [-1, 1]$ for plane Couette, and $c_r \in [0, 1]$ for plane Poiseuille) corresponding to the elastic continuous spectrum, (ii) a horizontal string of eigenvalues corresponding to the aforementioned HFGL modes, and (iii) a roughly ‘hourglass’ shaped structure that extends above and below the HFGL line; note that the length of the HFGL sequence obtained is a function of the numerical resolution of the spectral method, and is smaller for plane Poiseuille flow due to the lower $N$. Despite both spectra conforming to a common template, all modes remain stable for plane Couette flow, as mentioned above, while some of the eigenvalues belonging to the small-$c_r$ ‘arm’ of the hourglass become unstable at sufficiently high $Re$ and $E$, for plane Poiseuille flow; see Fig. 5.

Chaudhary et al. [54] further showed that plane Poiseuille flow of a UCM fluid is susceptible to an apparently infinite hierarchy of elasto-inertial wall mode instabilities. In contrast to the antisymmetric Newtonian TS mode, these unstable elastoinertial modes can have either symmetry (symmetry, here, is based on the variation of the streamwise velocity eigenfunction, about the centerline, in the wall-normal direction). The multiple unstable tongues in the $Re - k$ plane, for both the antisymmetric and symmetric wall-mode instabilities, are shown in Figure 6 for $E = 3.5 \times 10^{-3}$. The lowest critical Reynolds
number was found to be $Re_c \approx 1210.9$ for $E = 0.0066$; $Re_c$ was found to diverge in the limit $E \ll 1$, although the scalings differed for the symmetric ($Re_c \propto E^{-1}$) and antisymmetric ($Re_c \propto E^{-2}$) modes. Both the unstable wall modes above, that are part of the hour-glass structure, and the HFGL modes, are found to be strongly stabilized on introduction of a solvent viscosity component (non-zero $\beta$) [58, 46]. Thus, although relevant to the UCM limit, the wall-mode instabilities are not relevant to the dilute solutions on which most experiments have been performed.

In contrast to the many studies discussed above, that have focused on the stability of plane Poiseuille flow of an Oldroyd-B fluid, rather surprisingly, there had not been a single study, until recently (see [33, 59]), analyzing the stability of pipe flow of an Oldroyd-B fluid. The only stability analysis in the literature by Hansen [60, 18] had neglected the crucial convected nonlinearities in the Oldroyd-B model. The lack of emphasis on pipe flow could perhaps be attributed to the linear stability of Newtonian pipe flow for all $Re$ [61], in turn leading to the assumption of viscoelastic pipe flow also being linearly stable in $Re$–$Wi$–$\beta$ space; an assumption that has often found an explicit mention in the literature [62, 63, 12, 64]. This is despite the absence of a systematic exploration of the larger (three-dimensional) parameter space, and inspite of the recent pipe flow experiments of Samanta et al. [23] showing the existence of an perturbation-amplitude-independent threshold $Re$ for transition from the laminar state in sufficiently elastic polymer solutions. The latter observation is a clear signature of an underlying linear instability.

It is worth pointing out here that two protocols were adopted by Samanta et al. [23]: in the first protocol, the flow was forced by fluid injection (normal to the flow direction) near the inlet resulting in the oft-quoted threshold $Re \approx 2000$ for the Newtonian case. The second protocol did not involve any external forcing, corresponding therefore to ‘natural’ transition, and transition in this case occurred at $Re \approx 8000$ for Newtonian fluids.

With increase in polymer concentration, the threshold $Re$ for the natural transition decreased, while that for the forced transition is increased. However, for concentrations greater than 300ppm, the threshold $Re$ was independent of the experimental protocol. For the the 500ppm solution used, they found the threshold $Re$ could be as low as 800, and the transition was bereft of signatures such as turbulent puffs that accompany the onset of Newtonian turbulence. As mentioned in the Introduction, the flow state that resulted after the non-hysteretic transition was referred to as elasto-inertial turbulence, to distinguish it from both purely-elastic turbulence and inertial Newtonian turbulence. The subsequent experimental study of Choueiri et al. [25] showed that, at a fixed $Re < 3600$, as the polymer concentration is increased, frictional drag decreased and approached the maximum-drag-reduction asymptote, in accordance with the well-established paradigm of turbulent drag reduction. However, in a significant departure from this scenario, further increase in polymer concentration resulted in exceeding the MDR asymptote, with the flow relaminarizing completely for a range of polymer concentrations. This laminar state becomes unstable when polymer concentration is increased further, eventually again approaching the MDR asymptote. As alluded to in Ref. [33], the MDR regime could thus be viewed as a ‘drag-enhanced’ state directly accessible via an instability of the laminar state, rather than as a drag-reduced state accessible from Newtonian turbulence.

Unlike the wall-mode instabilities described above, the
center-mode instability is not restricted to small $\beta$ for either pipe or plane Poiseuille flow. In fact, for both flows, the instability appears to require a combination of fluid inertia, elasticity and solvent viscous effects. This may be seen in the limit $Re \gg 1$ when the threshold Reynolds number $Re_c \propto E^{-3/2}$ for both these flows, a scaling that can only be obtained by balancing fluid inertia, elasticity and solvent viscous effects in a thin layer near the pipe centerline/channel midplane [59]; see Figs 8a and 8b. Thus, in sharp contrast to the known irrelevance of linear (modal) stability theory vis-a-vis the Newtonian transition in the canonical shearing flows, a common modal mechanism is predicted to underlie the transition to EIT, in plane- and pipe-Poiseuille flows of an Oldroyd-B fluid, over a significant fraction of the $Re$-$\beta$ space, with supporting evidence from both simulations[64, 65] and experiments [24].

While the center-mode instabilities for pipe- and plane-Poiseuille flows share many similarities, there are crucial differences. The instability ceases to exist for $\beta < 0.5$ in channel flow [46], while continuing down to $\beta \sim 10^{-3}$ for pipe flow [33]. Recent work by Wan et al. [66] has found that the center-mode instability is present even in the UCM limit for some isolated regions in the $Re$-$Wi$ space. The more interesting limit is that corresponding to dilute solutions with $\beta \to 1$. As shown in Fig. 8a, for pipe flow, $Re_c \approx 63$ for $E \to \infty$, this being minimum Reynolds number for the instability onset. In contrast, as shown in Fig. 8b, $Re_c$ for plane Poiseuille flow behaves in a similar manner only for $\beta < \beta_c \approx 0.99052$; for $\beta > \beta_c$, the center-mode instability continues to down arbitrarily low $Re_c$, with $Re_c \propto E^{-1}$ (corresponding to a threshold $Wi$) for $E \to \infty$; in the process, the elastoinertial center-mode transitions to the purely elastic center-mode instability described in section 2.2 [48]. While the predictions for $Re_c$ in Figs 8a and 8b correspond to an Oldroyd-B fluid, using a nonlinear constitutive equation, such as the FENE-P model, is expected to lead to these curves closing up beyond a second larger critical $Re$, owing to shearing-thinning-induced stabilization; this feature will again be seen, in the context of the curvilinear instabilities, in section 3.

Finally, getting down to actual numbers, linear stability theory [33, 59] predicts a threshold $Re \approx 800$ for transition, similar to the observations of Samanta et al. [23], albeit at much higher $E$’s. The theoretical predictions are in better agreement with the pipe flow experiments of of Chandra et al. [67], an aspect that might have to do with the differing methods used to test the polymer relaxation times in the two efforts. For viscoelastic plane Poiseuille flow, the predictions [46] are in good agreement with the limited experimental data of Srinivas and Kumar [68] for channels with a cross-sectional aspect ratio of 10:1. From a structural viewpoint, the recent pipe-flow experiments of Choueiri et al. [24] have shown remarkable agreement between the structures seen immediately after transition and the linear center-mode eigenfunction; these experiments also point to a secondary transition to a wall-mode.

2.4. Transition scenarios in rectilinear viscoelastic shearing flows

Figures 9 and 10 illustrate the various possible transition scenarios in the $Wi$-$Re$ plane, for different fixed $\beta$, for pipe and plane Poiseuille flows. In these schematic illustrations, we bring together ideas based on the section above, on the centermode instability, and other hypotheses based on earlier non-modal and nonlinear analyses (sections 5 and 6 respectively); we also comment briefly on a recent independent line of work by Michael Graham and coworkers that proposes a new subcritical route to EIT based on elastoinertial TS-wave analogs [69, 70, 71]. In the aforementioned figures, the linearly unstable regions in the interior of the $Wi$-$Re$ plane correspond to the domain of the elastoinertial centermode instability, and are depicted using colored lines for different $\beta$. Regions adjacent to the $Re$ and $Wi$ axes correspond to the onset of predominantly inertial and elastic instabilities, respectively, with the former underlying the sub-critical Newtonian transition. Recall that Newtonian pipe flow is believed to be linearly stable at all $Re$ in sharp contrast to the observed transition at $Re \approx 2000$. Likewise, the presence of the classical TS instability in Newtonian channel flow, at $Re \approx 5772$ (see section 2.1), is now known to be irrelevant to the observed transition at $Re \approx 1000$. Thus, the inertial Newtonian transition for either flow configuration has a nonlinear subcritical character. Indeed, transition in these flows is now understood to be a complex process triggered by the emergence, via saddle-node bifurcations, of novel three-dimensional solutions called ‘exact coherent states’ (ECS) [72, 73, 74].

We begin with a brief discussion of the features common to the center-mode instability in both pipe and channel flows, before going on to describe those unique to channel flow in Figs. 10a and 10b. It has been shown that elasticity suppresses the 3D ECS solutions [31, 32, 75, 76, 77], making the nonlinear Newtonian-ECS-based mechanism irrelevant for weakly elastic flows. Although this suppression has been demonstrated specifically for plane Poiseuille flow, the prediction should be valid for pipe flow as well, on account of the similarity of the Newtonian ECSs across the different rectilinear shearing flows [78, 72, 73]. The elasticity-induced suppression of the ECS has been proposed to underlie delayed transition and eventual disappearance of the Newtonian turbulent state in the flow of polymer solutions. As a result, in the said figures, the Newtonian turbulent-like state is confined to a region between the $Re$-axis and a curve that corresponds to a critical $Re$-dependent $Wi$.

For smaller $\beta$, as shown in Fig. 9, the aforementioned Newtonian turbulent state likely gives way to a laminar one with increasing elasticity. Indeed, the vertical arrow shown on the extreme right in Fig. 9 corresponds to the experimental path of Choueiri et al. [25], who first accessed an intermediate laminar state, and then the MDR regime, with increasing $Wi$, as discussed above in Sec. 2.3. On the other hand, for very dilute solutions, as shown in Fig 10b, the intervening laminar state gives way to overlapping Newtonian and elastoinertial turbulent regions at higher $Re$. The vertical arrow shown in the figure, again on the extreme right, now corresponds to a ‘reverse’ transition where the Newtonian turbulent state exhibits an increasing degree of spatiotemporal intermittency with increasing $Wi$, before giving way to EIT; this was observed to be the pathway, at higher $Re$ values, in Ref. [25].
For sufficiently high elasticities, the linear center-mode instability, discussed in section 2.3 above, becomes relevant. Although the extent of the linearly unstable region depends sensitively on flow-type and $\beta$, the unstable regions for both pipe and channel flow exhibit qualitative similarities for $0.5 < \beta < 0.98$, with $Wi \propto Re^{3/2}$ along the lower branch of the unstable region (this scaling corresponds to the $Re \propto E^{-3/2}$ regime in Figs. 8a and 8b), and $Wi \propto Re$ along the upper one (this corresponds to the near-vertical divergence of $Re_c$ in Figs 8a and 8b). Note that $Wi \propto Re$, in also corresponding to a constant $E$, represents an experimental path of increasing flow rate for a given flow geometry and polymer solution. Thus, as shown in Figs 9, 10a and 10b, for both the plane and pipe Poiseuille geometries, the centermode eigenfunction is likely to lead to supercritical nonlinear structures that, either directly, or through secondary instabilities, might underlie the dynamics of the EIT state. The centermode instability, for both pipe and channel flows, therefore provides a continuous pathway from the laminar state to the EIT/MDR regime, a prediction that now has been confirmed in experiments [24].

Beyond the aforementioned range of $\beta$, as mentioned above in section 2.3, there exist significant differences between the pipe and plane Poiseuille cases. Specifically, in the limit $\beta \to 1$, while the center-mode instability appears to be restricted to $Re \gg 63$ for pipe flow (Figs. 8a and 9), it morphs into a purely elastic instability for channel flow, continuing to arbitrarily small $Re$ for $\beta > \beta_c \approx 0.990552$ (Fig. 8b). Correspondingly, in Fig 10b, the lower boundary of the linearly unstable envelope (with $Wi \propto Re^3$) opens out into a plateau with decreasing $Re$, approaching a threshold $Wi$ for $Re \to 0$. This purely elastic instability might in turn lead to an ET state, and the implied continuous (modal) pathway between the EIT and ET states is shown schematically in Fig 10b. Note that the blue curve in Figs 10a and 10b corresponds to the neutral boundary for $\beta = \beta_c$ that demarcates, within a linearized framework, the pure-EIT regime, and the one that exhibits the EIT-ET connection.

In regions of the $Re-Wi$ space where the centermode is linearly stable, and the originally Newtonian ECS are stabilized by elasticity, novel subcritical mechanisms are expected to dominate the transition process. For plane Poiseuille flow, at moderate $Re$, recent work [69, 70, 71] has identified a nonlinear mechanism based on elastoinertial wall modes closely related to the stable Newtonian TS mode (although still disconnected from it in phase space until a $Re$ of $10^4$). Such a pathway could be especially relevant in a direct transition between the Newtonian and elastoinertial turbulent states (as in Fig 10b), with the near-wall coherent structures in the former state acting as possible seeds for the aforementioned TS-wave analogs. However, the

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2Given that recent experimental evidence points to EIT and MDR states being one and the same, for low to moderate $Re$ values, it is worth mentioning here that the 2D TS-wave-analogs recently proposed to underlie EIT [69, 70, 71] stand in sharp contrast to an earlier interpretation that regarded the MDR regime as corresponding to a hibernating state of turbulence [79, 80] comprising 3D so-called edge-state solutions (lying on the basin boundary between the laminar fixed point and the turbulent attractor in an appropriate phase space). Such states already exist in Newtonian turbulence, and their frequency of occurrence is thought to be progressively enhanced with increasing polymer concentration (although, the hibernating periods have been found to be strongly box-size dependent [30]). The relation between this earlier edge-state-based hypothesis, and the more recent TS-analog-based hypothesis, needs further investigation.
Figure 10: Schematic representation of various transition scenarios in viscoelastic channel flow in the $Wi–Re$ plane.
fact that there is no analog (linear or nonlinear) of the TS-mode in the Newtonian pipe-flow spectrum, and that the centermode remains the least stable one even in the weakly elastic regime [59], suggests that the TS-analog-based subcritical mechanism may not be obviously applicable to pipe Poiseuille flow; more work is clearly required in this regard.

The recent subcritical termination of the unstable center mode, in viscoelastic channel flow, to a nonlinear EIT structure [81] implies that subcritical mechanisms based on the centermode might also be operative in certain regions of Re-Wi-β space, and thus the relevance of the centermode might extend outside of the linearly unstable regions indicated; see the dashed line in Fig 10b. The very recent weakly nonlinear analyses of Buza et al. [49] for channel flow and Wan et al. [66] for pipe flow further confirms that the center-mode instability is likely subcritical in large fractions of the parameter space. Despite these developments, it is relevant to point out that there still remain vast tracts of the viscoelastic parameter space where the mechanism of transition is not understood. As an example, Khalid et al. [46] have shown that for β = 0.97, and for 0.02 < E < 0.5, neither the center mode nor the wall mode is the least stable. Instead, it is the singular modes belonging to the continuous spectrum that are the least stable for these E values, and that might therefore dictate the nature of the transition. More work is therefore required to clarify the transition mechanisms in such parameter regimes.

