

A formula for frequency responses of distributed systems with one spatial variable[☆]

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Abstract

We develop a method for the exact determination of frequency responses for a class of infinite dimensional systems. In particular, we consider distributed systems in which a spatial independent variable belongs to a finite interval, and in which the inputs and outputs are spatially distributed over the same interval. We show that an explicit formula for the Hilbert–Schmidt norm of the operator-valued frequency response can be obtained whenever the underlying operators are represented by a forced two point boundary value state-space realizations (TPBVSR). This formula involves finite dimensional computations with matrices whose dimension is at most four times larger than the order of the underlying differential operator. Two examples are provided to illustrate the procedure.

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1. Introduction

We study frequency responses of distributed systems in which a spatial independent variable belongs to a finite interval. Computation of frequency responses for this class of systems is usually done numerically by resorting to finite dimensional approximations of the underlying operators. We show that the spatial discretization can be circumvented and that frequency responses can be determined explicitly whenever the underlying operators can be represented by forced two point boundary value state-space realizations (TPBVSR) which are well posed.

Our results build on the work presented in [3], where a formula for the trace of a class of differential operators defined by forced TPBVSR with constant coefficients was derived. This formula was used for computation of the \mathcal{H}_2 norm for a class of infinite dimensional systems in which the

dynamical generators are normal (or self-adjoint). Here we study the frequency responses of distributed systems with, in general, non-normal dynamical generators and non-constant coefficients in a spatially independent variable. The underlying partial differential equations (PDEs) are assumed to involve boundary conditions expressed in terms of a general linear homogeneous constraint. Our main result is an explicit formula for the *Hilbert–Schmidt norm* (that is, the *power spectral density*) of the frequency response operator. This formula involves only finite dimensional computations with matrices whose dimension is at most four times larger than the order of the underlying differential operator. In this way an exact reduction of an infinite dimensional problem to a finite dimensional problem is accomplished.

Our presentation is organized as follows: in Section 2, we formulate the problem and briefly discuss the notion of frequency response for distributed systems. In Section 3, we describe the notion of a TPBVSR for the underlying spatial operators and the corresponding frequency response operator. In Section 4, we show how the Hilbert–Schmidt norm of the frequency response can be determined in terms

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of the parameters of the original PDE. In Section 5, we provide two examples to illustrate the procedure, and we conclude with a summary in Section 6.

2. Preliminaries

We begin with an example. Our interest is to study the frequency responses of systems of the type

$$\begin{aligned}\partial_t \psi(y, t) &= \partial_{yy} \psi(y, t) + d(y, t), \\ \psi(\pm 1, t) &= 0, \\ \psi(y, 0) &= 0, \quad y \in [a, b], \quad a, b \in \mathbb{R},\end{aligned}$$

which is the diffusion equation with homogenous Dirichlet boundary conditions, zero initial conditions, and a *spatially distributed* input field d . Considering the state as an output, the operator-valued transfer function of this system is given by

$$(sI - \partial_{yy})^{-1},$$

where ∂_{yy} refers to the second derivative operator with homogenous Dirichlet boundary conditions. Since this operator generates an exponentially stable semigroup [5], it is a standard fact that the corresponding transfer function is well defined over the region $\Re(s) \geq 0$. Furthermore, it can also be shown that it is of the Hilbert–Schmidt class as an operator on $L^2[a, b]$ for each s in that same region. An operator \mathcal{F} is said to be in the Hilbert–Schmidt class if $\mathcal{F}\mathcal{F}^*$ has finite trace, i.e.

$$\|\mathcal{F}\|_{\text{HS}}^2 := \text{trace}(\mathcal{F}\mathcal{F}^*) < \infty.$$

We refer to the quantity $\|\cdot\|_{\text{HS}}$ as the Hilbert–Schmidt norm. It can be thought of as a generalization of the Frobenius norm for matrices.

For systems with infinite dimensional input and output spaces, the transfer function and frequency response are operator-valued. Just as for MIMO systems, one desires to investigate some aggregate quantities of these matrices and operators to characterize the frequency response. The most common ones are the maximum singular value and the square root of the sum of squares of the singular values (i.e. the Frobenius norm). In this work, we are interested in determining the Hilbert–Schmidt norm of the frequency response as a function of frequency, i.e. in determining (for the above example)

$$\|(j\omega I - \partial_{yy})^{-1}\|_{\text{HS}}$$

as a function of ω . We will provide a general formula for such functions in terms of the parameters of the underlying PDEs.

2.1. Problem setup

We now describe the problem in general. Consider distributed systems (over one spatial dimension) of

the form

$$\partial_t \mathcal{E} \psi(y, t) = \mathcal{F} \psi(y, t) + \mathcal{G} d(y, t), \quad (2.1a)$$

$$\phi(y, t) = \mathcal{H} \psi(y, t), \quad (2.1b)$$

where for each t , the vector valued functions $d(\cdot, t)$, $\psi(\cdot, t)$ and $\phi(\cdot, t)$ are in $L^2[a, b]$ (where the vector dimension is suppressed in our notation). \mathcal{E} , \mathcal{F} , \mathcal{G} , and \mathcal{H} are linear differential operators with a particular structure to be specified shortly (in the above diffusion equation example, $\mathcal{F} = \partial_{yy}$ with Dirichlet boundary conditions, whereas \mathcal{E} , \mathcal{G} , and \mathcal{H} are the identity operators). Signals d , ψ , and ϕ , respectively, represent spatially distributed time varying input, state, and output fields of system (2.1). Specifically, each of these fields is a scalar or vector valued function of both a spatial variable (in an interval) and time. For example, $d(y, t)$ with $(y, t) \in [a, b] \times [0, \infty)$, and $a, b \in \mathbb{R}$. The operators \mathcal{E} , \mathcal{F} , \mathcal{G} , and \mathcal{H} are matrices of differential operators. For example, the ij th entry of \mathcal{F} has the form

$$\sum_{k=0}^n f_{ij}^k(y) \frac{\partial^k}{\partial y^k},$$

where $\{f_{ij}^k(y)\}$ are smooth functions on the interval $[a, b]$. For a term of this form, n is referred to as the *order* of the operator, and we additionally assume for the highest order coefficient $f_{ij}^n(y) \neq 0$, for $y \in [a, b]$. The definition of the order of a *matrix* differential operator with the same structure as \mathcal{F} is more involved. We will instead define the order of the overall set of PDE operators (2.1) in terms of a TPBVS. R.