The above discussion of transition scenarios has been restricted to either new elastoinertial modal pathways, or the elastic modification of essentially Newtonian nonmodal pathways. In the opposite limit of Re ≪ 1, pipe and plane Poiseuille flows, as indeed all rectilinear shearing flows, are linearly stable for Wi ∼ O(1) [45, 59], since a linear instability at such Wi’s requires a hoop-stress-based mechanism (see section 3). The absence of a linear instability at moderate Wi’s has led to the exploration of novel nonmodal pathways due to elasticity alone [82, 83], or due to a non-trivial interplay of elasticity and inertia [84]. While such efforts are discussed in detail in Sec. 5 below, it is worth summarizing a few salient points that appear in Figs 9, 10a and 10b. The nonmodal pathways, in the inertialess limit in particular, point to the importance of spanwise varying disturbances (much like the Newtonian case) that are amplified by an elastic analog of the lift-up effect [85, 86], and by an amount that increases with increasing Wi; an elastoinertial nonmodal pathway, involving a reverse-Orr mechanism has also been examined [84].

An alternate transition scenario, again relevant to the elasticity-dominant limit, is that of a subcritical 2D nonlinear instability [87, 88] based on the classical Stuart-Landau amplitude expansion, an approach originally developed to describe the Newtonian transition [41, 42]. This approach, described in more detail in Sec. 6 below, has been demonstrated only for Wi ∼ O(1) and β → 0. Both the elastic nonmodal and nonlinear (modal) pathways above are shown in Figs 9, 10a and 10b, in the vicinity of the Wi-axis, and are believed to trigger transition to an ET state (‘ET1’ in the figures). The existence of an additional linear instability for Wi ∼ O(1000) and β → 1[48], discussed in section 2.2, also implies a possible bifurcation to a distinct elastic turbulent state. It is therefore possible to envisage (at least) two different ET states (ET1 and ET2 in Fig. 10b), in inertialless plane Poiseuille flow, depending on Wi. There, however, remains a wide intermediate range of β (0 < β < βc) for which the nature of the subcritical transition is not fully understood.

2.5. Interface Instabilities

We next focus on interfacial instabilities present in viscoelastic liquids, which are of major concern in many practical applications that involve multi-layer flows, i.e. the flow of immiscible liquids that are in contact but in distinct layers (for instance, in polymer processing such as coating, coextrusion and others). The main purpose of the applications that involve these kinds of flows is to obtain materials that can combine properties of two (or more) different components. For these materials, uniformity is desired and thus, instabilities must be avoided. For instance, one of the major problems in multi-layer flows is the formation of interfacial waves which can result in a significant deterioration of product properties. The instability typically occurs because of some stratification of fluid properties: a difference between the fluids in density, viscosity or elasticity (or some combination of those properties). For brevity, we will not discuss the role of density differences here. Even when all the fluid properties are the same, if two co-flowing fluid streams have different velocities, one has the classic Kelvin-Helmholtz (‘shear layer’) instability. Azaiez and Homsy [89] used the Oldroyd-B model (in addition to Giesekus and co-rotational Jeffereys models) to show that elasticity has a stabilizing effect on the shear layer instability, although it does not lead to a complete stabilization of the flow.

Interaural instabilities are well known to occur even in Newtonian fluids. Yih [90] analysed one of the simplest interfacial flows: two-layer stratified planar Couette flow between two horizontal plates, and found that viscosity stratification can cause a long wave (k → 0) inertial instability for any nonzero Reynolds number. This instability is seen, for instance, in lubricated pipelining [91], where a viscous core fluid (typically oil) is lubricated by a thin annulus of viscous fluid (water). If the thin layer is the less viscous, instability only occurs for low Reynolds numbers, a phenomenon is known as the thin-layer effect [92]; however, it can potentially extend beyond long waves to all wavelengths [93, 94].

2.5.1. Predicting interfacial instabilities using the Oldroyd-B model

Once we move beyond Newtonian fluids, there is the possibility of elasticity mismatch even between fluids having identical shear viscosities. The first study of this situation was made by Waters & Keeley [95] in 1987, using two Oldroyd-B liquids in a plane Couette flow; they found no instability, but this was due to an error in the interfacial boundary term. Chen [96] carried out a rather similar equivalent calculation for a core-annular coextrusion flow (though he restricted his study to the special case of the UCM model), corrected the error, and discovered a new instability. This instability to long waves – which
can occur even when the fluids are matched in viscosity and density – is due to a jump in the first normal stress difference \( N_1 \) across the fluid-fluid interface. However, the elastic stratification can be stabilizing or destabilizing, depending on the volume ratios of the two liquids. These predictions were experimentally verified by Bonhomme et al. [97]. Hinch et al. [98] gave a simple physical mechanism to show which fluid arrangements would be stable or unstable to both varicose waves (as studied by Chen) and snakelike sinuous modes, and showed that where both modes are unstable the sinuous modes are the more dangerous. Further analysis of interfacial instabilities of UCM liquids in Couette flow was carried out by Renardy [99], quantifying five interfacial modes in the short-wave limit.

These results were extended – still using the Oldroyd-B model to avoid the complications of shear-thinning – to all wavelengths in symmetric three-layer planar interfacial flows (essentially the 2D analogue of coextrusion) by Miller, Wilson & Rallison [100, 101, 102] and to general two-layer arrangements in plane Poiseuille flow by Su & Khomami [103]. More recent works related to the above problems have extended the range of applications to various directions: (i) to very high Weissenberg numbers in the more elastic band [115] to instabilities of the interface between the bands [116, 117, 118, 119]. Although these phenomena go well beyond what can be captured with the Oldroyd-B model, significant understanding of micellar systems can be gained from drawing analogies with purely elastic bulk and interfacial instabilities predicted by the Oldroyd-B model. In particular, the interfacial instabilities seen in shear-banded Couette and plane Poiseuille flow [120, 121, 122, 123] can be explained, at least in their long-wave limit, by Chen’s mechanism discussed above [96], adapted to allow for the fact that the two shear bands have mismatches of viscosity as well as of \( N_1 \). It has also been shown that the Pakdel-McKinley criterion, discussed below in Sec. 3.9, can be adapted to describe instabilities in shear-banded flows [123].

In a very recent paper, Castillo & Wilson [124] found a similar interfacial instability while analysing the stability of channel flow of a shear-banded thixotropic-viscoelastic-plastic fluid. In this case the first normal stress difference \( N_1 \) is matched across the interface, and only the viscosity jumps; they were able to reproduce the main points of their instability (seen in a highly shear-thinning fluid with many constitutive complications) by using two Oldroyd-B fluids, but this time matched in the interfacial value of \( N_1 \) and having different shear viscosities. The explanatory power of Oldroyd-B continues.

### 2.5.2. Beyond Oldroyd-B

In addition to introducing the upper-convected derivative, Oldroyd [34] also discussed the opposite extreme, the lower-convected derivative, which leads naturally to the Oldroyd A model; and implied the existence of everything between the two. The Johnson-Segalman model [108] has a slip parameter, \( a \), which transitions from Oldroyd-A \((a = -1)\) to Oldroyd-B \((a = 1)\) via the corotational derivative \((a = 0)\). It shear-thins for all slip values except in the Oldroyd-B and Oldroyd-A limits, and for some values of the parameters it has a nonmonotonic flow curve: in some parts of parameter space an increase in flow rate results in a decrease in shear stress, as shown in figure 11.

![Figure 11: Non-monotonic flow curve, as exhibited by the Johnson–Segalman model. If the applied stress is \( \sigma = \sigma_p \), flow can coexist at two shear-rates \( \gamma_1 \) (low-viscosity band) and \( \gamma_2 \) (high-viscosity band). The intermediate shear-rate \( \gamma_2 \) is thermodynamically unstable.](image)

One of the most striking phenomena associated with the nonmonotonicity of the flow curve is shear banding, in which a simple shear flow of a complex fluid spontaneously separates into high- and low-shear-rate bands [109, 110]. This behavior is often observed in solutions of worm-like micelles [111, 112], and there has been widespread interest in such banding flows since the phenomenon was first reported in the early 1990s.

Surprisingly, shear-banded flows of worm-like micellar solutions are themselves unstable [113, 114] and exhibit a variety of instabilities ranging from purely elastic instabilities localised in the more elastic band [115] to instabilities of the interface between the bands [116, 117, 118, 119]. Although these phenomena go well beyond what can be captured with the Oldroyd-B model, significant understanding of micellar systems can be gained from drawing analogies with purely elastic bulk and interfacial instabilities predicted by the Oldroyd-B model. In particular, the interfacial instabilities seen in shear-banded Couette and plane Poiseuille flow [120, 121, 122, 123] can be explained, at least in their long-wave limit, by Chen’s mechanism discussed above [96], adapted to allow for the fact that the two shear bands have mismatches of viscosity as well as of \( N_1 \). It has also been shown that the Pakdel-McKinley criterion, discussed below in Sec. 3.9, can be adapted to describe instabilities in shear-banded flows [123].

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### 3. Purely elastic instabilities in curvilinear flows

The term ‘purely elastic’ instabilities has traditionally been used to refer to the instabilities observed in flows of viscoelastic fluids, in geometries with curved streamlines, including in viscometric configurations such as the Taylor-Couette, cone-and-plate and parallel-plate setups, even when inertial effects are not significant. Excellent comprehensive reviews by the pioneers of this field, earlier ones by Larson [4] and Shaqfeh [7], and the more recent one by Muller [8], already exist in the literature; the goal of this section is to provide a brief and more recent summary of this important and novel class of instabilities. A precise prediction of the domain of existence of these instabilities is of immense importance to rheological characterization of polymeric liquids, since the inference of rheological properties presupposes the existence of viscometric flows in the aforementioned geometries; the occurrence of instabilities corrupts rheological measurements, precluding characterization. Further, the instabilities are of relevance to coating applications, and other polymer processing scenarios where flow configurations akin to the said viscometric flows occur. In fact, a discussion of these instabilities is all the more pertinent to the present review, because their prediction is one of the prominent success stories of the Oldroyd-B model.
3.1. Effect of viscoelasticity on the Newtonian Taylor-Couette instability

Purely azimuthal flow of a Newtonian fluid between concentric cylinders, (the Taylor-Couette configuration) becomes unstable due to (inertial) centrifugal effects when the inner cylinder or both inner and outer cylinders rotate [125]. The instability is absent when only the outer cylinder rotates, or when both cylinders rotate in the same direction with the angular velocity of the outer cylinder exceeding that of the inner cylinder by the ratio \((R_{\text{out}}/R_{\text{in}})^2\). \(R_{\text{out}}\) and \(R_{\text{in}}\) being the radii of the outer and inner cylinders, respectively. This is consistent with the Rayleigh criterion for inviscid instability that requires the base-state angular momentum to monotonically decrease (in magnitude) with increasing radius [2]. The domain of existence of both the primary linear instability, and higher order transitions has been well characterized in a parameter plane consisting of the Reynolds numbers based on the radii and angular velocities of the inner and outer cylinders [126]. Excluding the case of strong counter-rotation, the unstable mode, at onset, is axisymmetric and stationary (i.e., with zero frequency). A similar centrifugal instability is also present in ‘Dean flow’ entailing pressure-driven flow through a curved channel, and originally analyzed in the limit where the channel width is small compared to the radius of curvature [127]. When a combination of cylinder rotation and a streamwise pressure gradient drives the flow, the resulting centrifugal instability is dubbed the ‘Taylor-Dan’ instability, and may be achieved experimentally by inserting a meridional obstruction in the original Taylor-Couette geometry[128].

Early efforts by Ginn and Denn [129] probed the role of weak viscoelasticity on the centrifugal instability using a second-order fluid model. The analysis showed that positive values of the first normal stress difference \((N_1)\), corresponding to a tension along the base-state azimuthal streamlines, had a destabilizing effect. In contrast, a negative second normal stress difference \((N_2)\), corresponding to a tension along the base-state axial vortex lines, had a stabilizing effect. The latter effect could be interpreted as being due to the resistance of the tensioned vortex lines to bending caused by axially modulated perturbations. This is somewhat analogous to the work of Azaei and Homsy [89] mentioned in the subsection above, where the bending resistance of tensioned streamlines acts to stabilize the viscoelastic shear layer. In the narrow-gap limit of \(\varepsilon \ll 1\) (\(\varepsilon\) being the ratio of gap width between the cylinders and the inner cylinder radius), \(N_2\) appears at a lower order in \(\varepsilon\) than \(N_1\). Hence, although \(N_2\) is usually negative and much smaller in magnitude than \(N_1\) (typically 10-30\% for polymer melts, and smaller for polymer solutions [130]), the stabilizing or destabilizing action of viscoelasticity can nevertheless be expected to depend sensitively on \(N_2\) for \(\varepsilon \ll 1\).

The second-order model used in Ref. [129], being restricted to weak flows in the quasi-steady limit, is only valid for \(Wi \ll 1\) [131]; experiments, particularly those examining instabilities, are most often performed outside this regime, especially in the narrow-gap limit. Using the more realistic upper-convected Maxwell (UCM) model, the limiting form of the Oldroyd-B model for \(\beta \to 0\), Walters and coworkers [132, 133, 134] again found that \(Re_c\) for the stationary Newtonian mode decreased with \(E\) for small \(E\), this decrease being consistent with the destabilizing role of a positive \(N_1\) mentioned above. A new oscillatory ‘inertio-elastic’ mode was found to become more unstable for higher \(E\). Note that this new oscillatory unstable mode is not connected to the Newtonian limit, and is therefore not captured by the second-order fluid model. Beard et al. [134] found the \(Re_c\) for this mode to also decrease monotonically with \(E\) in the range \(0 < E < 1\), although the authors did not extend their computations all the way down to the inertialess limit (\(Re = 0\) or \(E = \infty\)). The destabilizing effect of weak elasticity on the Newtonian centrifugal instability also holds for the Dean and Taylor-Dan configurations [135].

3.2. The purely elastic Taylor-Couette instability

While Giesekus reported evidence for the onset of a cellular instability in Taylor-Couette flow of polymer solutions as early as 1966 [7] at Reynolds numbers of \(O(10^{-2})\), it is the pioneering theoretical-cum-experimental efforts of Larson, Muller and Shaqfeh [136, 137, 138] that led to the unequivocal establishment of an inertial instability in viscoelastic Taylor-Couette flow. In addition to carrying out a classical modal stability analysis, using the Oldroyd-B model in the inertialess limit, the authors also characterized the transition experimentally using a Boger fluid. The latter refers to a class of fluids prepared by dissolving small amounts of high-molecular weight polymer in a very viscous solvent [139], which leads to a high elasticity (owing to the long relaxation time) but negligible shear-thinning (in the viscosity); these fluids have served as model systems reasonably well described by the Oldroyd-B model. The theoretical predictions, obtained for axisymmetric disturbances, were in qualitative agreement with experimental observations. Unlike the Newtonian case, the unstable mode existed with rotation of either cylinder, and was found to be oscillatory at onset, with the dominant measured frequency in good agreement with theory. However, experiments showed the vertical length scale of the cellular pattern at onset to correspond to an axial wavenumber smaller than the theoretical prediction. Interestingly, the cellular pattern in the experiments continued to evolve over times much longer than the nominal polymer relaxation time, with the cell height eventually shrinking to half its initial value. The authors attributed this discrepancy to the relatively flat neutral curve, implying the excitation of unstable modes across a broad spectrum of wavenumbers in the immediate vicinity of the threshold, and the resulting nonlinear interactions then contributing to the aforementioned evolution (as discussed below, consideration of nonaxisymmetric disturbances leads to better agreement). Further, the measured critical Weissenberg number was typically found to be between 0.5–0.9 times the predicted value. Plausible reasons behind this discrepancy are discussed below. Analogous elastic instabilities have also been predicted in the Dean flow [140] and Taylor-Dan flow configurations [128], with the unstable mode in the latter case changing from an oscillatory to a stationary one, as the pressure gradient becomes dominant in relation to cylinder rotation.
Both the elastic Taylor-Couette and Dean instabilities owe their origin to either the base-state or perturbation hoop stress fields that arise on account of tension along the curved base-state streamlines. For the former flow, Larson et al. [137] proposed a physical mechanism based on a dumbbell model for a polymer molecule, consistent with the microscopic picture underlying the Oldroyd-B equation used for the stability analysis. The toroidal circulation associated with the axisymmetric unstable eigenmode leads to an extensional flow in the meridional plane that stretches the dumbbell in the radial direction. This stretched dumbbell is now acted upon by the base-state azimuthal shear, which tilts it, leading to an increased separation between the beads along the azimuth (see Fig. 12 of Ref. [137]). This drives a perturbation normal stress in the $\theta \theta$ direction (the hoop stress), which in turn produces a radial perturbation pressure gradient. The radial flow driven by this pressure gradient is out of phase with the original extensional flow, but for sufficiently high $Wi$, overwhelms the original radial perturbation, leading to overstability, and a growing oscillatory response.