The class of systems we consider is defined indirectly. They are systems of the form (2.1) that can be rewritten in the TPBVS (3.3) to be described in the next section.

The frequency response of system (2.1) is given by [5]

$$\mathcal{T}(\omega) := \mathcal{H}(j\omega \mathcal{E} - \mathcal{F})^{-1} \mathcal{G},$$

where ω denotes temporal frequency. If we define the operator

$$\mathcal{P}(\omega) := \mathcal{T}(\omega) \mathcal{T}^*(\omega),$$

then from the definition above the Hilbert–Schmidt norm of $\mathcal{T}(\omega)$ is just the trace of $\mathcal{P}(\omega)$. As we will see in the next section, for the class of operators we consider, operators $\mathcal{T}(\omega)$ and $\mathcal{P}(\omega)$ have so-called kernel representations. This implies that we can express, for example, $\mathcal{P}(\omega) : f \mapsto g$ as

$$g(y) = \int_a^b P(\omega, y, \eta) f(\eta) d\eta, \quad (2.2)$$

where $P(\omega, \cdot, \cdot)$ denotes the kernel function of operator $\mathcal{P}(\omega)$. We assume that for any ω , $P(\omega, \cdot, \cdot)$ is a bounded matrix valued function on $[a, b] \times [a, b]$. It is a standard fact [13] that under these conditions $\mathcal{P}(\omega)$ is trace class and

its trace (and thus the Hilbert–Schmidt norm of $\mathcal{F}(\omega)$) can be determined from the ‘trace-like’ integral

$$\text{trace}(\mathcal{P}(\omega)) = \int_a^b \text{tr}(P(\omega, y, y)) dy, \quad (2.3)$$

where $\text{trace}(\cdot)$ is the operator trace and $\text{tr}(\cdot)$ is the matrix trace.

The computation of $\|\mathcal{F}(\omega)\|_{\text{HS}}$ is typically done numerically after finite dimensional approximations of the underlying operators using some spatial discretization scheme. In this paper, we show how the kernel functions and the integral in (2.3) can be determined explicitly using a formula that involves only the coefficients of the original PDE (2.1).

3. Conversion to a spatial state-space realization

In this section, we describe how the original system (2.1) can be converted to a standard form that we refer to as a TPBVSR. This is accomplished by first transforming (2.1) using a temporal Fourier transform. Temporal differentiation then becomes multiplication by $j\omega$, and the equation becomes an ordinary differential equation (ODE) in the spatial variable y with mixed boundary conditions. Standard procedures for converting a high-order ODE to a first-order vector ODE (i.e. a state-space realization) are then applied. The fact that the independent variable is a spatial one rather than time does not alter the standard procedures. However, care must be exercised to keep track of how the original boundary conditions are transformed into the relevant ones for the TPBVSR. We note that in this paper, we do not provide the most general or even the most efficient procedure for such a transformation. This would be analogous to procedures for obtaining minimal realizations from matrix fraction descriptions of linear time invariant (LTI) systems [1,4]. It is beyond the scope of this paper to recreate this procedure for the class of systems we consider. Instead, we illustrate how this can in general be carried out, and present two examples, one of which involves a high order PDE from fluid dynamics, to illustrate our main result, which is the exact determination of frequency responses *after* such TPBVSR are obtained.

We first describe how a TPBVSR can be found for the frequency response operator $\mathcal{F}(\omega)$, from which we can also obtain realizations for its adjoint $\mathcal{F}^*(\omega)$ and consequently for the composition $\mathcal{F}(\omega)\mathcal{F}^*(\omega)$. First note that we can obtain a description of the frequency response operator of system (2.1) by simply applying the temporal Fourier transform, i.e. define

$$\hat{\psi}(y, \omega) := \int_{-\infty}^{\infty} \psi(y, t) e^{-j\omega t} dt$$

and \hat{d} and $\hat{\phi}$ similarly. In this representation, system (2.1) becomes

$$(j\omega\mathcal{E} - \mathcal{F})\hat{\psi}(y, \omega) = \mathcal{G}\hat{d}(y, \omega), \quad (3.1a)$$

$$\hat{\phi}(y, \omega) = \mathcal{H}\hat{\psi}(y, \omega). \quad (3.1b)$$

Note that this represents a *family* of *independent* ODEs parameterized by temporal frequency ω . We can thus regard these as individual ODEs (in the spatial variable y) in which ω enters simply as a parameter. For simplicity of notation, we omit the ω -dependence and the hat-designate on each signal, and regard system (3.1) a matrix ODE of a certain order determined by the order of operators \mathcal{E} , \mathcal{F} , \mathcal{G} , and \mathcal{H} . To emphasize this point, we rewrite system (3.1) in this new notation

$$(j\omega\mathcal{E} - \mathcal{F})\psi(y) = \mathcal{G}d(y), \quad (3.2a)$$

$$\phi(y) = \mathcal{H}\psi(y), \quad (3.2b)$$

where now \mathcal{E} , \mathcal{F} , \mathcal{G} , and \mathcal{H} are matrices of differential operators whose entries are linear combinations of terms of the form $f(y)(d^k/dy^k)$.

Under mild conditions, that are satisfied in our examples, it is a standard fact of linear systems realization theory [1,4] that (3.2) can be rewritten in the first order form (i.e. a state-space realization) by introducing ‘state variables’ $\{x_i(y)\}$. Each of these state variables is a linear combination of the individual entries of $\psi(y)$ and its derivatives up to a certain order. For example, if d and ϕ are scalar valued, then one can immediately use either the controllable canonical form or the observable canonical form realizations. In the case of vector valued functions d and ϕ , more elaborate realization procedures are needed. Ultimately, we will arrive at

$$\mathcal{F} : \begin{cases} x'(y) = A_0(y)x(y) + B_0(y)d(y), \\ \phi(y) = C_0(y)x(y), \\ 0 = N_1x(a) + N_2x(b), \quad y \in [a, b], \end{cases} \quad (3.3)$$

where $x'(y) := dx(y)/dy$, and $x(y)$ is a vector of all the state variables. We define the *order* n_0 of this realization as the dimension of vector $x \in \mathbb{C}^{n_0}$. A_0 , B_0 , and C_0 are now matrices with constant or spatially varying entries (depending on whether the original PDE operators have constant or spatially varying coefficients, respectively). Boundary conditions on the original PDE translate into boundary conditions of the form above, which is the most general form for linear homogeneous boundary conditions. The boundary value matrices N_1 and N_2 are constant and are such that $[N_1 \ N_2]$ has a full row rank, to prevent a redundancy in boundary conditions. We remark that TPBVSR (3.3) is well-posed (that is, it has a unique solution for any input $d(y)$) if and only if [7] $\det(N_1 + N_2\Phi_0(b, a)) \neq 0$, where $\Phi_0(y, \eta)$ denotes the state transition matrix of $A_0(y)$.

Remark 3.1. There are some conditions required on the original system (3.2) to guarantee it can be rewritten in the TPBVSR form (3.3). In this paper, we assume that the original system can be rewritten as such. We remark that exact conditions on the operators \mathcal{E} , \mathcal{F} , \mathcal{G} and \mathcal{H} that guarantee

the existence of a TPBVSR can be derived. These are similar to the ‘properness’ conditions for matrix transfer functions. Roughly, the order of differentiations in the operators \mathcal{G} and \mathcal{H} should not exceed that in \mathcal{F} . The derivation of these exact conditions and developing a systematic procedure to arrive at a minimal realization of the form (3.3) from a given PDE (2.1) is beyond the scope of this paper. However, it is often possible to derive such realizations by inspection and some insight into the underlying PDEs.

3.1. State-space realizations of \mathcal{F}^* , and $\mathcal{F}\mathcal{F}^*$

As we have illustrated, the operator $\mathcal{F}(\omega) : d(y, \omega) \mapsto \phi(y, \omega)$ has a TPBVSR given by (3.3) (where the dependence on ω is suppressed, a notation we adopt from now on). From this realization, it is straightforward to find a realization for its adjoint \mathcal{F}^* , and then by combining these two realizations, one arrives ultimately at a realization for $\mathcal{F}\mathcal{F}^*$.

Given that the operator $\mathcal{F} : d(y) \mapsto \phi(y)$ is described by realization (3.3), it is a standard fact [7] that its adjoint $\mathcal{F}^* : f(y) \mapsto g(y)$ also has a realization which can be derived as

$$\mathcal{F}^* : \begin{cases} z'(y) = -A_0^*(y)z(y) - C_0^*(y)f(y), \\ g(y) = B_0^*(y)z(y), \\ 0 = M_1z(a) + M_2z(b), \quad y \in [a, b], \end{cases} \quad (3.4)$$

where A_0^* , B_0^* , and C_0^* , respectively, represent the adjoints (complex-conjugate-transpose matrices) of A_0 , B_0 , and C_0 . On the other hand, the constant boundary value matrices M_1 and M_2 are determined by

$$[M_1 \ M_2] \begin{bmatrix} N_1^* \\ -N_2^* \end{bmatrix} = 0. \quad (3.5)$$

We note that a variety of different M_1 and M_2 for which $[M_1 \ M_2]$ is a full row rank matrix can be chosen to satisfy (3.5). For example, a singular value decomposition of a full column rank matrix $\begin{bmatrix} N_1^* \\ -N_2^* \end{bmatrix}$:

$$\begin{bmatrix} N_1^* \\ -N_2^* \end{bmatrix} = USV^* = [U_1 \ U_2] \begin{bmatrix} S_1 \\ 0 \end{bmatrix} V^* \\ = U_1S_1V^*, \quad U_1, U_2 \in \mathbb{C}^{2n_0 \times n_0}$$

can be used to select these two matrices as: $[M_1 \ M_2] = U_2^*$.

Finally, a well-posed state-space realization of $\mathcal{F}\mathcal{F}^* : f(y) \mapsto \phi(y)$ can be obtained by combining (3.3) and (3.4) (by equating the output of \mathcal{F}^* to the input of \mathcal{F} , i.e. $d(y) = g(y)$) to arrive at

$$\mathcal{F}\mathcal{F}^* : \begin{cases} q'(y) = A(y)q(y) + B(y)f(y), \\ \phi(y) = C(y)q(y), \\ 0 = L_1q(a) + L_2q(b), \quad y \in [a, b], \end{cases} \quad (3.6)$$

where

$$q(y) := \begin{bmatrix} x(y) \\ z(y) \end{bmatrix} \in \mathbb{C}^{2n_0}, \quad L_1 := \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix},$$

$$L_2 := \begin{bmatrix} N_2 & 0 \\ 0 & M_2 \end{bmatrix},$$

$$A(y) := \begin{bmatrix} A_0(y) & B_0(y)B_0^*(y) \\ 0 & -A_0^*(y) \end{bmatrix},$$

$$B(y) := \begin{bmatrix} 0 \\ -C_0^*(y) \end{bmatrix}, \quad C(y) := [C_0(y) \ 0].$$

We finally remark that the well-posedness of (3.3) guarantees the well-posedness of the adjoint realization (3.4). Furthermore, (3.6) is well posed by definition since it is the cascade connection of two well-posed realizations.

4. Determination of frequency responses from state-space realizations

In this section, we derive an explicit formula for the Hilbert–Schmidt norm of $\mathcal{F}(\omega)$ as a function of ω . To that end we derive a formula for the kernel function of $\mathcal{P}(\omega) = \mathcal{F}(\omega)\mathcal{F}^*(\omega)$, and then an expression for its integral along the diagonal coordinates.

4.1. Determination of traces from state-space realizations

We consider a well-posed TPBVSR of the form

$$\mathcal{P} : \begin{cases} q'(y) = A(y)q(y) + B(y)f(y), \\ \phi(y) = C(y)q(y), \\ 0 = L_1q(a) + L_2q(b), \quad y \in [a, b], \end{cases} \quad (4.1)$$

with $q(y) \in \mathbb{C}^n$, $f(y) \in \mathbb{C}^m$, $\phi(y) \in \mathbb{C}^m$, and derive a formula for the trace of operator $\mathcal{P} : f \mapsto \phi$. This formula is expressed in terms of matrices $\{A(y), B(y), C(y), L_1, L_2\}$, and it requires the computation of two state transition matrices: the state transition matrix of $A(y)$, and the state transition matrix of a matrix that depends on $A(y)$, $B(y)$, and $C(y)$. The size of the latter matrix is $2n \times 2n$.