Note that, on account of the underlying hoop stress, the mechanism above is relevant only for flows with curved streamlines, and is absent, at linear order, in the rectilinear shearings flows discussed in section 2. However, the mechanism can operate at a nonlinear order, where the streamline curvature itself arises on account of the perturbation, and this scenario is discussed in more detail in section 6.

### 3.3. Finite-gap effects and nonaxisymmetric disturbances

The early theoretical efforts [136, 137] were restricted to small gap-widths ($\epsilon \ll 1$) and axisymmetric disturbances. In order for the effects of curvature, in the Oldroyd-B model, to remain important for $\epsilon \ll 1$, one requires $Wi \sim \epsilon^{-1/2}$, and expectedly, the linear analysis [137] yields a threshold value of $Wi \epsilon^2$ for instability (note that, when inertial effects are negligible, the limit $Wi \epsilon^{1/2} \ll 1$ corresponds to plane Couette flow which, as already seen in section 2, is linearly stable [44, 50, 54]). The above $Wi = \epsilon$ scaling, and similar scalings found for the other curvilinear flows (the parallel-plate and cone-and-plate geometries, with $\epsilon$ being replaced by the pertinent geometric factor), are discussed below in Sec. 3.9, in the context of the Pakdel-McKinley criterion.

The restriction to axisymmetric disturbances, for $\epsilon \ll 1$, is often justified based on the assumption of the azimuthal variation of the perturbation being on scales comparable to the cylinder radii; as a result, for $\epsilon \rightarrow 0$, both axisymmetric and nonaxisymmetric modes are governed by the same leading order eigenvalue problem. Working within the axisymmetric disturbance framework, Shaqfeh et al. [138] showed that, as $\epsilon$ increases, $Wi \epsilon$ from both experiments and the finite-gap theory decreases much more slowly with $\epsilon$ than the aforementioned prediction ($Wi \epsilon \sim \epsilon^{-1/2}$) of small-gap theory. Significant (positive) deviations of the threshold are already predicted for $\epsilon = 0.05$, pointing to the restrictive range of validity of the small-gap assumption, with $Wi \epsilon$ becoming nearly independent of $\epsilon$ in the range 0.1 to 0.25, this being consistent with experimental observations (Fig. 11 of Ref. [128]).

As pointed out above, the effect of nonaxisymmetry is only expected to enter the analysis at a higher order for $\epsilon \ll 1$. Indeed, Joo and Shaqfeh [128] showed that the terms differentiating the various nonaxisymmetric modes are $O(\epsilon^2 Wi^3)$. Since $\epsilon Wi^2 \sim O(1)$ for curvature effects to remain finite, these additional terms are seen to be $O(\epsilon^{1/2})$, and therefore, asymptotically small compared to the leading order axisymmetric ones. However, detailed calculations showed that the thin gap assumption was again very restrictive. Thus, Joo and Shaqfeh, even for $\epsilon \sim 10^{-4}$, found the additional terms (characterizing departure from axisymmetry) to be important; nonaxisymmetric modes, in fact, turned out to be more unstable than the axisymmetric mode. Joo and Shaqfeh [128] found the results obtained using nonaxisymmetric disturbances to be in good qualitative agreement with experimental observations of the variation of $Wi$, with $\beta$ and $\epsilon$. Further, the wavenumber of the axial vortices developing at onset, determined using an image analysis, was found to be in good agreement with the prediction for the most unstable nonaxisymmetric mode (about 5.5 inverse gap thicknesses). The results of Joo and Shaqfeh were also consistent with the earlier findings of Avgousti and Beris [141], who again found a nonaxisymmetric oscillatory mode in viscoelastic Taylor-Couette flow to be more unstable than the aforementioned axisymmetric mode, even for $\epsilon = 0.1$.

### 3.4. The dominant instability in the $Wi$–$Re$ plane

The discussion above began with a brief description of the Newtonian Taylor-Couette instability ($Wi = 0$), and thereafter, shifted to an examination of the purely elastic instability ($Re = 0$) arising in the same flow configuration. The initial efforts referred to above [129, 132, 133, 134] emphasized the destabilizing role of weak elasticity on the Newtonian (centrifugal) instability (for $N_2/N_1 \rightarrow 0$). A study of the opposite limit, that is, the role of weak inertia on the purely elastic instability, discovered by [137], was first done by Joo and Shaqfeh [135]. Expectedly, the rotation of the outer cylinder was found to have a stabilizing effect, and that of the inner cylinder a destabilizing one; note that this breaks the symmetry present in the purely elastic limit, for $\epsilon \ll 1$, where the instability is independent of the particular cylinder being rotated. The destabilizing nature of weak elasticity on the Newtonian instability holds even for the Dean and Taylor-Dean configurations [135]. On the other hand, the role of weak inertia on the purely elastic instabilities in different configurations can be both stabilizing or destabilizing.

It is also of interest to examine the nature of the dominant unstable mode in the $Re$–$Wi$ plane as a whole. Recall from section 2 that, within the Oldroyd-B framework of a shear-independent viscosity and first normal stress coefficient, the elasticity number $E = Wi/Re$ is a convenient measure of the relative magnitudes of elasticity and inertia, with $E = 0$ and $\infty$ corresponding to the purely inertial and purely elastic limits, respectively.

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3It is perhaps worth noting that the analogous question for the other viscometric flow configurations viz. the cone-and-plate and parallel-plate geometries, is complicated by the fact that the purely azimuthal flow is no longer an exact solution for finite $Re$; see section 3.5.
Avgousti and Beris [142], using both linear stability analyses and symmetry arguments that remain valid beyond the linear regime, reported the existence of three different eigenmodes in the $Re \rightarrow \infty$ plane for axisymmetric perturbations. For $E \ll 1$, the unstable mode is stationary and its structure analogous to the well-known Taylor vortices; For $E \gg 1$, the unstable mode is oscillatory. For finite values of $E$, however, a distinct inertio-elastic oscillatory mode becomes most unstable with a wavelength and flow structure intermediate between the purely elastic and inertial modes.

Avgousti and Beris [141] showed that the above trend, of new modes becoming dominant with increasing $E$, persisted even for non-axisymmetric disturbances [141], although this finding was restricted to the UCM limit ($\beta = 0$). Joo and Shaqfeh [135] showed that the inertio-elastic mode was the most unstable only over a small, intermediate range of $Re$, and this range disappears entirely for $\beta$ slightly greater than zero, suggesting that the inertio-elastic mode may not be relevant for polymer solutions; the said mode is also not relevant when the flow is driven only by the motion of the inner cylinder. In contrast to this rather complicated scenario in Taylor-Couette flow, wherein different modes become dominant as $E$ is increased, the same unstable axisymmetric mode continues all the way from $E = 0$ to $E = \infty$ for Taylor-Dean flow [135]. This simple picture may not, however, hold for nonaxisymmetric disturbances, although this remains to be investigated.

Interestingly, the nature of the dominant instability in the $Re \rightarrow Wi$ plane in the limit $Re, Wi \rightarrow \infty$, with $E$ fixed, is of relevance to the astrophysical scenario. Note that this distinguished limiting scenario arises only for the Oldroyd-B model since shear thinning in the nonlinear constitutive models leads to elastic stresses scaling sub-quadratically, and thereby, becoming asymptotically small in relation to the inertial ones in the aforementioned limit. The limit may be termed the elastic Rayleigh limit since the equation governing linearized evolution is a second order ODE similar to the Rayleigh equation in the inviscid limit [2], but with $E(1 - \beta)$ as a parameter that measures the importance of elastic stresses (the reduction of order points clearly to the singular nature of this limit). It has been shown [143] that the equations governing polymer solutions in the elastic Rayleigh limit are identical to the magnetohydrodynamics (MHD) equations in the relaxationless limit (that of infinite magnetic Reynolds number), with the dumbbell end-to-end vector field being analogous to the magnetic field. A consequence is that the instabilities of a polymer solution in this limit have corresponding astrophysical analogs; in particular, an inertio-elastic instability in this limit should map onto the so-called magnetorotational instability (MRI). The axisymmetric version of the latter instability was originally predicted by Vehlikov [144] and Chandrasekhar [145], and rediscovered much later by Balbus and Hawley [146]. In the polymer solution literature, the elastic Rayleigh limit was first considered by Aziaiz and Homsy [89], in the context of a viscoelastic shear layer, although elasticity had a stabilizing influence in this case, as already discussed in section 2.5. Rallison and Hinch [147] were the first to discover an inertio-elastic instability for a submerged jet configuration, arising from a novel mechanism involving a resonant interaction of elastic shear waves (the analogs of Alfvén waves in the MHD context) in the aforementioned limit. More recently, an analogous shear-wave-driven instability has been shown to destabilize an elastic vortex column [148]; unlike the jet, this inertio-elastic instability exists in isolation for the vortex case, and is therefore of greater relevance. Importantly, the vortex column configuration bears an obvious relation to the Taylor-Couette geometry. There have indeed been experiments in the Taylor-Couette setup motivated by the above analogy [149], and also numerical stability calculations for relatively modest Re and Wi [150]. It is certainly of interest to theoretically investigate the instabilities of the Taylor-Couette configuration in the elastic Rayleigh limit, and in particular, examine the relation between any unstable modes arising from the aforementioned shear-wave-resonance mechanism, and the inertio-elastic mode, at finite Re and Wi, that has been discussed in the elastic instability literature[141, 135].

3.5. Purely elastic instability in cone-and-plate and parallel-plate flows

The mechanism underlying the purely elastic Taylor-Couette instability is suggestive of a similar instability in the two other rheometric flow configurations, viz., the cone-and-plate and parallel-plate geometries. Magda and Larson [151] observed an anomalous increase in the shear stress above a threshold rate of shear, in both these geometries, even for a vanishingly small Re. Although similar observations had been made by earlier researchers [152, 153], the aforementioned authors were the first to ascribe the anomalous increase to a flow transition, rather than to the intrinsic rheological character of the fluid. Subsequent visualization experiments by McKinley et al. [154] showed the onset of a secondary flow at the critical shear rate, corroborating the inference of Magda and Larson.

The first theoretical studies to demonstrate an instability of the flow of an Oldroyd-B fluid, in the above geometries, were due to Phan-Thien [155, 156]. Motivated by the form of the Newtonian flow in these geometries, the author assumed a self-similar so-called von-Karman ansatz for the base-state flow at finite Re and De, with $De = \lambda \Omega$. Despite the base-state flow reducing to a purely azimuthal one for zero Re (regardless of De)\(^4\), the author again invoked the Karman ansatz for the perturbation, and demonstrated the onset of an stationary and axisymmetric linear instability beyond a critical De. The absence of a length scale in the assumed ansatz implied a disturbance flow field in the form of a single ‘roll’, in the meridional plane, that ‘closes at infinity’. Thus, the only relevant

\(^4\)In the Newtonian case, the (inertial) centrifugal forces driving the meridional secondary flow scale with the velocity, which is larger near the rotating plate, and therefore give rise to a secondary flow. In contrast, the elastic stresses scale with the velocity gradient (the square of the gradient for Oldroyd-B), and are therefore uniform across the gap. The divergence of these stresses may be balanced with a radial pressure gradient, allowing the base-state flow to have only an azimuthal component. This argument is applicable, however, only in the limit of small cone angles, which is when the velocity gradient across the gap is a constant. The existence of an elastic instability, of course, implies solution multiplicity, and the azimuthal flow is not the only possible axisymmetric flow beyond a threshold De.
dimensionless group is $De$ above, with the threshold criterion therefore coming out to be independent of the gap-angle ($θ$) for the cone-and-plate geometry, and of the non-dimensional plate spacing ($H/R$, $H$ and $R$ being the inter-plate spacing and plate radii) for the parallel-plate geometry; note that $Wi = De/θ$, and $De(R/H)$ for the two geometries.

The aforementioned visualization experiments of McKinley et al. [154], for the parallel plate geometry, showed that the growing disturbance flow field was not of the similarity form; instead, the observed patterns were characterized by a radial scale of $O(H)$. The first analysis to account for this scale was that of Oztekin and Brown [157], who used the Oldroyd-B model to conduct a local stability analysis of the parallel-plate flow in the vicinity of a particular radius ($R^*$). While the analysis correctly showed the unstable modes to be time dependent non-axisymmetric spiraling patterns, the prediction of the threshold was qualitatively incorrect. The threshold $De$, obtained from the Oldroyd-B analysis, bore an inverse relation to $R^*$, so the torsional flow configuration was predicted to be unstable for any finite $De$ beyond a certain $R^*$ (in contrast to the Phan-Thien analysis above). This, however, contradicted the later experiments of McKinley and coworkers [158], who showed that the unsteady secondary motion, just beyond the threshold, was restricted to an annular region between a pair of critical radii. The restabilization of the flow beyond the second critical radius arises from effects of shear thinning which decrease the effective relaxation time at higher shear rates. This was confirmed by a linear stability analysis using the FENE-CR model [159] which incorporates the effects of a shear-thinning first normal stress coefficient[158]. The aforementioned restabilization also translates to the stability of the torsional flow configuration below a threshold inter-plate spacing; the resulting shape of the unstable region in parameter space is discussed in section 3.9 below in the context of the Pakdel-Mckinley criterion.

An analogous scenario was shown to hold for the cone-and-plate geometry [160]. Use of the Oldroyd-B model again reproduced the unstable spiral patterns in a qualitative sense; unlike the parallel-plate case, the homogeneity of the base-state shear flow and the resulting absence of a characteristic length scale implies that the spirals in this case are no longer radially localized. Use of the FENE-CR model was needed to capture the correct nature of the threshold condition, which involved a stabilization of the flow for small enough cone angles; this is discussed in section 3.9. The use of a multimode Giesekus model, in fact, yielded quantitative agreement with experimental observations [161].

3.6. Experiment vs Theory (Non-isothermal effects)

The comparison between theory and experiment, for even the primary transition in the viscoelastic case, is nowhere as quantitative as for the Newtonian case. The first and perhaps obvious reason is the uncertainty surrounding the constitutive equation used. The Oldroyd-B equation, used in the stability analyses, is only an approximation for the Boger fluids used in the experiments. The latter have a nontrivial spectrum of relaxation times, and consistent modeling of both their steady state and transient rheology requires a nonlinear multimode constitutive equation [162]. The underlying relaxation spectrum manifests as a dependence of the apparent relaxation time on the method of measurement. The resulting arbitrariness in the choice of time scale, to be used in the Oldroyd-B model, leads one to expect an ambiguity in the theory-experiment comparison. A second reason has already been mentioned above in the context of the first experiments: the shallow nature of the viscoelastic neutral curve, in comparison to the Newtonian one (compare Figs.III-2 and III-11 of Ref. [163]), has been speculated to lead to strong nonlinear interactions even close to onset, that then manifest as a slow drift of the observed pattern with the appearance of progressively smaller-scaled structure over times much longer than the nominal relaxation time [137, 164]. A second factor favoring the aforementioned importance of nonlinearity is that the bifurcation to the primary non-axisymmetric mode is likely a subcritical one [165], in contrast to the Newtonian case. Thus, as briefly mentioned in Sec. 3.2, while the theoretical predictions based on axisymmetric disturbances over-predict the threshold for instability, consideration of nonaxisymmetric disturbances and the associated subcritical nature of the bifurcation leads to a narrowing of the gap between the theoretical and experimental thresholds.