It is well known that if an operator $\mathcal{P} : f \mapsto \phi$ is given by a continuous kernel function, i.e. its action can be represented by

$$\phi(y) = \int_a^b P(y, \eta)f(\eta) d\eta, \quad (4.2)$$

then its trace is given by the integral

$$\text{trace}(\mathcal{P}) = \int_a^b \text{tr}(P(y_0, y_0)) dy_0 \\ = \int_a^b \text{tr} \left(\sum_{k=1}^m P(y_0, y_0) e_k e_k^T \right) dy_0,$$

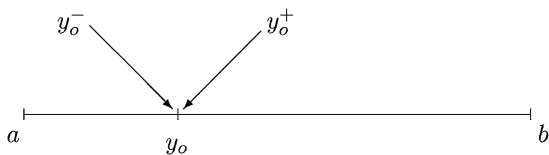


Fig. 1. Boundary conditions with delta function as input.

where e_k denotes the k th unit vector in \mathbb{R}^m . Each ‘column’ of the kernel can be obtained by using the input $f(y) = \delta(y - y_0)e_k$, for then the corresponding output is $\phi(y) = P(y, y_0)e_k$. The effect of forcing (4.1) with $f(y) = \delta(y - y_0)e_k$ is given by

$$q'(y) = A(y)q(y) + B(y)\delta(y - y_0)e_k$$

$$\Rightarrow \begin{cases} q'(y) = A(y)q(y), \\ q(y_0^+) - q(y_0^-) = B(y_0)e_k, \end{cases}$$

where $q(y_0^+)$ and $q(y_0^-)$, respectively, represent the limits of $q(y)$ as y_0 is approached from above and below (see Fig. 1). Hence,

$$\left. \begin{aligned} q'(y) &= A(y)q(y), \\ q(y_0^+) - q(y_0^-) &= B(y_0)e_k, \end{aligned} \right\}$$

$$\Rightarrow \Phi(y_0, b)q(b) - \Phi(y_0, a)q(a) = B(y_0)e_k, \quad (4.3)$$

where $\Phi(y, \eta)$ denotes the state transition matrix of $A(y)$. By combining the boundary conditions from (4.2) with (4.3) we obtain

$$\begin{bmatrix} L_1 & L_2 \\ -\Phi(y_0, a) & \Phi(y_0, b) \end{bmatrix} \begin{bmatrix} q(a) \\ q(b) \end{bmatrix} = \begin{bmatrix} 0 \\ B(y_0)e_k \end{bmatrix},$$

which yields

$$\begin{bmatrix} q(a) \\ q(b) \end{bmatrix} = \begin{bmatrix} -(L_1 + L_2\Phi(b, a))^{-1}L_2\Phi(b, y_0)B(y_0)e_k \\ (\Phi(b, y_0) - \Phi(b, a)(L_1 + L_2\Phi(b, a))^{-1}L_2\Phi(b, y_0))B(y_0)e_k \end{bmatrix}. \quad (4.4)$$

We note that the invertibility of matrix $L_1 + L_2\Phi(b, a)$ is equivalent to the well-posedness of the TPBVSR (4.1) [7].

Using (4.4), we express vector $P(y_0, y_0)e_k$ as

$$P(y_0, y_0)e_k = \frac{1}{2}C(y_0)(q(y_0^-) + q(y_0^+))$$

$$= \frac{1}{2}C(y_0)[\Phi(y_0, a) \quad \Phi(y_0, b)] \begin{bmatrix} q(a) \\ q(b) \end{bmatrix}$$

$$= \frac{1}{2}C(y_0)(I - 2\Phi(y_0, a))$$

$$\times (L_1 + L_2\Phi(b, a))^{-1}$$

$$\times L_2\Phi(b, y_0)B(y_0)e_k,$$

which yields

$$\text{trace}(\mathcal{P})$$

$$= \text{tr} \left(\frac{1}{2} \int_a^b C(y)B(y) dy - (L_1 + L_2\Phi(b, a))^{-1}L_2 \right.$$

$$\left. \times \int_a^b \Phi(b, y)B(y)C(y)\Phi(y, a) dy \right). \quad (4.5)$$

We have arrived at (4.5) using the commutativity property of the matrix trace.

The integral in (4.5) can be evaluated in terms of the response of the unforced dynamical system

$$\begin{bmatrix} X_1'(y) \\ X_2'(y) \end{bmatrix} = \begin{bmatrix} A(y) & 0 \\ B(y)C(y) & A(y) \end{bmatrix} \begin{bmatrix} X_1(y) \\ X_2(y) \end{bmatrix},$$

$$Y(y) = [0 \quad I] \begin{bmatrix} X_1(y) \\ X_2(y) \end{bmatrix},$$

$$\begin{bmatrix} X_1(a) \\ X_2(a) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad y \in [a, b] \quad (4.6)$$

as

$$Y(b) = \int_a^b \Phi(b, y)B(y)C(y)\Phi(y, a) dy$$

$$= [0 \quad I]\Psi(b, a) \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad (4.7)$$

where $\Psi(y, \eta)$ denotes the state transition matrix of system (4.6). Therefore, we obtain an expression for $\text{trace}(\mathcal{P})$ by combining (4.5) and (4.7)

$$\text{trace}(\mathcal{P}) = \text{tr} \left(\frac{1}{2} \int_a^b C(y)B(y) dy \right.$$

$$\left. - (L_1 + L_2\Phi(b, a))^{-1} \right.$$

$$\left. \times L_2[0 \quad I]\Psi(b, a) \begin{bmatrix} I \\ 0 \end{bmatrix} \right). \quad (4.8)$$