A third and rather unexpected reason that has led to a stark difference between theory and some experiments is the deviation from isothermal conditions arising due to viscous heating (Boger fluids have very high viscosities). Despite the critical parameters for instability onset being sensitive to the particular rheological model used, the elastic instability for Taylor-Couette flow, within the usual isothermal formulation, is predicted to be always non-axisymmetric and oscillatory at onset. However, many of the experimental observations [168, 169, 166, 167] have revealed a primary transition to a weak stationary axisymmetric mode on time scales much longer than the polymer relaxation time, and at $Wi$ much lower than the theoretical threshold. This discrepancy was addressed by Al-Mubaiyedh et al. [170, 171] who showed that the inclusion of viscous heating in the stability analysis leads to a good agreement between experiment and theory. That viscous heating leads to destabilization is somewhat counter-intuitive since both viscosity and the relaxation time should decrease with an increase in temperature, and this ought to lead to a higher threshold rotation rate for a given $Wi_c$. This expected stabilization effect has indeed been found for other curvilinear geometries such as the parallel-plate and cone-and-plate configurations [172, 173]. The destabilizing mechanism [170, 171] for Taylor-Couette flow is argued to arise due to the stratification of the hoop stress in the gap between the cylinders, which drives a radial secondary flow, and the convection of base-state temperature gradients by the radial perturbation velocity then leads to the instability. Figure 12 shows the slow development of the stationary vortices on a space-time plot [166], with the adjoining plot showing the experimentally observed decrease in $Wi$ with the Nahme number, the latter being a dimensionless measure of viscous heating [167].
3.7. Secondary instabilities and elastic turbulence

The nature of the secondary instabilities and associated flow patterns that ensue after the linear instability in Taylor-Couette flow was studied in a series of papers by Groisman and Steinberg [174, 175, 176, 169], for a range of $E$, in the $Re-E$ plane. For $E < 0.15$, the primary instability of the azimuthal flow results in the Taylor-Vortex flow (TVF) (similar to the Newtonian case), which then becomes unstable with further increase in $Re$ to give rise to a new oscillatory state termed ‘rotating standing wave’. This state itself becomes unstable at higher $Re$, resulting in another new oscillatory state termed ‘disordered oscillations’. For $E > 0.15$, the rotating standing wave disappears, and the TVF transitions to disordered oscillations, while for $E > 0.22$, the purely azimuthal flow directly undergoes a transition to disordered oscillations. The latter state is characterized by broad peaks in the frequency spectra, and by the appearance of patches of standing and traveling waves.

At higher elasticities ($E \sim 20$), the purely azimuthal base flow becomes unstable to disordered oscillations in a rather abrupt, hysteretic manner. When the angular velocity is decreased, the disordered oscillations resulted in stationary vortex structures referred to as ‘diwhirls’. The value of $De$ for which the diwhirls completely decay to recover the azimuthal Couette flow could be even 45% smaller than that required for the onset of disordered oscillations. At large $E$, the disordered oscillations transition to a purely elastic mode, with fluid inertia becoming irrelevant.

In the context of torsional flow between two parallel plates, Groisman and Steinberg [9], while using relatively large gap-to-radius ratios (viz., $\frac{H}{R} = 0.263$ and 0.526), observed a direct transition from the laminar base flow to a disordered state termed ‘elastic turbulence’, even at negligibly small $Re$. The shear stress in the disordered flow state could be 20 times larger than a hypothetical laminar flow under the same flow conditions. They used imaging as well as Doppler velocimetry to characterize the disordered state, and showed that the velocity spectra showed a broad range of spatio-temporal scales, with a steep power-law decay. Finally, mixing rates of passive scalars in the elastic-turbulent state were orders of magnitude larger than the diffusion rates. The review by Steinberg [11] provides a more detailed and up-to-date summary of elastic turbulence. In the context of secondary instabilities in torsional flow between to parallel plates, Schianberg et al. [177] used a smaller gap ratio compared to Groisman and Steinberg, and reported a rich sequence of novel flow states that connect the azimuthal laminar state to the elastic turbulent state.

3.8. Beyond Oldroyd-B

Here, we comment briefly on how features not captured by the Oldroyd-B model affect the prediction of the purely elastic instabilities in curvilinear flows. These include (i) a nonzero $N_2$, (ii) the shear-rate dependence of the viscosity and first normal stress difference, and (iii) the existence of a nontrivial relaxation time spectrum.

Most entangled polymer solutions have negative $N_2$'s [130], which are predicted to strongly stabilize the purely elastic instability in Taylor-Couette flow [138], especially for small gap
widths ($\epsilon \ll 1$); as alluded to in section 3.1, in the context of the second-order fluid model, the effects of $N_2$ enter at a lower order in $\epsilon$ (compared to $N_1$) in this limit. Indeed, Shaqfeh et al. [138] noted a significant discrepancy between the prediction from the Oldroyd-B model ($Wi = 47$) and experiment ($Wi = 71$) for the smallest gap ratio ($\epsilon = 0.03$), where the small-gap theory should otherwise have been accurate. To address this discrepancy, Shaqfeh et al. used a modified Oldroyd-B model with an additional contribution to the stress tensor that, albeit of the second-order fluid form, gave rise to a nonzero $N_2$, and showed that this indeed had a stabilizing effect. Similarly, using the Giesekus model, Beris et al. [178] also concluded that negative second normal stress differences were strongly stabilizing. Indeed, for $-N_2/N_1 > 0.1$ (typical for entangled polymer solutions), $Wi > 100$, and this could be one reason why the purely elastic Taylor-Couette instability is not reported for the flow of entangled polymeric solutions [4].

Larson et al. [179] used the K-KBZ equation to incorporate both shear thinning and a relaxation time spectrum to analyze the elastic Taylor-Couette instability, and concluded that the critical conditions are a function of both the longest (the Oldroyd-B) and the average relaxation times. This immediately points to the arbitrariness inherent in the choice of a single-time-scale (either linear or nonlinear) model, as has already been pointed out in section 3.6. In a later effort, a multimode Giesekus model, with parameters tuned so as to best fit the rheological properties of the Boger fluid used in the experiments, has been shown to quantitatively predict the critical Weissenberg number as a function of gap width or cone angle [161]. Larson et al. found shear thinning to have a monotonically stabilizing effect on the instability, on account of the decrease in the first normal stress coefficient with the shear rate. As mentioned in Sec. 3.5, shear-thinning has, in fact, a profound effect on the neutral boundary demarcating the stable and unstable regions [154, 160, 157, 158], and this is discussed in more detail in the next subsection.

3.9. The Pakdel-McKinley criterion

We now discuss a heuristic argument developed by Pakdel and McKinley [180, 181] which unifies the threshold criteria for the different curvilinear shearing flows examined above. As will be seen below and in section 4, it allows one to arrive at sensible stability thresholds (to within a numerical factor of order unity) even when closed form results from linear stability analysis are not available (owing, for instance, to the geometrical complexity of the flow configuration). The criterion incorporates the two key ingredients required for a purely elastic instability viz. streamline curvature in the base-state laminar flow as well as the magnitude of the streamwise (tensile) normal stresses, and expresses the threshold for elastic instability in terms a product of the Deborah and Weissenberg numbers. Recalling the definition $De = \lambda/T$, $T = \mathcal{R}/U$ now being the residence time with $\mathcal{R}$ the radius of curvature, and writing $Wi$ in terms of the ratio of the normal and shear stresses as $Wi = N_1/2\tau$, the Pakdel-McKinley criterion for elastic instability may be written as:

$$[(AU/\mathcal{R})(N_1/\tau)]_c > M^2,$$

where $M$ being an order unity number in the simplest cases (curvilinear viscometric flows), or a field (when the flow configuration is more complicated, and in particular, inhomogeneous, as is the case for the non-viscometric examples considered in section 4). Note that the ratio $AU/\mathcal{R}$ may also be interpreted as the ratio of the distance (along a streamline), over which disturbances relax elastically, to the characteristic radius of curvature. Replacing $AU$ by an appropriate viscous boundary layer length scale recovers the known stability criteria associated with Newtonian instabilities driven by curvature; these include the classical Taylor-Couette instability itself, and the Gortler instability associated with laminar boundary layers on curved surfaces [182].

We now obtain the explicit form for this criterion for the Taylor-Couette and cone-and-plate configurations. For the former, $U = \Omega R_0$, corresponding to the inner cylinder (say) rotating with angular velocity $\Omega$, and an obvious choice for the radius of curvature is the cylinder radius, viz., $\mathcal{R} = R_0$. The streamwise normal stress for the Oldroyd-B model is $\tau_{0\nu} = 2\eta_0\lambda\gamma^2$, and the total shear stress $\tau = \eta_0\gamma$, $\eta_0$ being the solution viscosity. Substituting these expressions leads to

$$(\sqrt{De Wi})_c \geq \frac{M}{\sqrt{2(1 - \beta)}},$$

for instability in the thin-gap limit, where $De = \lambda\Omega$, $Wi = \lambda\gamma$, and $\eta_0/\eta = (1 - \beta)$. Using $\gamma = \Omega R_0/\mathcal{R}$ and the dimensionless gap width $\epsilon = d/R_0$, with $Wi = De/\epsilon$, the above equation reduces to

$$Wi_\epsilon^{1/2} \geq \frac{M}{\sqrt{2(1 - \beta)}},$$

in agreement with the original thin-gap analysis of Larson et al. [137] (note that these authors refer to the Weissenberg number, $Wi$, of the present review as the Deborah number $De$). In Eq. 4, $M$ is in general a complicated function of $\beta$, and asymptotes to a constant only in the limits $\beta \rightarrow 0$ and 1. In the former limit, the stability analysis yields $M \approx 6$ for axisymmetric disturbances [137], and for the latter case, the threshold $Wi$ diverges as the reciprocal square root of $(1 - \beta)$ in the limit of vanishing elasticity.

For the cone-and-plate configuration, the velocity at any radial position is $U = r\Omega$, and for $\theta_0 \ll 1$, the shear rate, which is now uniform across the gap, is $\Omega/(r\theta) = \Omega/\theta_0$. Using $\mathcal{R} = r$, and the expression for the hoop stress mentioned for Couette flow above, one obtains:

$$Wi_0^{1/2} \geq \frac{M}{\sqrt{2(1 - \beta)}},$$

The same comments as above apply with regard to the $\beta$-dependence. For $\beta = 0.5$, linear stability analysis yields $M = 4.602$ [181].

The real utility of the criterion above is evident when accounting for thinning effects which may be done by using the shear-rate-dependent analogs of the quantities that appear in Eq. 2; these include the relaxation time $\lambda(\dot{\gamma})$, and the
first normal stress coefficient $\Psi_1(\dot{\gamma}) = N_1/\dot{\gamma}^2$. This generalization is particularly important since, as already seen in section 3.5, shear-thinning associated with nonlinear constitutive models can lead to qualitatively different shapes of the unstable region in the relevant parameter space. Considering the FENE-CR model, which predicts a shear-rate-independent (polymeric contribution to the) viscosity given by $\eta_0 = \eta_1(1 - \beta)$, but a shear-thinning first normal stress coefficient that is given by $\Psi_1(\dot{\gamma}) = \Psi_{10} \sqrt{\frac{L^2-1}{2} \frac{1}{\gamma_0}}$ for $\dot{\gamma} \gg 1$; here, the subscript ‘0’ denotes the zero-shear-rate limit, and $L$ is the fully extended length of the polymer. Using the above, one obtains $\epsilon^2 \frac{L^2}{2} (1 - \beta) > M_2^2$ for instability, which points to a shear-rate-independent geometrical threshold. Thus, for a given $\beta$, the instability will vanish for sufficiently small $\epsilon$ due to shear thinning. Note that the above threshold applies to different configurations by appropriate choice of $\epsilon$; for instance, $\epsilon = d/R_m$ for Taylor-Couette flow, $\epsilon = \theta_0$ for the cone-and-plate geometry, and $\epsilon = H/R$ for the parallel-plate configuration. Consideration of a FENE-P model, which predicts shear thinning of both the first normal stress coefficient and viscosity [183], leads to a stronger stabilizing role of shear thinning, with the Pakdel-McKinley criterion yielding $\epsilon^2 \frac{L^2}{2M_2^2} (1 - \beta) > M_3^2$, $W_{00}$ being the Weissenberg number based on the zero-shear relaxation time. A sketch of the boundaries demarcating the unstable region on the $\epsilon - W_i$ plane, obtained using the Oldroyd-B and nonlinear (FENE-CR and FENE-P) constitutive models, are shown in Fig. 13. While the Oldroyd-B captures the lower branch of this boundary, the upper branch arises solely due to effects of shear thinning, and as seen above, is dependent on the details of the nonlinear terms (the version of Fig. 13, on the $D_0 = 1/\epsilon$ plane appears in [181], without the trend for the FENE-P model).

In summary, while the Oldroyd-B model and its refinements do provide a first-cut prediction of viscoelastic instabilities in flows with curvilinear streamlines, the comparison between experimental observations and theoretical predictions for instability in viscoelastic Taylor-Couette (and other viscometric) flows is not nearly as quantitative as their Newtonian counterparts for the following reasons: (i) the sensitivity of the threshold conditions to details of fluid rheology such as shear thinning, spectrum of relaxation times, and nonzero $N_2$, (ii) the relatively shallow neutral curve that results in multiple modes getting excited at the onset, and (iii) viscous heating effects.

4. Instabilities in non-viscometric flows

There is a wide variety of flow situations beyond the viscometric flows discussed in the previous two sections, which conform to a local simple-shear topology $^5$, and many of them are again susceptible to elastic instabilities. Poole [185] constructed a purely elastic instability ‘flow map’ which categorizes flows as being viscometric, shear- or extension-dominated, and those with mixed kinematics. By way of illustration, flows in the Taylor-Dean, Dean, and serpentine-channel configurations are predominantly shear-dominated, while the flow in a cross-slot geometry or a T-channel (in the vicinity of the stagnation points) would be largely extensional, with a fluid element experiencing elongation or compression in the streamwise direction. Flow past a cylinder, or a periodic array of cylinders, and that through a contraction-expansion geometry would be characterized by mixed kinematics. In this section, we do not aim to be comprehensive in terms of coverage of the aforementioned nonviscometric flows, but instead provide a few canonical examples to show the range of real instabilities for which the Oldroyd-B model is nevertheless an important foundation.

Before venturing into a discussion on stability, it is worth emphasizing that, for both the rectilinear and curvilinear viscometric flow configurations in sections 2 and 3, the base-state is independent of $W_i$ for the Oldroyd-B model, being identical to that for a Newtonian fluid. This is no longer true for a non-viscometric flow. There is a strong dependence of the base-state flow itself on $W_i$, for both the Oldroyd-B and nonlinear constitutive models, as also evident in experiments and computations [186, 187]. In fact, in contrast to the aforementioned viscometric cases, the dependence on $W_i$ is usually the strongest for the Oldroyd-B model, on account of the extensional stresses having the maximum magnitude (this in turn being due to the underlying divergence of the extensional viscosity, at an order unity $W_i$, in a homogeneous extensional flow). Thus, calculation of the base-state itself is often a non-trivial task. An example is the flow past a cylinder (or a sphere), a problem whose stability is examined in section 4.3, where numerical computations using the Oldroyd-B model invariably break down at a modest $W_i$. This breakdown was originally thought to be

$^5$Note that the local linear flow topology for Taylor-Couette flow, depending on the ratio of the cylinder angular velocities, may range over the entire one-parameter family of planar linear flows which include the hyperbolic and elliptic linear flows [184]; however, the unsteadiness arising from the changing orientations of the principal axes leads to simple-shear-like kinematics.
on account of a physical singularity, being dubbed the large-Weissenberg-number problem [188]. Although it is now known that the breakdown is owing to the eventual inability of any numerical protocol to resolve the stress boundary layer that develops in the neighborhood of the rear stagnation streamline with increasing \( Wi \) (manifesting in experiments as a birefringent strand), computations for moderate \( Wi \) nevertheless remain a challenge for Oldroyd-B in particular. Thus, nonlinear models, where the extensional thickening is alleviated by choosing appropriate choice of the finite extensibility parameter (the FENE-CR model, for instance), are often chosen for computational tractability [159].