In particular, for systems with constant coefficients in y formula (4.8) simplifies to

$$\text{trace}(\mathcal{P}) = \text{tr} \left(\frac{c}{2}CB - (L_1 + L_2e^{cA})^{-1}L_2[0 \quad I] \right.$$

$$\left. \times \exp \left\{ c \begin{bmatrix} A & 0 \\ BC & A \end{bmatrix} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} \right), \quad (4.9)$$

where $c := b - a$. In the case that only the B or C matrices are spatially varying with an exponential dependence, we can also state something more explicit than (4.8). For example, if

$$A(y) := A, \quad B(y) := e^{-\alpha(y-a)}B, \quad C(y) := C,$$

formula (4.8) simplifies to

$\text{trace}(\mathcal{P})$

$$\begin{aligned} &= \frac{1 - e^{-\alpha c}}{2\alpha} \text{tr}(CB) \\ &\quad - \text{tr} \left((L_1 + L_2 e^{cA})^{-1} L_2 \begin{bmatrix} 0 & I \end{bmatrix} \right. \\ &\quad \left. \times \exp \left\{ c \begin{bmatrix} A - \alpha I & 0 \\ BC & A \end{bmatrix} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} \right), \end{aligned} \quad (4.10)$$

Note that in the limit as $\alpha \rightarrow 0$, (4.10) approaches (4.9).

4.2. Explicit formula for Hilbert–Schmidt norm

We now combine the results of Sections 3 and 4.1 to derive a formula for the Hilbert–Schmidt norm of the frequency response operator.

Theorem 4.1. *Let the frequency response operator $\mathcal{T}(\omega)$ be given by a well-posed TPBVSR (3.3) at each ω . Its Hilbert–Schmidt norm can be calculated from*

$$\begin{aligned} &\|\mathcal{T}(\omega)\|_{\text{HS}}^2 \\ &= -\text{tr} \left((L_1 + L_2 \Phi(b, a))^{-1} \right. \\ &\quad \left. \times L_2 \begin{bmatrix} 0 & I \end{bmatrix} \Psi(b, a) \begin{bmatrix} I \\ 0 \end{bmatrix} \right), \end{aligned} \quad (4.11)$$

where

$$L_1 := \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix}, \quad L_2 := \begin{bmatrix} N_2 & 0 \\ 0 & M_2 \end{bmatrix},$$

$$A(y) := \begin{bmatrix} A_0(y) & B_0(y)B_0^*(y) \\ 0 & -A_0^*(y) \end{bmatrix},$$

$$B(y) := \begin{bmatrix} 0 \\ -C_0^*(y) \end{bmatrix}, \quad C(y) := [C_0(y) \ 0].$$

The matrices M_1 and M_2 are determined from the following singular value decomposition:

$$\begin{aligned} \begin{bmatrix} N_1^* \\ -N_2^* \end{bmatrix} &= USV^* = [U_1 \ U_2] \begin{bmatrix} S_1 \\ 0 \end{bmatrix} V^* \\ &= U_1 S_1 V^*, \quad U_1, U_2 \in \mathbb{C}^{2n_0 \times n_0} \end{aligned}$$

by

$$[M_1 \ M_2] = U_2^*.$$

The matrices $\Phi(y, \eta)$ and $\Psi(y, \eta)$ are the state transition matrices of $A(y)$ and

$$\begin{bmatrix} A(y) & 0 \\ B(y)C(y) & A(y) \end{bmatrix},$$

respectively. For systems with constant coefficients in y formula (4.11) simplifies to

$$\begin{aligned} &\|\mathcal{T}(\omega)\|_{\text{HS}}^2 \\ &= -\text{tr} \left((L_1 + L_2 e^{cA})^{-1} L_2 \begin{bmatrix} 0 & I \end{bmatrix} \right. \\ &\quad \left. \times \exp \left\{ c \begin{bmatrix} A & 0 \\ BC & A \end{bmatrix} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} \right), \end{aligned} \quad (4.12)$$

where $c := b - a$.

Note that the matrices in the TPBVSR (3.3) of a frequency response operator are functions of ω as well as any other parameters in the problem. Thus, the Hilbert–Schmidt norm (4.11) will also be a function of ω and these variables. This dependence has been suppressed in the statement of the theorem for notational simplicity.

The above theorem converts the problem of evaluating the Hilbert–Schmidt norm of an infinite dimensional object to computations with finite matrices. For systems with non-constant coefficients in y , at any given ω , we need to solve a matrix differential equation whose order is four times larger than the order of the original differential operator to obtain the state transition matrix $\Psi(b, a)$. On the other hand, for systems with constant coefficients in y , this computation amounts to determination of the corresponding matrix exponential. Both computations can be easily performed using commercially available software such as MATLAB or MATHEMATICA.

5. Examples

We now illustrate the use of our technique using two examples: a one-dimensional diffusion equation, and a system that describes the dynamics of velocity fluctuations in channel fluid flow. In both cases, we first derive the TPBVSR from the original PDEs and then apply Theorem 4.1 to the resulting realization.

5.1. A one-dimensional diffusion equation

We consider a one-dimensional diffusion equation on $L^2[-1, 1]$ with Dirichlet boundary conditions

$$\begin{aligned} \partial_t \psi(y, t) &= \partial_{yy} \psi(y, t) + d(y, t), \\ \psi(\pm 1, t) &= 0. \end{aligned}$$

The application of the temporal Fourier transform yields

$$\psi(y, \omega) = [(j\omega I - \partial_{yy})^{-1} d(\omega)](y) =: [\mathcal{T}(\omega)d(\omega)](y).$$

Considering this as an ODE in the independent variable y , and ω as a parameter, we can use the controller

canonical form realization to obtain

$$\mathcal{F}(\omega) : \begin{cases} \begin{bmatrix} x'_1(y, \omega) \\ x'_2(y, \omega) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ j\omega & 0 \end{bmatrix} \begin{bmatrix} x_1(y, \omega) \\ x_2(y, \omega) \end{bmatrix} \\ \quad + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(y, \omega), \\ \psi(y, \omega) = [1 \ 0] \begin{bmatrix} x_1(y, \omega) \\ x_2(y, \omega) \end{bmatrix}, \\ 0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(-1, \omega) \\ x_2(-1, \omega) \end{bmatrix} \\ \quad + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(1, \omega) \\ x_2(1, \omega) \end{bmatrix}, \\ y \in [-1, 1]. \end{cases}$$