4.1. Cross-slot flow

The instability that has garnered substantial attention in recent years is the family of flows seen in a symmetric planar cross-slot experiment; the geometry for this experiment may be regarded as a canonical one for an extensional flow topology. Although first observed by Gardner et al. in 1982 [189], the recent experiments of Arratia and coworkers [190], using a high-molecular weight polyacrylamide solution in a microfluidic channel, have unambiguously established this instability as having an elastic origin. The classical cross-slot geometry consists of four bisecting rectangular channels, with pairs of opposing inlets and outlets, which result in a flow field with a ‘free’ stagnation point nominally located at the center of the cross slot. The fluid velocity at this point is, of course, zero by definition, but the velocity gradient, characterizing the local linear extensional flow, remains finite. Similar free stagnation points also occur, for instance, on the surface of a bubble. In light of the modified cross-slot geometry discussed below, it is worth contrasting the nature of these stagnation points to that on a rigid particle; the latter is only one of a continuum of zero velocity points on the particle surface, and corresponds to a local non-linear extensional flow with the streamwise velocity gradient going to zero right at the particle surface. As mentioned above, a characteristic feature of stagnation point flows is the formation of ‘birefringent strands corresponding to highly localized regions of stretched polymers (associated with large stresses), extending downstream from the stagnation point. The structure of such strands, induced by a flow, that conforms to the symmetry of the cross-slot, has been studied in some detail in earlier efforts [191, 192]. In contrast, the focus here is on the onset of a symmetry-breaking instability.

The instability observed by Arratia and coworkers above is best understood visually: see figure 14. At low \( Wi \), the incoming flow along the left and right channels in the figure) divides symmetrically into the “up” and “down” outflows, in the manner expected of a Newtonian fluid in the inertialess limit. As \( Wi \) is increased, this symmetric flow becomes unstable and the system transitions to an alternate steady state. This is, in fact, not the only instability to be seen in the cross-slot – unstable instabilities also occur, as described by Arratia et al. [190] and, more recently, by Sousa et al. [193]; but, perhaps because of its appealing simplicity, this steady flow has seen the most intense modeling activity.

As mentioned in the Introduction, the Oldroyd-B model is at its best in flows that are shear-dominated. In extensional flows, however, it is flawed. While it captures the strain-hardening behavior seen in real polymeric liquids, the infinite extensibility of the underlying dumbbells implies that a steady extensional flow leads to divergent stresses for flow rates approaching the inverse relaxation time; further, if the critical flow rate corresponding to this divergence is exceeded, the mathematical prediction is that of a negative extensional viscosity. Thus, it is by no means obvious to choose the Oldroyd-B to model the flow in the cross-slot geometry. We note in passing that, for stagnation points on no-slip surfaces, alluded to above, the Oldroyd-B model does not fail in an obvious manner, since the extensional flow is a nonlinear one, with the rate of strain going to zero at the stagnation point.

Nevertheless, Oldroyd-B, and its limiting form, the UCM are exceptionally simple models, with very few parameters, which makes them attractive for a first attempt at modeling any new phenomenon, including ones that owe their origin to an underlying (transient) extensional flow. Indeed, the first simulations to replicate the cross-slot instability were carried out by Poole et al. in 2007 [195] using the UCM model. These were later [196] made more robust using the FENE model. As a result, there was a renewed interest in the stability of pure extensional flows of the Oldroyd-B fluid. In fact it had been known since 1985 that planar linear flows of an Oldroyd-B fluid were susceptible to a linear instability [197] to 2D perturbations; but, this instability is essentially Newtonian in nature, unlike the cross-slot asymmetry which only occurs above a critical \( Wi \). More recently, Cruz & Pinho [198] have found a new analytical approximation to the extensional flow of the Oldroyd-B model. Despite this, the exclusive focus on the vicinity of the stagnation point did not yield any progress in understanding this instability.

The aforementioned focus on the extensional flow topology is, in fact, not representative of the classical cross-slot geometry. While the fluid elements passing in the immediate vicinity of the cross-slot stagnation point do experience a predominantly extensional flow, those passing between the stagnation point and the re-entrant corners also experience shear; in fact,

---

6Even so, and as will be mentioned in Sec. 4.3, the nonlinear extensional flow leads to a breakdown of numerical analyses, based on the Oldroyd-B model, beyond a modest \( Wi \); early signatures of this breakdown led to dubbing this as the ‘large-\( Wi \)’ problem, as mentioned earlier [194].

![Figure 14: Dye advection patterns for a cross-channel flow with two inputs and two outputs at low Re (< 10^2) for (a) Newtonian fluid, and (b) PAA flexible polymer solution (strain rate \( \epsilon = 0.36 \text{s}^{-1} \), Weissenberg number \( Wi = 4.5 \)). The interface between dyed and undyed fluid is deformed by an instability. Reproduced with permission from Ref. [190].](image)
the flow topology in the immediate vicinity of the re-entrant corners is singular, and very different from that close to the stagnation point [199, 200]. In light of the heterogeneity in the local flow topologies, it is important to find out if the observed instability was dominated by the extensional flow dynamics close to the stagnation point, or otherwise. The breakthrough in this regard came when Davoodi et al. [201] modified the flow by placing a small solid cylinder, with the cylinder center at the point where the stagnation point would otherwise be located. Based on the description of the flow topology, in the vicinity of a no-slip surface, above, one no longer expects an extensional-dominant character for the flow away from the corners. Nevertheless, and contrary to expectations, the presence of the cylinder made little difference to the instability, indicating that the stagnation point was not, in fact, the critical area of the flow. The authors went on to show that the onset of instability can be predicted rather well using the Pakdel-McKinley criterion discussed in section 3.9 above. While the value of $M$ required for instability, in the aforementioned criterion, is known from linear stability calculations for the viscometric flows in section 3, it must be regarded as a scalar field, either determined experimentally or from a detailed computation of the base-state cross-slot flow; the maximum value of this ‘$M$-field’ would then determine the threshold. Davoodi et al. [201] plotted contours of the $M$-field, and found it to be the largest near the four corners, both in the absence and presence of the cylindrical insert at the center. The larger magnitude of the $M$ parameter was attributed to the strong streamline curvature and high deformation rate prevalent near the corners. In hindsight, this explains why the Oldroyd model, even with its problematic response to extension, turned out to reproduce experimental observations so well. A consequence, of course, is that the utility of the classical cross-slot flow configuration, as an extensional flow rheometer, is limited by the instability.

A closely related line of research is that of Haward, Alves, McKinley and collaborators [202], who have focused on a shape-optimized cross-slot flow geometry, in order to overcome the limitation of the aforementioned classical cross-slot geometry where the re-entrant corners likely decrease the neighborhood in which the flow is predominantly extensional, in turn leading to an instability that is (re-entrant) shear-dominated. In this modified cross-slot, abbreviated as ‘OSCER’, the sharp corners are replaced by smoothly varying contours, which result in a homogeneous elongation rate over a larger neighborhood of the stagnation point. The OSCER configuration was again found to susceptible to an instability that led to an elasticity-induced asymmetry, with some similarities to that observed by Arratia and coworkers for the classical cross-slot.

To rationalize their findings, Haward et al. [203] again plotted the ‘$M$-field’ (of the Pakdel-McKinley criterion) for the OSCER geometry, and found that maximum values of this field compared well with the critical $M$ for the onset of instabilities in viscometric torsional flows discussed in the previous section. Crucially, the spatial location of the maximum $M$ occurred close to the stagnation point, in marked contrast to the classical cross-slot, suggesting an extensional origin for the elastic instability instead. In this regard, it is worth noting that the Pakdel-McKinley criterion was developed for viscometric (or nearly viscometric) shearing flows. That it works well for both the classical cross-slot and OSCER elastic instabilities suggests that the observed instability relies only on the coupling between the streamline curvature and streamwise tensile stresses, regardless of whether the underlying kinematics is shear or extension dominated. Nevertheless, the success of the criterion is not a substitute for the actual physical mechanism, and more work is needed in this regard that would likely benefit from the focus on the dynamics in the vicinity of the birefringent strand.

Unlike the original experiments in the classical cross-slot geometry, the OSCER device has also been analyzed in regimes which allow effects of both inertia and elasticity to become important [204], with the ratio of the two (as already seen) being characterized by the elasticity number $E$. The discussion in the preceding paragraphs pertain to the elasticity dominant limit. In the opposite limit of $E < 1$, the instability in the OSCER device occurred only beyond a critical $Re$, manifesting as an oscillatory motion of the birefringent strand alluded to above. Owing to the importance of both elasticity and inertia, the authors have referred to this mode as an ‘inertio-elastic’ mode. The experimental results have been summarized in the form of a stability diagram on the $Re-Wi$ plane, demarcating regions of occurrence of stable flows conforming to the cross-slot symmetry, and those corresponding to purely elastic and inertio-elastic instabilities.

### 4.2. Contraction–expansion flow

One of the original benchmark flows for numerical simulations in viscoelastic fluids was the 4:1:4 contraction-expansion flow (or, earlier, just contraction flow) of an Oldroyd-B fluid. The reason is that the (relatively simple) flow encapsulates so many complex aspects: large elastic stresses, varying shear rates, and curved streamlines – and captures some of the physics of industrial flows like extrusion. Early work (e.g., Refs.[205, 206]) focused on accurate calculations of the pressure drop, but in time it became clear that instabilities could feature in this flow, as in so many. Indeed, there is now a much wider interest in the variety of different flow behaviour in contraction–expansion flows than in the simple tracking of pressure drops.

The theory of these flows was, like so many others, initially founded in the Oldroyd-B fluid. Its correspondence with experiment made a huge leap forward in the 1980s with Boger’s discovery of a class of constant-viscosity viscoelastic liquids, allowing us to separate the effects of shear-thinning from those of elasticity. The resultant progress is reviewed in Boger’s seminal work of 1987 [207], describing the flow transitions (at low Reynolds number, and with no shear-thinning) in the upstream contraction region. At low flow rates there is a simple Newtonian-like contraction, which progresses through the appearance of a lip vortex, elastic vortex growth, and finally the appearance of elastic instabilities.

It is possible to achieve a high elasticity number using Boger fluids because of their high viscosity. Fluids which shear-thin at high flow rates tend to have their elastic effects lost through
shear-thinning in laboratory-scale experiments for the following reason. Because inertial effects scale as the inverse of viscosity, fluids which shear-thin at high flow rates can easily have their phenomenology dominated by inertia, and any elastic instabilities become harder to observe. While the Reynolds number is proportional to the relevant flow lengthscale, the elasticity number scales as the inverse of the square of the flow lengthscale. This implies that inertial effects can be suppressed by the use of microfluidic experiments, while simultaneously allowing the effects of elasticity to become prominent in the flow. Thus, with the advent of microfluidic experimentation, researchers now have the ability to attain moderate Reynolds numbers with the effects of viscoelasticity remaining important. These advances have resulted in a good qualitative understanding of the different flow patterns which emerge upstream of a contraction [208, 209], across the whole range of Re–Wi parameter space. Here the dynamics are more complex: inertioelastic turbulence (section 6) can be seen at the contraction inlet even before the appearance of a large lip vortex. This upstream vortex can appear on one side of the flow only, or fluctuate from side to side. Most simulations of axisymmetric contractions [210], however, have imposed axisymmetry (to avoid simulating a fully 3D flow). It seems natural to believe that the equivalent to the fluctuating single upstream vortex seen in a planar contraction (e.g., Ref. [211]) would be a vortex which migrates around the central axis in 3D, resulting in helical fluctuations in the downstream flow.

There has been one direct study of the stability of contraction–expansion flows [212], which found a primary mode of instability which was downstream, rather than upstream, of the contraction – but this was work associated with polymer melts, so it used a more appropriate constitutive equation (the Rolie-Poly model) and found the mechanism to be critically dependent on chain stretch, so it is no surprise that this downstream instability has not been seen in work on Oldroyd-B nor in experiments on solutions.

4.3. Viscoelastic flow past a cylinder

Fluid flow around a circular cylinder is one of the most widely studied external flow configurations in fluid mechanics. This configuration is also one of the benchmark problems in viscoelastic fluid mechanics, and has been used to test how well constitutive models and numerical schemes are able to predict experimental data. With the advent of microfabrication and 3D printing technologies, Haward, Chen and collaborators have shown that it is possible to fabricate cylinders with radii of $O(10) \mu m$ [213, 214] in a microfluidic channel. The fabrication of slender cylinders with high length-to-radius (aspect) ratio ($\sim 50$) in a channel with relatively low cylinder diameter to channel width (‘blockage’) ratio ($\sim 0.1$) enables the experiments to mimic the 2D flow past a circular cylinder. For flow past cylinders of such small radii, the typical Reynolds numbers will be significantly lower than unity ($Re < 10^{-4}$ in Ref. [214]), and inertial effects are therefore negligible. At the same time, the Weissenberg numbers can be very high ($Wi \sim O(4000)$ in Ref. [214], with $Wi$ defined using the cylinder radius and the average velocity in the microfluidic channel) implying that elastic forces can be significantly enhanced ($E \sim O(10^6)$ in the experiments). Flow over cylinders of such small radii thus provide ideal platforms for studying purely elastic instabilities in non-viscometric flows, which are characterized by features of both shear and extensional kinematics.

At lower Weissenberg numbers, the fluid flow around the cylinder, although fore-aft asymmetric, exhibits a lateral symmetry. Note that the fore-aft asymmetry is on account of an elastic wake that develops along the rear stagnation streamline, and whose length increases with $Wi$, as has been shown for the case of a sphere [186]. Beyond $Wi \approx 60$, however, the lateral symmetry is broken due to a purely elastic instability which also leads to a distortion of the downstream elastic wake [214]. The resulting flow is characterized by the preferential passage of the fluid around one side of the cylinder, the preferred side being determined by experimental conditions (theoretically, speaking, passage around either side is possible, mathematically corresponding to a pitchfork bifurcation).

The efforts of Haward, Shen and coworkers have shown that the instability is initiated in the vicinity of the downstream wake, on account of the high streamline curvature and tensile stresses near the downstream stagnation point. Numerical simulations using the linear Phan-Thien and Tanner (l-PTT) model were again used to compute the $M$-field of the Pakdel-McKinley criterion [215]. This exercise showed the variation of the onset $Wi$ with blockage ratio to be in excellent agreement with predictions. However, for a fixed $Wi$ and different blockage ratios, asymmetric flows are present only when the characteristic shear rate near the cylinder lies in the shear-thinning region of the flow curve, suggesting that both shear thinning and elasticity play essential roles in the downstream asymmetric flow. For shear rates in the high-shear rate plateau of the flow curve, symmetric flow is again recovered. The authors further constructed a stability diagram using the Weissenberg number and a parameter that characterizes the extent of shear thinning, for different values of extensional rate-hardening parameter in the l-PTT model.

4.4. Extrudate instabilities and die-swell

Extrusion is such a ubiquitous industrial process, that there has been a sustained interest in its stability. The flow within the die is shear-dominated, so these flows are a suitable candidate for modelling with Oldroyd-B; earlier reviews on this topic can be found in Refs. [3] and [6]. In this section, we briefly review various phenomena associated with extrusion, and the current understanding of them.

Die swell

Not technically an instability, die swell is a steady phenomenon in which the extrudate (especially immediately downstream of the nozzle) becomes wider than the nozzle. Empirically, it has long been understood that die swell is at least partly a viscoelastic phenomenon, as polymeric materials exhibit a much larger die swell ratio than Newtonian liquids.

An early physical interpretation of die swell was proposed by Tanner in 1970 [216]: the swelling ratio (the ratio of downstream to upstream radius) is calculated from the recoverable
stress $R_s$, defined as the ratio of $N_1$ to shear stress, both at the
die wall. This formula – which is a much simplified version of
the real problem – has been remarkably robust over 50 years;
it is particularly effective at low to moderate values of $R_s$ (be-
low, say 2). It was derived from the K-BKZ model, which has
a single time constant, so is more appropriate to UCM than to
Oldroyd-B with its relaxation and retardation times.