This is indeed a TPBVSR of the form (3.3). From this we can obtain a TPBVSR of $\mathcal{F}(\omega)\mathcal{F}^*(\omega)$ using (3.6)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ j\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & j\omega \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

$$C = [1 \ 0 \ 0 \ 0],$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We now calculate $\|\mathcal{F}(\omega)\|_{\text{HS}}$ using formula (4.12). With the help of MATHEMATICA we obtain

$$\begin{aligned} \|\mathcal{F}(\omega)\|_{\text{HS}}^2 &= \frac{1}{2\omega^2} \left\{ -1 + \frac{\sqrt{2\omega}(\sinh(\sqrt{8\omega}) + \sin(\sqrt{8\omega}))}{\cosh(\sqrt{8\omega}) - \cos(\sqrt{8\omega})} \right\}. \end{aligned} \quad (5.1)$$

It is noteworthy that in this simple example, $\|\mathcal{F}(\omega)\|_{\text{HS}}^2$ can be also obtained by doing a spectral decomposition of the operator ∂_{yy} with Dirichlet boundary conditions. It is well known (see, for example [5]) that this operator has the following set of orthonormal eigenfunctions $\{\varphi_n\}$ and corresponding eigenvalues $\{\gamma_n\}$:

$$\varphi_n(y) := \sin\left(\frac{n\pi}{2}(y+1)\right), \quad \gamma_n := -\frac{n^2\pi^2}{4}, \\ n = 1, 2, \dots$$

Using this eigenfunction expansion we can directly calculate

$$\|\mathcal{F}(\omega)\|_{\text{HS}}^2 = \sum_{n \in \mathbb{N}} \frac{1}{\omega^2 + (n\pi/2)^4}.$$

Using various series formulae for trigonometric functions, this series can be shown to be equal to (5.1).

It is not often possible to do full eigenvalue/eigenfunction expansions as can be done for this simple example. Such expansions actually provide more information than

what is needed to compute the Hilbert–Schmidt norm. Note that formula (4.11) does not require such expansions. Our second example has no known explicit description for the eigenvalues, but we will show that formula (4.11) yields rather explicit expressions for the Hilbert–Schmidt norm.

5.2. A system from fluid dynamics

We consider the dynamics of streamwise constant perturbations in channel flows using the linearized Navier–Stokes (LNS) equations. For background on this model and the use of system norms in transition to turbulence studies, we refer the reader to [2,8–10,12] and the references therein.

The forced LNS equations for streamwise constant flow perturbations are described by [10,8]

$$\begin{aligned} \partial_t \begin{bmatrix} \Delta\psi_1 \\ \psi_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{R}\mathcal{F}_{11} & 0 \\ \mathcal{F}_{21} & \frac{1}{R}\mathcal{F}_{22} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \mathcal{G}_y & \mathcal{G}_z \\ \mathcal{G}_x & 0 & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}, \end{aligned} \quad (5.2a)$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{H}_u \\ \mathcal{H}_v & 0 \\ \mathcal{H}_w & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad (5.2b)$$

where the output fields u , v , and w are the three components of the velocity perturbation fields. The input fields d_x , d_y , and d_z represent the three components of body force fields. The state of the system is captured by two fields, ψ_1 and ψ_2 (namely, vertical velocity v and vertical vorticity). Although this is a simplified model of *three-dimensional* flow perturbations, there is only one spatial dimension in the PDEs, namely the vertical coordinate $y \in [-1, 1]$. Dynamics in the spanwise direction (the z coordinate) appear through the wave-number parameter k_z . Another important parameter that appears in the equations is the Reynolds number R . The scaling of system norms with respect to these parameters gives important information about flow structures that appear in transition to turbulence [10]. The boundary conditions on the state in the above equations are: $\psi_1(\pm 1, t) = \partial_y\psi_1(\pm 1, t) = \psi_2(\pm 1, t) = 0$. Furthermore, the underlying operators in (5.2) are defined by: $\{\mathcal{F}_{11} := \Delta^2, \mathcal{F}_{22} := \Delta, \mathcal{F}_{21} := -jk_z U'(y)\}$, $\{\mathcal{G}_x := jk_z, \mathcal{G}_y := -k_z^2, \mathcal{G}_z := -jk_z\partial_y\}$, $\{\mathcal{H}_u := -j/k_z, \mathcal{H}_v := I, \mathcal{H}_w := (j/k_z)\partial_y\}$, where $U(y)$ denotes the nominal velocity profile, $U' := dU/dy$, $\Delta := \partial_{yy} - k_z^2$ with homogenous Dirichlet boundary conditions, and $\Delta^2 := \partial_{yyyy} - 2k_z^2\partial_{yy} + k_z^4$ with homogenous Dirichlet *and* Neumann boundary conditions. For a more complete description of the underlying spaces and domains of these operator, we refer the reader to [6,11]. It is well known that system (5.2) is exponentially stable for any pair (k_z, R) [12].

For notational convenience, we set R in (5.2) to one. Under this assumption, application of the temporal Fourier