Indeed, because of the difficulties in simulating this flow,
with its transition from shear to extension and a free surface,
much of the early work was confined to the UCM fluid be-
cause of its simplicity [217, 218, 219, 220]. In both planar
and axisymmetric geometries, researchers found that there was
a major and sudden divergence from Tanner’s formula above
some critical Weissenberg number. Extension to Oldroyd-
B [221, 222, 223, 224] brought no great surprises (either in pla-
nar or axisymmetric geometries): the die swell ratio increases
with increasing Weissenberg number, and decreases with an in-
crease in either solvent viscosity $\beta$ or inertia $Re$.

In practical applications, die swell is much more pronounced
in polymer melts than in dilute solutions. The Oldroyd-B model
ceases to adequately capture flow behaviour, which depends
critically on the level of polymer chain stretch as captured by,
for example, the Rolie-Poly model [225].

Sharkskin

We mention the commonly-observed sharkskin instability
here for completeness. So called because of its visual similarity
to the small-scale structure of a shark’s skin, this phenomenon
occurs in extrusion of polymer melts, and is critically depen-
dent on the interaction between the polymer molecules and the
channel walls, especially near the die exit. The Oldroyd family
of models is not best designed for simulation of the details at
this flow singularity, with its high extension rates; what is re-
quired goes beyond the pure continuum approach and includes
wall-slip, the disentanglement of wall-adsorbed polymers from
the bulk flow, and therefore much more advanced constitutive
modelling [226].

Helical instability

In extrusion from an axisymmetric die, or even a zero-length
die (just a circular hole), a common form of failure is a heli-
cal deformation of the extrudate. While the observations take
place in the extrudate, in face the instability here happens up-
stream of the die (or the abrupt contraction, in the case of a sim-
ple hole). These beautiful extrudate shapes are nothing more
than the downstream consequence of the instabilities discussed
above in section 4.2.

Melt fracture

Melt fracture is a term used to describe total failure of ex-
trusion [5, 6] an extrudate which comes out of the die having
lost all coherence. This behaviour can come from a wide range
of physical causes, many of which begin with one of the more
moderate failure modes described above, but some of which are
rooted in chemistry rather than physics. While the onset of the
difficulties may be predictable using the simplicity of a model
like Oldroyd, this eventual downstream state is probably too
far gone for any theoretical treatment: only numerical simula-
tions [227] can hope to reproduce them.

5. Nonmodal stability

Modal stability analysis has been phenomenally successful
at explaining experimental observations in large variety of fluid
flows. However, and perhaps surprisingly, it fails spectacu-
larly for rectilinear flows of Newtonian fluids. Standard lin-
ear stability analysis predicts plane Couette flow and circular
Poiseuille (i.e., pipe) flow to be linearly stable for all Reynolds
numbers. Yet, instability is observed experimentally. Although
plane Poiseuille flow is predicted to be linearly unstable above
a certain Reynolds number, in practice instability is observed at
Reynolds numbers much lower than this critical value.

While these discrepancies may be attributed to the presence
of finite-amplitude perturbations, which would cause the as-
sumption of linearization to fail, it is now widely recognized
that the discrepancies are likely due to other assumptions in-
herent in modal analysis [228, 229, 230, 231]. As briefly men-
tioned in Section 1, modal analysis provides predictions about
asymptotic stability, i.e., whether perturbations grow or decay
at long times. It says nothing about behavior at short times.
However, two stable modes can interact in such a way that a
disturbance grows at short times before decaying at long times.
This growth at short times could put the flow into a regime
where nonlinear effects are no longer negligible, causing a tran-
sition to another flow state. Thus, an initially small-amplitude
disturbance can be amplified through a purely linear mechanism
that is overlooked by modal analysis.

Nonmodal analysis yields information about this alternative
type of perturbation growth [231, 232]. In this section, we will
provide an overview of some basic ideas, discuss their rele-
vance to viscoelastic channel flows, highlight some recent re-
sults related to amplification of external disturbances, and iden-
tify some important open issues in the area. Our discussion is
not intended to be a comprehensive tutorial or review, but is in-
stead aimed at providing non-expert readers a brief introduction
to some fundamental concepts and selected results.

5.1. Nonmodal amplification: Basic ideas

To illustrate the basic ideas of nonmodal amplification, it is
sufficient to consider the following coupled pair of constant-
coefficient linear ordinary differential equations (adapted from
[233]; see also [234]):

\[
\begin{pmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{pmatrix} = \begin{pmatrix}
  \lambda_1 & 0 \\
  R & \lambda_2
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix},
\]

where the dot denotes a time derivative and $\lambda_1, \lambda_2, \text{and} R$ are all
real constants. We will denote the matrix appearing in (6) as $A$.
The solution to this equation system for $\lambda_1 \neq \lambda_2$ is

\[
x_1 = x_1^0 e^{\lambda_1 t},
\]

\[
x_2 = x_2^0 e^{\lambda_1 t} + \frac{R x_1^0}{\lambda_1 - \lambda_2} \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right),
\]
where $x_0^0$ and $x_1^0$ are the initial conditions for $x_1$ and $x_2$.

With $\lambda_1, \lambda_2 < 0$, both $x_1$ and $x_2$ decay to zero as $t \to \infty$. However, if $R \neq 0$, $x_2$ can actually grow before decaying if $x_1^0 \neq 0$. Note that the difference of the two decaying exponentials appearing in the expression for $x_2$ is zero at $t = 0$ and approaches zero at long times, but is non-zero at intermediate times. The influence of this term increases as the magnitude of $R$ does, and so does the amplification rate, since $dx_2/dt \sim R x_0^0$ for $R \gg 1$. Thus, there exist initial conditions such that $x_2$ can exhibit large transient growth. Such growth is also referred to as nonmodal amplification because it would be missed by standard modal analysis, which focuses on eigenvalues, and thus long-time behavior. We note that terms like $te^{-t}$ that arise when $\lambda_1 = \lambda_2$ reflect a resonant interaction and also exhibit transient growth, but such terms are not required for transient growth as the above example illustrates.

Clearly, the parameter $R$ is causing this behavior. If $R = 0$, there is no transient growth and both $x_1$ and $x_2$ decay monotonically to zero. When $R \neq 0$, the two stable modes interact, leading to nonmodal amplification. Having $R \neq 0$ significantly changes the properties of the matrix $A$ in (6). In particular, $A$ no longer commutes with its adjoint, which in this case is simply the transpose.

When a linear operator $L$ commutes with its adjoint $L^*$, then $LL^* = L^*L$ and we refer to $L$ as being a normal linear operator [235, 236]. Normal linear operators have orthogonal eigenfunctions. However, if $L$ does not commute with its adjoint, then $LL^* \neq L^*L$, and $L$ is non-normal. Non-normal linear operators produce eigenfunctions that are non-orthogonal.

In the context of the present example, the eigenvectors are

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{1 + R^2/(\lambda_1 - \lambda_2)^2}} \begin{pmatrix} 1 \\ \frac{R}{\lambda_1 - \lambda_2} \end{pmatrix}. \tag{9}$$

These eigenvectors are orthogonal only when $R = 0$. As $R \to \infty$, these eigenvectors become parallel. In that situation, an initial condition that is nearly orthogonal to the eigenvectors would be “misfit”, and the coefficients involved in the solution would have a very large magnitude (e.g., $(1 \; 0)^T = -e^{-1}(0 \; 1)^T + e^{-1} (\epsilon \; 1)^T$, where $\epsilon \ll 1$).

If system (6) represents the linearization of a nonlinear system around a steady state, then simply focusing on the eigenvalues is misleading. The steady state is asymptotically stable, but there may be large growth of perturbations at short times. This could put the system into a regime where nonlinear terms are no longer negligible, and the system may transition to another state rather than returning to the steady state one started with.

Although the above discussion has focused on an unforced linear system with non-zero initial conditions, the same ideas apply to a forced linear system with zero initial conditions. If the underlying linear operator is non-normal, then perturbations in the problem variables created by the forcing can have magnitudes considerably larger than the magnitude of the forcing (e.g., see [231, 232, 237]).

### 5.2. Relevance to viscoelastic channel flows

The above example is directly relevant to channel flows of Newtonian and viscoelastic fluids [82]. We consider channel flows driven by a constant pressure gradient (plane Poiseuille flow) or a constant boundary velocity (plane Couette flow). The streamwise direction (direction of mean flow) is $x$, the wall-normal direction is $y$, the spanwise direction is $z$, and $t$ is time. The equations are linearized around the base state and Fourier transforms are applied in the $x$- and $z$-directions to obtain a system of partial differential equations for the velocity, pressure, and stress fluctuations where $y$ and $t$ are the independent variables. The Fourier transforms introduce the wavenumbers $k_x$ and $k_z$, characterizing variations in the streamwise and spanwise directions, respectively.

For the case of streamwise-constant disturbances ($k_x = 0$), the linearized governing equations can be put into forms very similar to the example problem (6), allowing us to make powerful analogies [82]. For channel flows of Newtonian fluids, the linearized governing equations can be written as

$$\begin{pmatrix} \dot{\psi} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \bar{A}_{11} & 0 \\ Re \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} \psi \\ u \end{pmatrix}, \tag{10}$$

where $\psi$ is the streamfunction in the $yz$-plane, $u$ is the streamwise velocity component, and the dot now denotes a partial derivative with respect to time. Here, $A_{11} = \Delta^2 \Delta^2$ is known as the Orr-Sommerfeld operator, and $\bar{A}_{22} = \Delta$ is known as the Squire operator. The operator $A_{21} = -ik_xU'(y)$ is known an the vortex-tilting or lift-up term [238, 239], with $\Delta = \partial_{y^2} - k_x^2$ and $\Delta^2 = \partial_{y^2} - 2k_x^2\partial_{y^2} + k_x^4$. In this term, $U(y)$ is the base-state velocity and the prime denotes a derivative. The Reynolds number $Re = \rho U_0 L/\eta_s$, where $\rho$ is the density, $U_0$ is the maximum magnitude of the base-state velocity, $L$ is the channel half-height, and $\eta_s$ is the viscosity.

Comparing (10) with (6), we see that the Reynolds number $Re$ in (10) plays the role of the parameter $R$ in (6). When $Re$ is non-zero, the Orr-Sommerfeld and Squire modes become coupled, and the problem becomes increasingly non-normal as $Re$ increases. Note that the analogy can be made more direct by recognizing that numerical discretization of the derivatives with respect to $y$ will convert the operator in (10) into a standard matrix, and the example problem (6) can be generalized to higher dimensions.

The coupling term $A_{21} = -ik_xU'(y)$ involves interaction between the mean shear, or vorticity, and three-dimensional velocity perturbations (This term is often referred to as vortex tilting, but it really involves the mean vorticity.) It gives rise to alternating regions of high and low streamwise velocity, often referred to as streamwise streaks [238, 239].

For inertialess channel flows of Oldroyd-B fluids, the linearized governing equations for the components of the polymer stress fluctuations $\tau_{ij}$ can be written as [82]

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ Wi A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix}, \tag{11}$$

where $\tau_1 = (\tau_{22} \tau_{33} \tau_{33})^T$ and $\tau_2 = (\tau_{12} \tau_{13} \tau_{13})^T$ with $(1, 2, 3)$ representing $(x, y, z)$. The Weissenberg number is given by $Wi = \lambda U_0/L$, where $\lambda$ is the fluid relaxation time. The Weissenberg number is the ratio of the fluid relaxation time to the characteristic flow time, and provides a measure of fluid elasticity.
The operators $A_{11}$, $A_{22}$, and $A_{21}$ are all independent of Wi; the definitions of these operators can be found in [82]. A static-in-time relationship [82] (not shown here) connects the polymer stress fluctuations to the velocity fluctuations. The operators $A_{11}$ and $A_{22}$ involve the spanwise wavenumber $k_z$, derivatives with respect to $y$, and the viscosity ratio $\beta = \eta_s/\eta_p$, where $\eta_s$ and $\eta_p$ are, respectively, the solvent and polymer contributions to the viscosity. The operator $A_{21}$ shares these features as well, but it also involves the first and second derivatives of the base-state velocity. This operator embeds the interaction between base-state velocity gradients and polymer stress fluctuations, and the interaction between base-state polymer stress gradients and velocity fluctuations. Physically, such interactions produce polymer stretching.

Comparing (11) with (10) reveals a remarkable analogy between creeping flows of Oldroyd-B fluids and inertial flows of Newtonian fluids for streamwise-constant disturbances [82]. Polymer stretching and the Weissenberg number in elasticity-dominated flows of viscoelastic fluids play the role of vortex tilting and the Reynolds number in inertia-dominated flows of Newtonian fluids. When $\text{Wi} = 0$, the operators $A_{11}$ and $A_{22}$ are decoupled. However, when $\text{Wi} \neq 0$, these operators become coupled and the problem becomes increasingly non-normal as Wi increases.

The discussion above lays bare the relevance of example problem (6) to nonmodal amplification in streamwise-constant channel flows of Newtonian and viscoelastic fluids. In the case of Newtonian fluids, initially small-amplitude perturbations can become highly amplified through a nonmodal mechanism. This puts the flow into a regime where nonlinear terms are no longer negligible, triggering a transition to turbulence even when no eigenvalue of the linearized problem has a positive real part [240, 241, 242, 243, 244]. In the case of viscoelastic fluids, such disturbance amplification could trigger a transition to elastic turbulence. Further discussion of the linearized dynamics in viscoelastic fluids can be found in [245].

In analogy with example problem (6), the discussion above implies that there exist initial conditions for systems (10) and (11) that lead to large transient growth of flow fluctuations. This raises the question of how such initial conditions can be generated [246, 247, 237], a topic we turn to next.

5.3. Amplification of external disturbances

Fluid flows in practice are subject to disturbances that arise from sources such as vibrations and pressure fluctuations. If these disturbance are amplified by the flow, they could produce the initial conditions that lead to significant transient growth, or could themselves induce a flow transition [246, 247, 237].

While the exact form of disturbances that arise in experimental settings may be unknown, it is still useful to consider how a flow model responds to various types of well-defined disturbances. This is closely related to the topic of frequency response, which is often covered in undergraduate courses on control systems. There, one typically considers systems of ordinary differential equations with disturbances that are time-periodic. Of course, disturbances can be localized in time as well (e.g., an impulse), and with the partial differential equations that arise in flow models, the disturbances can be spatially varying. Disturbance amplification can also provide insight into model robustness, or how sensitive a model is to neglected terms [246, 237, 231, 234].

Some of the most basic disturbances that can be considered in fluid flows are those that are random, e.g., white noise. A simple way to account for these disturbances is to include them as body forces in the linearized equations. Because disturbances are easily characterized by streamwise and spanwise wavenumbers, it is particularly useful to consider disturbances that are harmonic in those directions but stochastic in the wall-normal direction and time. It then of interest to know how the disturbances to the base flow behave as a function of the streamwise and spanwise wavenumbers, and other parameters such as the Reynolds number, Weissenberg number, and viscosity ratio.

In control-systems courses, transfer functions relate input and output variables. The same idea can be applied to fluid flows, with the input being the body force and the output measured in terms of scalar quantities like the kinetic energy of the velocity fluctuations [232]. Below, we discuss some important results concerning this topic, which draw heavily upon ideas and tools from linear systems theory and control theory.

We begin with inertial flows of Newtonian fluids subject to a body force that is harmonic in the streamwise and spanwise directions but stochastic (white noise) in the wall-normal direction and time [237]. Figure 15 shows a plot of the ensemble average energy density, $E$. This quantity, which we will refer to as the energy density for brevity, is simply the kinetic energy of the velocity fluctuations averaged over the wall-normal direction and time [246, 237].

![Figure 15: Plot of $(1/2)\log_{10}E$ for plane Poiseuille of a Newtonian fluid at $Re = 2000$; adapted (with permission) from Fig. 5 of [237].](image-url)

Figure 15 shows the energy density as a function of $k_z$ and $k_i$ for plane Poiseuille of a Newtonian fluid at $Re = 2000$. The largest energy density occurs for $k_z \approx 0$ and $k_i \sim 1$, which corresponds to streamwise-constant disturbances. In this limit, an analytical expression can be obtained for the energy density [246, 237]

$$E = Re f_N(k_z) + Re^4 g_N(k_z).$$

The function $f_N$ is a monotonically decreasing function of $k_z$, whereas the function $g_N$ has a peak at $k_i \sim 1$. The function $f$ reflects the influence of viscous dissipation and the function $g$ reflects the influence of vortex tilting. Thus, for $Re \gg 1$,
the term involving \( g \) dominates, and the energy density will be largest for \( O(1) \) values of \( k_z \). This highlights the importance of three-dimensional disturbances in inertial channel flows of Newtonian fluids.

Figure 16 shows the energy density as a function of \( k_z \) and \( k_z \) for plane Poiseuille of an Oldroyd-B fluid at \( Re = 1000 \), and two different elasticity numbers, \( E = Wi/Re = 0.1 \) and 10 [248]. Note that \( E = Wi/Re = \lambda v/L^2 \) is the ratio of the fluid relaxation time to the vorticity diffusion time, with \( v \) being the kinematic viscosity. It is seen that the energy density increases with increasing elasticity number. In addition, the most amplified disturbances become increasingly streamwise constant as the elasticity number increases.

For streamwise-constant disturbances at large elasticity numbers, the energy density is found to obey the following expression [249]

\[
E \approx Re f_{VE}(k_z, \beta) + E Re g_{VE}(k_z, \beta),
\]

(13)

As in the Newtonian case, the function \( f_{VE} \) is a monotonically decreasing function of \( k_z \), whereas the function \( g_{VE} \) has a peak at \( k_z \sim 1 \). The function \( f \) again reflects the influence of viscous dissipation but the function \( g \) now reflects the interaction of the polymer stresses with the velocity field. When \( Re \ll 1 \), the term involving \( g \) will dominate if \( E \) is large enough, and

the energy density will be largest at \( O(1) \) values of \( k_z \). This highlights the importance of three-dimensional disturbances in strongly elastic channel flows of viscoelastic fluids.

In the creeping-flow limit, the problem of energy amplification from stochastic forcing becomes ill-posed since the forcing affects the velocity and pressure directly without being “filtered” by the unsteady inertial term [83]. This issue can be addressed by applying singular perturbation methods to treat the case of large elasticity numbers [83]. Such an analysis shows that for inertialess streamwise-constant flows of Oldroyd-B fluids, the linearized dynamics of the wall-normal vorticity \( \eta \) are governed by

\[
\partial_t \Delta \eta = -Wi(1/\beta - 1)(\partial_{z}(U'(y)\tau_{22}) + \partial_{z}(U'(y)\tau_{23})) - (1/\beta)\Delta \eta
\]

(14)

In contrast, for streamwise-constant inertial flows of Newtonian fluids, the linearized dynamics of \( \eta \) are governed by

\[
\partial_t \eta = -Re U'(y)\partial_y \eta + \Delta \eta
\]

(15)

where \( \eta \) is the wall-normal velocity fluctuation. The first term on the right-hand side of (15) corresponds to vortex-tilting and acts as a source term. The spanwise vorticity of the base flow, \( U'(y) \) gets “tilted” in the wall-normal direction by spanwise changes in \( \eta \). This leads to the amplification of \( \eta \) and thereby the streamwise velocity \( (\eta = \partial_x u) \), giving rise to streamwise streaks. This is the viscoelastic analog of the lift-up effect, which was briefly mentioned in Sec. 2.4.

We now consider the physical interpretation of (14) [83]. Here, the first term on the right-hand side represents spanwise variations in stretching of polymer stress fluctuations by the background shear. This stretching, which acts as a source term, produces amplification of \( \eta \) and, consequently, \( u \), leading to streamwise streaks. Thus, there is again a remarkable analogy between creeping flows of Oldroyd-B fluids and inertial flows of Newtonian fluids for streamwise-constant disturbances. Polymer stretching and the Weissenberg number take the roles of vortex tilting and the Reynolds number.

Further analysis reveals that the energy associated with the velocity fluctuations is \( O(Wi^2) \) [83]. One can also define an energy associated with the stress fluctuations, and this is \( O(Wi^4) \) [83]. The same scalings are obtained if one considers creeping flows with disturbances that are harmonic in time as well as in the streamwise and spanwise directions, and deterministic in the wall-normal direction [234]. Thus, in the creeping-flow limit, the more elastic the fluid is, the larger the disturbance amplification.

Although the above discussion focuses on work conducted over approximately the past decade, it must be pointed out that the potential importance of nonmodal amplification in channel flows of viscoelastic fluids was recognized much earlier [250, 251, 252, 253, 254]. However, that prior work mainly focused on two-dimensional (i.e., spanwise-constant) flows and did not consider amplification of external disturbances.

5.4. Open issues

The results highlighted above indicate that channel flows of viscoelastic fluids are exceedingly sensitive to external disturbances. Amplification of these disturbances could create the
Figure 17: Three-dimensional streamtubes of the velocity fluctuation vector arising from an impulsive disturbance in plane Poiseuille flow at $Re = 50$. For the FENE-CR fluid, $Wi = 50$, $\beta = 0.5$, and $L = 100$ (where $L$ is the maximum polymer extensibility). Reproduced with permission from Fig. 9 of [255].

initial conditions needed for transient growth, or could itself put the flow into a regime where nonlinear effects are no longer negligible. In this way, amplification of initially small-amplitude disturbances by a purely linear but nonmodal mechanism may eventually trigger a transition to a more complicated flow state such as elastic turbulence.

Despite this progress in our fundamental understanding of nonmodal amplification, there remain important open issues, some of which are identified and briefly discussed here.

Localized disturbances

The discussion in §5.3 focused on disturbances in the form of body forces distributed throughout the flow domain. However, this is unlikely to be the case in most experiments, where localized disturbances are easier to realize. A simple way to introduce a localized disturbance in experiments is via obstacles, which exert a drag force on the fluid (e.g., [256, 13, 257]). Although these obstacles are of finite size, consideration in nonmodal analysis of disturbances that are localized at a point reveals rather rich behavior in the linearized dynamics.

An example of the dramatic influence viscoelasticity can have on the linearized dynamics of localized disturbances is shown in Fig. 17 [255] (Other examples of the evolution of localized disturbances are discussed in [258, 259]). Here, the disturbance occurs in plane Poiseuille flow and takes the form of an impulse in space and time. The FENE-CR constitutive equation, which accounts for the finite extensibility of polymer molecules, is used for these calculations.

It is seen from Fig. 17 that the presence of vortical structures is more pronounced in viscoelastic fluids than in Newtonian fluids. The curved streamlines associated with these vortical structure could be susceptible to additional instabilities [4, 7] if the amplitude of the velocity disturbance becomes sufficiently large. Determining whether such a transition occurs will require nonlinear calculations, another open issue we discuss below. Nonlinear calculations involving finite-sized obstacles similar to those used in experiments (e.g., [256, 13, 257]) would also help bridge the gap between theory and experiment, as would additional experiments in which the disturbances are more localized (e.g., by using a small actuator).

As another example of the richness of the linearized dynamics, recent calculations using localized time-periodic disturbances in plane Poiseuille flow show that polymer-stress fluctuations can be amplified by an order of magnitude while there is only negligible amplification of velocity fluctuations [260]. This appears consistent with experimental observations of elastic turbulence in microchannel flows of viscoelastic fluids, where the magnitude of velocity fluctuations decreases downstream before increasing [256, 13]. Notably, the large stress amplification is highly localized in space, occurs for spanwise-constant disturbances, and was overlooked in prior studies that used square-integrated measures of disturbance amplification, which are typically applied in nonmodal analysis [260]. The large stress amplification could put the flow into a regime where nonlinear terms are no longer negligible, and this could trigger a transition to elastic turbulence.

Finite extensibility

As noted above, the calculations shown in Fig. 17 use the FENE-CR constitutive equation. Accounting for the finite extensibility of polymer molecules will be important for strengthening connections between theory and experiment. Some progress has already been made on this front. Lieu et al. considered the influence of harmonic body forces on creeping plane Couette flow of FENE-CR fluids [234]. It was found that the velocity and polymer stress fluctuations are proportional to $\hat{L}^2$ and $\hat{L}^4$, respectively, as $Wi \rightarrow \infty$, where $\hat{L}$ is the maximum polymer extensibility. In contrast, as $L \rightarrow \infty$, the velocity and polymer stress fluctuations are bounded by $Wi^2$ and $Wi^4$, respectively (see also §5.3). Clearly, finite extensibility places bounds on the achievable level of nonmodal amplification. Nonlinear calculations will thus be critical to ascertaining how important nonmodal amplification is in triggering flow transitions to complex states such as elastic turbulence.

Nonlinear calculations
6. Nonlinear stability of parallel shear flows

The analysis of viscoelastic flow instabilities presented in the previous Sections was largely restricted to the linear theory, either modal or non-modal. Here, we demonstrate that the Oldroyd-B model can also be used to successfully describe strongly non-linear states that emerge beyond a linear instability or even in its absence. As an example, we treat the case of parallel shear flows of dilute polymer solutions, like pressure-driven channel flows or by the relative sliding of the plates (plane Couette flow). We select a Cartesian coordinate system with \((x, y, z)\) being along the streamwise, velocity gradient, and spanwise directions, respectively. As in the previous Sections, our starting point is the dimensionless version of the Oldroyd-B model. In the absence of inertia, it is given by

\[
\tau + Wi \frac{\partial \tau}{\partial t} + \nu \cdot \nabla \tau - (\nabla \nu)^T \cdot \tau - \tau \cdot (\nabla \nu) = (\nabla \nu) + (\nabla \nu)^T, \tag{16}
\]

\[
- \nabla p + \beta \nabla^2 \nu + (1 - \beta) \nabla \cdot \tau = 0, \tag{17}
\]

\[
\nabla \cdot \tau = 0, \tag{18}
\]

where \(p\) is the pressure, \(\tau\) is the polymeric contribution to the total stress, and \(\nu\) is the velocity of the fluid assumed to satisfy the appropriate no-slip boundary conditions. These equations are rendered dimensionless by using the maximum value of the laminar fluid velocity \(U\) as a unit of velocity, half the distance between the plates \(d\) as a unit of length, and their ratio \(d/U\) as a unit of time. This yields \(\tilde{W} = \lambda U/d\) and \(\tilde{\beta} = \eta_s/(\eta_s + \eta_p)\), where \(\eta_s\) and \(\eta_p\) are the solvent and polymer contributions to the total viscosity of the solution; \(\lambda\) is the Maxwell relaxation time of the model. Equations \((16)-(18)\) can be written concisely as

\[
\tilde{L} V + \tilde{A} \frac{\partial V}{\partial t} = N(V, V), \tag{19}
\]
where the perturbation vector \( V = (v', \tau', p')^T \), comprises deviations of the velocity, stress, and pressure from their laminar values. Here, \( \hat{L} \) and \( N \) are the linear operator and the quadratic non-linear operator, respectively, while the constant diagonal matrix \( \hat{A} \) encodes the fact that only some equations in (16)-(18) contain time-derivatives. Their explicit expressions can be found in [261].

In the most general terms, a solution of Eq.(19) can be written as a Fourier series in the streamwise and spanwise directions, i.e.

\[
V(x, y, z, t) = \sum_{n,m=0}^{\infty} U_{n,m}(y, t)e^{imk_x x + ink_z z}, \tag{20}
\]

where the wavenumbers \( k_x \) and \( k_z \) set the dominant length-scales in the corresponding directions. The weakly non-linear analysis of Morozov and van Saarloos [261] approximates this expression with

\[
V(x, y, z, t) = \Phi(t)e^{ik_x x + ik_z z}\Phi_{\text{lin}}(y) + \Phi^*(t)e^{-ik_x x - ik_z z}\Phi^*_{\text{lin}}(y)
+ U_0(y, t) + \sum_{n=2}^{\infty} \left[ U_n(y, t)e^{imk_x x + ink_z z} + U^*_n(y, t)e^{-imk_x x - ink_z z} \right], \tag{21}
\]

where "*" denotes complex conjugation. This expansion is built around the eigenfunction \( e^{ik_x x + ik_z z}\Phi_{\text{lin}}(y) \) of the linear operator,

\[
\hat{L}e^{ik_x x + ik_z z}\Phi_{\text{lin}}(y) = -\chi \hat{A}e^{ik_x x + ik_z z}\Phi_{\text{lin}}(y), \tag{22}
\]

where \( \chi \) is an eigenvalue. In what follows, we use the least stable eigenvalues as the basis for our analysis. In plane Couette flow, these are given by the extension to the Oldroyd-B model of the Gorodtsov-Leonov modes [44] discussed in Section 2.2, while in pressure-driven channel flows, we use the least stable of the three leading eigenvalues. (e.g. see Fig. 6 of [45]).

The higher Fourier harmonics \( U_n \) are assumed to be forced by the dynamics of the time-dependent amplitude of the first Fourier mode \( \Phi(t) \) and are thus produced by the recursive non-linear interaction of the first Fourier mode with itself and other modes produced in the process. This yields

\[
\begin{align*}
U_0(y, t) &= |\Phi(t)|^2 u_0^{(2)}(y) + |\Phi(t)|^4 u_0^{(4)}(y) + \cdots, \\
U_2(y, t) &= \Phi^2(t)u_2^{(2)}(y) + \Phi^2(t)|\Phi(t)|^2 u_2^{(4)}(y) + \cdots, \\
U_3(y, t) &= \Phi^3(t)u_3^{(3)}(y) + \cdots,
\end{align*}
\tag{23}
\]

where the unknown functions \( u_n^{(m)}(y) \) are to be determined from the analysis. One can view Eq.(21) as a version of the Fourier expansion, Eq.(20), with an extra assumption about the form and interrelation between the coefficients. Unlike the usual amplitude equation technique [267, 268], there is no guarantee that this procedure yields a converging, meaningful solution; this has to be checked \textit{a posteriori}.

The goal of the theory is thus to determine the time-evolution of the amplitude \( \Phi(t) \), which is obtained by projecting the dynamics of Eq.(19) on to the slow manifold with the help of the adjoint operator; see [261] for details. Its main result is the derivation of the following equation for the amplitude:

\[
\frac{d\Phi}{dt} = \chi \Phi + C_3 \Phi|\Phi|^2 + C_3 \Phi|\Phi|^4 + C_7 \Phi|\Phi|^6 + C_5 \Phi|\Phi|^8 + C_{11} \Phi|\Phi|^{10} + \cdots, \tag{24}
\]

where the complex coefficients \( C \)'s are functions of \( k_x, k_z, Wi, \beta, \) and the particular eigenmode selected for the analysis. For sufficiently small values of \( \Phi(t) \), the amplitude equation reduces to the long-time decay predicted by the linear stability analysis, i.e. \( \Phi(t) \sim e^{\Omega t} \), while for larger values of \( \Phi(t) \) it can exhibit non-trivial behaviour. Although the type of solutions it can support varies from steady states and periodic orbits to chaotic dynamics, below we focus on travelling waves in the form \( \Phi(t) = \Psi e^{\Omega t} \), where \( \Psi \) and \( \Omega \) are real numbers.

The asymptotic nature of Eq.(24) implies that only converging series can represent a physical solution. To lie within the radius of convergence, defined by the coefficients \( C \)'s, the solution amplitude \( \Psi \) has to be sufficiently small. In turn, this implies that the solutions that can be found by this method have to be sufficiently close to the original eigenmode used as the starting point of the theory. Convergence of the series for \( \Psi \) can be assessed by studying the travelling wave solutions of Eq.(24) with a progressively increasing number of terms, i.e.

\[
\begin{align*}
0 &= Re(\chi) + Re(C_3)\Psi^2, \\
0 &= Re(\chi) + Re(C_3)\Psi^2 + Re(C_5)\Psi^4, \\
\cdots
\end{align*} \tag{25-27}
\]

In Fig.18 we plot the consecutive approximation to the travelling wave amplitude \( \Psi \) using this procedure for plane Couette and channel flows. Although we had no reason to expect this \textit{a priori}, the low-branch amplitude values and the position of the saddle-node appear to converge; see [261] for the details of convergence tests. As can be seen from Fig.18, the upper branches, which set the saturated amplitude of the travelling wave solutions, diverge rapidly close to the saddle-node of the bifurcation, indicating that the corresponding values of \( \Psi \) lie outside the radius of convergence of the asymptotic series in Eq.(24).