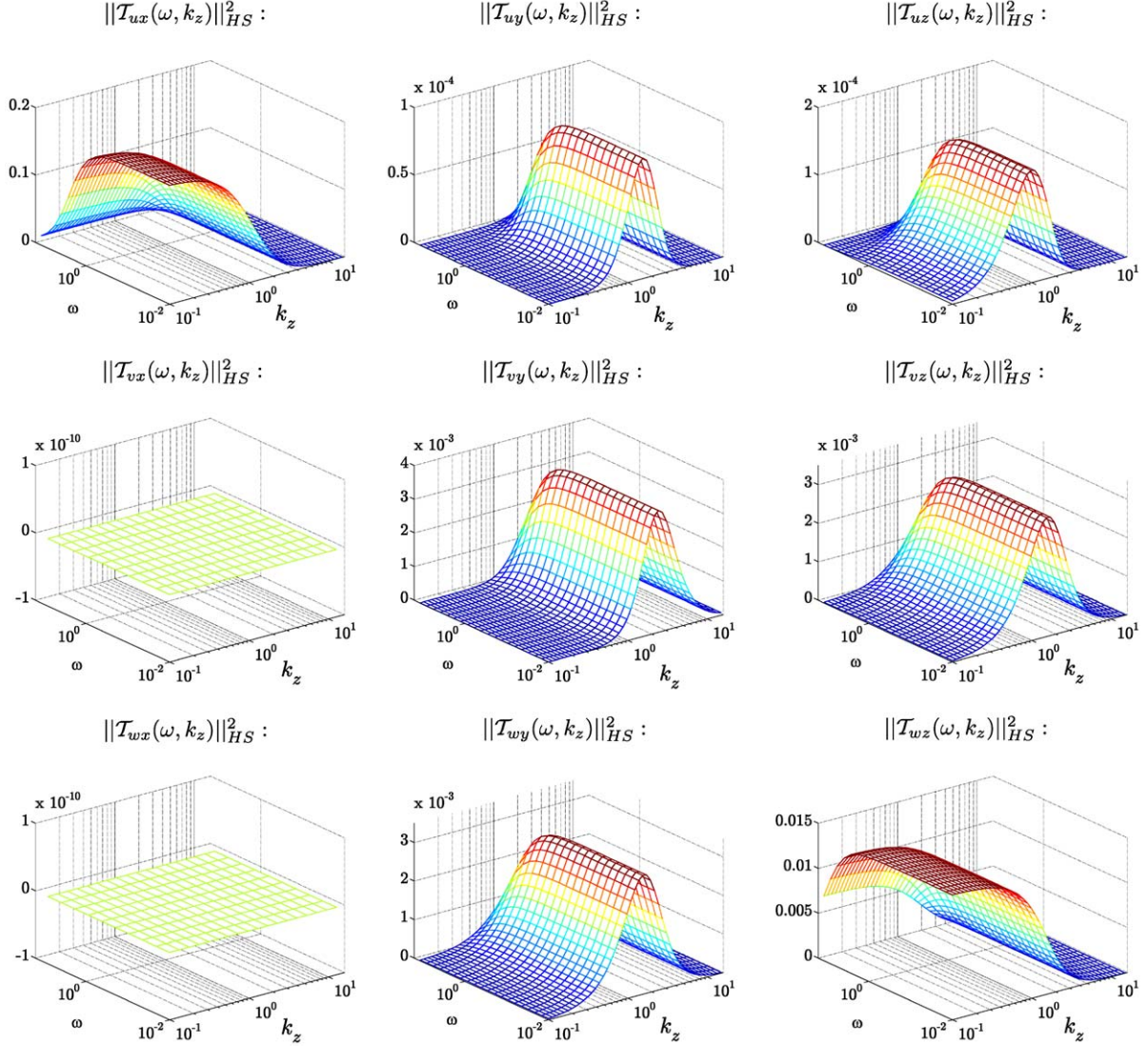


Fig. 2. Plots of $\|\mathcal{T}_{rs}(\omega, k_z)\|_{HS}^2$, $\{r = u, v, w; s = x, y, z\}$. $\|\mathcal{T}_{uy}(\omega, k_z)\|_{HS}^2$ and $\|\mathcal{T}_{uz}(\omega, k_z)\|_{HS}^2$ are determined for Couette flow.

transform to (5.2a) yields

$$(j\omega\Delta - \mathcal{F}_{11})\psi_1 = \mathcal{G}_y d_y + \mathcal{G}_z d_z, \quad (5.3a)$$

$$(j\omega I - \mathcal{F}_{22})\psi_2 = \mathcal{F}_{21}\psi_1 + \mathcal{G}_x d_x. \quad (5.3b)$$

Note that (5.3a) represents an equation for ψ_1 with inputs d_y and d_z . On the other hand, (5.3b) is an equation for ψ_2 driven by ψ_1 and d_x . Using definitions of operators appearing in (5.3), we, respectively, rewrite (5.3a) and (5.3b) as

$$\begin{aligned} \psi_1^{(4)} - (2k_z^2 + j\omega)\psi_1'' + k_z^2(k_z^2 + j\omega)\psi_1 \\ = [k_z^2 \ 0] \begin{bmatrix} d_y \\ d_z \end{bmatrix} + [0 \ jk_z] \begin{bmatrix} d_y' \\ d_z' \end{bmatrix}, \end{aligned} \quad (5.4a)$$

$$\psi_2'' - (k_z^2 + j\omega)\psi_2 = jk_z U'(y)\psi_1 - jk_z d_x. \quad (5.4b)$$

Eqs. (5.4a) and (5.4b) are ODEs in y parameterized by k_z and ω with the boundary conditions: $\psi_1(\pm 1) =$

$\psi_1'(\pm 1) = \psi_2(\pm 1) = 0$. For notational convenience, we have suppressed the dependence of ψ_i and d_s on (k_z, ω) , e.g. $\psi_i(y) := \psi_i(y, \omega, k_z)$, $i = 1, 2$. Similarly, $\psi_i^{(r)}(y) := d^r \psi_i(y, \omega, k_z) / dy^r$. It is now possible to find TPBVS for each of (5.4a) and (5.4b) separately. Two particular such realizations similar to observer canonical forms are given in Appendix A, where we also combine those realizations with the output (5.2b) to obtain TPBVS of the full operator $\mathcal{T}(\omega, k_z) : [d_x \ d_y \ d_z]^T \mapsto [u \ v \ w]^T$ as well as its nine subcomponents $\mathcal{T}_{rs}(\omega, k_z) : d_s \mapsto r$, $\{r = u, v, w; s = x, y, z\}$.

With the realizations in Appendix A, we compute all nine components $\|\mathcal{T}_{rs}(\omega, k_z)\|_{HS}^2$ of the overall frequency response. We note that for each pair (ω, k_z) , these computations involve a finite matrix computation rather than a spatial discretization of the differential operator. The dimension of the matrices involved are related to the order of the

realizations. As shown in Appendix A, the operator $\mathcal{F}_{ux}(\omega, k_z)$ has a second-order realization, the operators $\mathcal{F}_{rs}(\omega, k_z)$, $\{r = v, w; s = y, z\}$, can be represented by a fourth-order realization, whereas $\mathcal{F}_{uy}(\omega, k_z)$ and $\mathcal{F}_{uz}(\omega, k_z)$ can be described by minimal realizations with six states.