Nevertheless, the highest-order upper-branch amplitude values of the amplitude \( \Psi \) in Fig.18 allow us to study the spatial structure of the solution predicted by the theory. As an illustration, in Fig.19 we plot the velocity profile at \( z = 0 \) in the \( xy \)-plane, where arrows trace the in-plane components of the deviation of the velocity from its laminar profile, while the colour gives the spanwise velocity. Additional plots of the stress and velocity fields can be found in [261].

The main conclusion to be drawn from these results is that both plane Couette and channel flows of model Oldroyd-B fluids exhibit travelling-wave solutions above the saddle-node value of the Weissenberg number \( Wi_{\text{lin}} \approx 3 \). We now summarise how these results compare against recent studies of perturbed viscoelastic channel flows.

6.2. Experimental evidence

First indication that parallel shear flows of viscoelastic fluids can exhibit sub-critical instabilities can already be seen in the
early turbulence experiments [23] already mentioned in Section 2.3. There, it was observed that addition of polymers to Newtonian pipe flows before the onset of Newtonian turbulence often leads to drag-enhancement, betraying a transition that is different in nature. These novel instabilities can be observed as long as the pipe diameter is small enough (typically a few millimetres), yielding high values of \( Wi \), in which case they exist at Reynolds numbers significantly smaller than the onset of Newtonian turbulence. It is natural to suggest that these instabilities have a purely elastic origin, as was proposed by Samanta et al. [23], with their region of existence spreading from \( Re = 0 \) to the values observed in early turbulence experiments [25, 67, 270, 24].

A more convincing, yet still indirect, evidence for the existence of sub-critical instabilities in parallel shear flows comes from the phenomenon of melt fracture (mentioned earlier in Sec. ), observed when concentrated polymer solutions or melts are extruded from a thin capillary [271]. Above a critical extrusion speed, the extrudate develops unwanted long-wave undulations of its surface and even breaks entirely. The origin of melt fracture, which limits virtually every industrial process that involves extrusion, is hotly debated, with possible explanations ranging from bulk phenomena to the stick-slip behaviour at the capillary exit [6]. By studying a wide range of viscoelastic fluids, Bertola et al. [62] have shown that extrudate undulations appear through a sub-critical instability with the saddle-node Weissenberg number being around \( Wi = 5 \). Supported by an early version of the weakly non-linear analysis developed by Meulenbroek et al. [265] that predicted the same saddle-node value of the Weissenberg number, Bertola et al. [62] concluded that their observations are consistent with a sub-critical instability originating inside the capillary and being advected downstream by the flow.

To demonstrate the bulk origin of such instabilities, Bonn et al. [272] conducted a series of experiments with dilute and semi-dilute polymer solutions that were fed from a thin capillary into a capillary of a larger radius. The resulting sudden-expansion flow at the entry to the large capillary provided a high level of flow fluctuations capable of inducing the instability. For low \( Wi \), Bonn et al. [272] observed that the inlet perturbations decayed along the large capillary, while for \( Wi \geq 4 \) perturbations persisted far downstream. While being consistent with the scenario proposed here, that study has not demonstrated the sub-critical nature of the instability.

The issue was finally settled by Arratia and co-workers [12, 13, 14] who simultaneously demonstrated the existence of large three-dimensional flow fluctuations inside a microfluidic channel flow of dilute polymer solutions and the sub-critical nature of the transition. Their experimental setup consisted of a long microfluidic channel partially blocked at the entrance by a row of cylindrical obstacles along the flow direction. Flows of polymer solutions around cylinders have been extensively studied, both experimentally and numerically, and are known to exhibit a linear instability above a critical \( Wi \); we refer to [12, 13, 14] for relevant references. This instability was used by Arratia et al. to generate flow perturbations and they observed their development far downstream of the cylinders. Above
\( \frac{\nu}{\kappa} \approx 5 - 6 \), they reported that the inlet perturbations stayed at a constant level far downstream from the cylinders, indicating a sustained non-linear state. By reducing the flow rate from that state, Arratia and co-workers observed that the fluctuations disappeared at lower values of \( \frac{\nu}{\kappa} \) than at their onset, thus confirming their sub-critical nature. The instability around the cylinders has been shown to be a supercritical bifurcation and, thus, is not responsible for the phenomenon observed. Later work by Qin et al. [13, 14] has demonstrated the three-dimensional nature of the non-linear flow state thus created. An interesting feature of the experiments by Bonn et al. [272] and by Arratia and co-workers [12, 13, 14] is that a large inlet perturbation is required to drive the transition. The saddle-node value of the Weissenberg number reported by Arratia et al. is somewhat higher than the one predicted in Fig.18, which can be due to a different value of \( \beta \) for the solutions used in the experiments, their shear-thinning nature not accounted for in the Oldroyd-B model, and the approximate nature of the weakly non-linear analysis presented above. Nevertheless, these experiments convincingly demonstrate the existence of non-trivial flow states in channel flows of viscoelastic fluids, in line with the original suggestion of Bonn, Morozov, van Saarloos, and collaborators [265, 62, 266, 88]; see the summary in Table 1.

It is important to stress that the secondary flow structures predicted above might be difficult to detect experimentally. In the purely elastic regime, or at low \( Re \), the dynamical variable is the polymeric stress, as can be seen from Eqs. (16)-(18), while the velocity field adiabatically adjusts to its evolution. As the weakly non-linear analysis presented above suggests, even weak velocity fields can develop sufficient gradients to reinforce non-linear dynamics of the stress. In the absence of reliable techniques to measure three-dimensional profiles of polymeric stresses, this might lead to a situation where a strongly non-linear state is only weakly manifested through the available observables, i.e. the mean-velocity fluctuations as used in [272, 12]. The weakly non-linear analysis further suggests that the streamwise vortices and streaks in a plane perpendicular to the flow direction might be the best candidates for experimental detection [261]. Such structures are a hallmark of Newtonian coherent structures [273]; they also feature prominently in the non-normal growth analysis by Kumar and Jovanović [82, 83]; see also Section 5. Recent experiments by Qin et al. [13] and Jha and Steinberg [15] present preliminary evidence for the existence of such coherent structures.

In view of the potential difficulties in resolving three-dimensional velocity fields associated with purely elastic instabilities in straight channels, it would be natural to address this question in direct numerical simulations. Unfortunately, such calculations are made notoriously difficult by the so-called High-Weissenberg Number Problem [274], that renders simulations unphysical at sufficiently high values of \( \nu \). In the absence of shear-thinning, the Oldroyd-B model often suffers from the High-Weissenberg Number Problem even at very low \( \nu \approx 1 - 2 \) [274]. Atlık and Keunings [251] performed numerical simulations of two-dimensional parallel shear flows of various constitutive models and reported large fluctuations of vorticity at low \( Re \) and \( \nu \approx 0.5 \). Unfortunately, that study did not provide the spatial profiles of the associated velocity field. Also the low value of \( \nu \) needed to generate oscillations and the low numerical resolutions used in those simulations bring in question whether they are indeed related to the instabilities discussed here. Sadanandan and Sureshkumar performed similar simulations of an Oldroyd-B fluid in channel flows and observed large velocity fluctuations above \( \nu \sim 3 \) (private communication). Unfortunately, those simulations ultimately suffered from the High-Weissenberg Number Problem and did not yield a turbulent-like steady-state. In the past years, there emerged a class of numerical techniques to ensure positive-definiteness of the conformation tensor (absence thereof was implicated as a cause of the High-Weissenberg Number Problem) [275, 276, 277]. Their use in unsteady parallel shear flows should provide the ultimate argument for the existence of sub-critical instabilities in such flows.

### 6.3. Elastic turbulence in parallel shear flows

The sub-critical transition in viscoelastic fluids presented above echoes the instability scenario in parallel shear flows of Newtonian fluids [278, 279]. Recently, the transition to Newtonian turbulence has been understood to be organised by the exact solutions of the Navier-Stokes equations. These three-dimensional coherent structures in the form of travelling waves or periodic orbits comprise streamwise vortices, streaks, and three-dimensional flows connecting them dynamically [280, 281, 282, 283], and appear through a sub-critical bifurcation from infinity. Importantly, they are linearly unstable forming saddle-like structures in phase space: A turbulent trajectory passing by in the vicinity of such a structure is attracted towards it only to be eventually repelled along one of the few unstable directions [278, 279]. In the presence of a sufficient number of such solutions in the phase space, after overcoming a critical threshold, a turbulent trajectory performs a random walk in the phase space, being trapped among a large number of such structures. This scenario, coupled with the process of spatial splitting and merging of the localised coherent structures, was recently shown to play the transition to Newtonian turbulence within the directed percolation universality class [284].

The experimental results by Bonn et al. [272] and by Arratia and co-workers [12, 13, 14] demonstrate that while parallel shear flows of dilute polymer solutions are linearly stable, they exhibit sub-critical transitions that lead directly to a chaotic state related to purely elastic turbulence previously only observed in shear flows with curved streamlines [9, 285, 286]. The results of the weakly non-linear analysis suggest that that transition might also be guided by unstable coherent structures: While the non-linear states described in Section 6.1 were inherently unobservable, they exhibited sub-critical transitions that lead directly to a chaotic state related to purely elastic turbulence previously only observed in shear flows with curved streamlines [9, 285, 286]. The results of the weakly non-linear analysis suggest that that transition might also be guided by unstable coherent structures: While the non-linear states described in Section 6.1 were inherently unobservable, they exhibited sub-critical transitions that lead directly to a chaotic state related to purely elastic turbulence previously only observed in shear flows with curved streamlines [9, 285, 286].
importance of (quasi) two-dimensional coherent structures. Future work is needed to uncover the exact nature of purely elastic turbulence in parallel shear flows and its similarities and differences to the Newtonian transition scenario.

7. Conclusions and Outlook

The present review, written on the occasion of the birth centenary of James Oldroyd, has provided a summary of various instabilities encountered in viscoelastic flows. The Oldroyd-B model, originally proposed in the context of rheology of dilute polymer solutions, has been quite versatile in its ability to predict, at least qualitatively, a host of instabilities in rectilinear, curvilinear and nonviscometric flows. It has also been used to predict novel interfacial instabilities in multilayer flows.

Owing to the implicit assumption of infinite extensibility of the polymer molecules, the Oldroyd-B model cannot predict shear thinning of viscosity and first normal stress coefficient. In the dilute regime, shear thinning is naturally accounted for by the FENE-P model which incorporates finite extensibility of the polymer molecules [183]. While the Oldroyd-B model is originally intended for dilute polymer solutions, the pioneering experimental efforts that led to the discovery of EIT in pipe flow (Samanta et al. [23]), and subsequent efforts that characterized this transition [67], use polymer concentrations much below and around the overlap value. Some of the solutions used in these experiments may not be deemed as truly dilute. As discussed in Sec. 2.3, although the Oldroyd-B model has been used to predict a linear instability in pipe and channel flows in accordance with the onset of EIT, an accurate prediction of the elasto-inertial instability in the semi-dilute regime will require a more detailed constitutive model that extends across the overlap concentration, while accounting for the dynamics in both dilute and semidilute regimes [289].

Important questions also remain pertaining to the nature of post-transition scenarios in rectilinear flows of polymer solutions after the onset of instability. For instance, what is the nature of the bifurcation at onset? What is(are) the pathway(s) that lead to the EIT state from the onset? In this regard, there is a real need for direct numerical simulations and weakly nonlinear analysis of channel and pipe flows to resolve some of these issues. Recent efforts have taken a step towards answering these questions by showing that the center-mode instability is subcritical, especially in the limit of dilute solutions ($\beta > 0.8$), both in channel [81, 49] and pipe [66] flows. Thus, the center-mode eigenfunction could be relevant even in parameter regimes where the flow is linearly stable. Further, it is important to ascertain how finite extensibility will affect the various transition scenarios. In this context, the recent work of Buza et al. [49] has shown that the center-mode instability is present in viscoelastic channel flow, even when the FENE-P model is used. Finite extensibility will likely lower the growth rates predicted by modal analysis (for parameter regimes where the flow is linearly unstable), and it will also put bounds on the amount of nonmodal amplification that can be achieved. Again, numerical simulations would be very helpful in throwing more light on this issue.

In the context of curvilinear flows, once again the Oldroyd-B model provides a first-cut qualitative prediction of purely elastic instabilities, while the inclusion of multiple relaxation modes, shear thinning, and non-isothermal effects lead to a more accurate, quantitative prediction of the same. The Pakdel-McKinley criterion for the onset of the purely elastic instability, heuristically developed in the context of curvilinear viscometric flows, works quite well for nonviscometric flows with curved streamlines, and surprisingly, even in the case of the instabilities in flows with mixed shear-elongation kinematics such as the cross-slot flow [201], the OSCER device [202], and in flow past a cylinder [203].

We end with the mention some instabilities encountered in polymeric flows that cannot be captured by the Oldroyd-B model. There have been both experimental [290, 291, 292, 293, 294] and theoretical [295, 296, 297] efforts that have studied instabilities at very low $Re$ in the flow of concentrated polymer solutions that exhibit strong shear thinning. Such instabilities are now understood to be driven by the combined action of fluid elasticity and shear thinning. Their prediction is outside the purview of the Oldroyd-B model; the (phenomenological) White-Metzner model, where the degree of shear thinning can be independently specified, has been used to predict such instabilities. An important class of instabilities exhibited by some viscoelastic fluids such as worm-like micellar solutions, termed ‘constitutive instabilities’, arise due to the nonmonotonic nature of the constitutive curve [109]. In such systems, it is not possible to maintain a plane shear (i.e. Couette) flow when the shear rate is in the regions of the constitutive curve where the stress decreases with the shear rate, and the flow evolves to a shear-banded state, as discussed in Sec. 2.5.2. Clearly, the Oldroyd-B model cannot predict such instabilities because of the monotonic nature of its constitutive curve. Owing to the lack of a second normal stress difference [130], the Oldroyd-B fluid cannot predict instabilities that rely on this feature. For example, a class of instabilities, termed ‘edge fracture’, arises when a viscoelastic fluid is sheared, for example, in the cone-and-plate geometry. The free surface present at the rim of the flow gets destabilized due to the second normal stress difference, and this can lead to a complicated edge profile that resembles fracture in elastic solids. The Johnson-Segalman and Giesekus models,

| Table 1: Saddle-node value of $Wi$ for various flow geometries |
|-----------------|----------------|----------------|
| Plane Couette Flow | Pipe Flow | Channel Flow |
| Experiment | $4^a - 5^b$ | $5 - 6^c$ |
| Weakly non-linear analysis | $3^d$ | $5^e$ |

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[a] Buza et al. [272]; [b] Barletta et al. [62]; [c] Pan et al. [296]; [d] Memmes and van Saarloos [47]; [e] Meulenbroek et al. [205]; [f] Memmes and van Saarloos [281].
both of which predict a negative second normal stress difference, have been used [298] to predict edge fracture. Finally, the Oldroyd-B model cannot predict phenomena where coupling between the flow and polymer concentration plays a central role, since the model assumes a uniform (and dilute) polymer concentration. This coupling has been shown to lead to an instability in plane Couette flow, even when the constitutive curve of the fluid is monotonic [299, 300, 301, 302, 303, 304, 305].

To conclude, one may regard the Oldroyd-B model as the ‘hydrogen atom’ in viscoelastic fluid mechanics – a sufficiently simple, yet realistic, model that allows for detailed mathematical analysis and numerical computations. It can, therefore, be argued that, even after seven decades since it has been proposed by James Oldroyd, the Oldroyd-B model still remains relevant, and happens to be first ‘go-to’ model when one is faced with the prediction of a novel instability in viscoelastic flows.

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References

[38] G. H. McKinley, Dimensionless groups for understanding free surface flows of complex fluids, SOR Bull. 74 (2) (2005) 6.
[41] J. T. Stuart, On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows part 1, the basic behaviour in plane Poiseuille flow, J. Fluid Mech. 9 (3) (1960) 353370. doi:10.1017/S002211206000116X.
[42] J. Watson, On the non-linear mechanics of wave disturbances in stable and unstable parallel flows part 2, the development of a solution for plane...