Plane Couette flow is specified by the laminar flow profile $U(y) := y$. \mathcal{F}_{21} simplifies to $\mathcal{F}_{21} = -jk_z$ in this case, and all operators in (5.2) have constant coefficients in y . Thus the matrix A_0 in (A.1) becomes y -independent, and the functions $\|\mathcal{F}_{rs}(\omega, k_z)\|_{\text{HS}}^2$ can be determined by (cf. (4.12) in Theorem 4.1)

$$\begin{aligned} & \|\mathcal{F}_{rs}(\omega, k_z)\|_{\text{HS}}^2 \\ &= -\text{tr} \left((L_1 + L_2 e^{2A})^{-1} L_2 \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \right) \\ & \quad \times \exp \left\{ 2 \begin{bmatrix} A & 0 \\ B_s C_r & A \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \right\}. \end{aligned} \quad (5.5)$$

Fig. 2 illustrates the (ω, k_z) -parameterized plots of $\|\mathcal{F}_{rs}(\omega, k_z)\|_{\text{HS}}^2$, for every $\{r = u, v, w; s = x, y, z\}$. These computations are performed in MATLAB for any given pair (ω, k_z) using the formula above. The nominal-flow-dependent quantities $\|\mathcal{F}_{uy}(\omega, k_z)\|_{\text{HS}}^2$ and $\|\mathcal{F}_{uz}(\omega, k_z)\|_{\text{HS}}^2$ are determined for Couette flow.

The spatial frequency responses evaluated at $\omega = 0$ reveal interesting information about maximally amplified flow structures in channel flows [10]. It turns out that we can obtain explicit analytical forms for these frequency responses. The expressions for $\|\mathcal{F}_{rs}(0, k_z)\|_{\text{HS}}^2$ are determined using formula (4.12), and they are given in Appendix B. The underlying matrix exponentials are evaluated symbolically in MATHEMATICA.

6. Concluding remarks

We develop a procedure for the explicit determination of the frequency responses for a class of distributed systems. This procedure avoids the need for spatial discretization and provides an exact reduction of an infinite dimensional problem to a problem in which only matrices of finite dimensions are involved. The order of these matrices is at most four times larger than the order of the differential operator at hand.

We also illustrate application of this technique by providing two examples: a one-dimensional diffusion equation, and a system obtained by linearization of the Navier–Stokes equations in channel flows around a given nominal velocity profile.

Appendix A. TPBVS of the streamwise constant linearized Navier–Stokes equations

Here, we describe the TPBVS that are used for the frequency response determination of the streamwise constant

LNS equations at $R = 1$. In particular, we choose the observable canonical form realizations of (5.4a) and (5.4b)

$$(5.4a) : \begin{cases} \zeta'(y) = A_1 \zeta(y) + B_1 \begin{bmatrix} d_x(y) \\ d_y(y) \\ d_z(y) \end{bmatrix}, \\ \psi_1(y) = C_1 \zeta(y), \\ 0 = S_1 \zeta(-1) + S_2 \zeta(1), \quad y \in [-1, 1], \end{cases}$$

$$(5.4b) : \begin{cases} \eta'(y) = A_2 \eta(y) + B_2 \begin{bmatrix} d_x(y) \\ d_y(y) \\ d_z(y) \end{bmatrix} \\ \quad + B_3(y) \psi_1(y), \\ \psi_2(y) = C_2 \eta(y), \\ 0 = T_1 \eta(-1) + T_2 \eta(1), \quad y \in [-1, 1], \end{cases}$$

where

$$A_1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2k_z^2 + j\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_z^2(k_z^2 + j\omega) & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 := \begin{bmatrix} 0 & 1 \\ k_z^2 + j\omega & 0 \end{bmatrix},$$

$$B_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & jk_z \\ 0 & k_z^2 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 0 & 0 \\ -jk_z & 0 & 0 \end{bmatrix},$$

$$B_3(y) := \begin{bmatrix} 0 \\ jk_z U'(y) \end{bmatrix},$$

$$C_1 := [1 \ 0 \ 0 \ 0], \quad C_2 := [1 \ 0],$$

$$S_1 := \begin{bmatrix} I_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}, \quad S_2 := \begin{bmatrix} 0_{2 \times 2} & 0_{2 \times 2} \\ I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix},$$

$$T_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We now combine these two realization with (5.2b) to obtain a TPBVS of operator $\mathcal{F}(\omega, k_z): [d_x \ d_y \ d_z]^T \mapsto [u \ v \ w]^T$:

(5.4, 5.2b):

$$\begin{cases} x'(y) = A_0(y)x(y) + B_0 \begin{bmatrix} d_x(y) \\ d_y(y) \\ d_z(y) \end{bmatrix}, \\ \begin{bmatrix} u(y) \\ v(y) \\ w(y) \end{bmatrix} = C_0 x(y), \\ 0 = N_1 x(-1) + N_2 x(1), \quad y \in [-1, 1], \end{cases} \quad (A.1)$$

where $x^T(y) := [\xi^T(y) \ \eta^T(y)]$, and

$$\begin{aligned} A_0(y) &:= \begin{bmatrix} A_1 & 0 \\ B_3(y)C_1 & A_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2k_z^2 + j\omega & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k_z^2(k_z^2 + j\omega) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ jk_z U'(y) & 0 & 0 & 0 & k_z^2 + j\omega & 0 \end{bmatrix}, \end{aligned}$$

$$B_0 := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & jk_z \\ 0 & k_z^2 & 0 \\ 0 & 0 & 0 \\ -jk_z & 0 & 0 \end{bmatrix}$$

$$=: [B_{x0} \ B_{y0} \ B_{z0}], \quad N_1 := \begin{bmatrix} S_1 & 0 \\ 0 & T_1 \end{bmatrix},$$

$$C_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{j}{k_z} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{j}{k_z} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$=: \begin{bmatrix} C_{u0} \\ C_{v0} \\ C_{w0} \end{bmatrix}, \quad N_2 := \begin{bmatrix} S_2 & 0 \\ 0 & T_2 \end{bmatrix}.$$

Thus, we have rendered TPBVSR of system (5.4) and (5.2b) into (A.1), which is a suitable form for the application of the procedure developed in Section 4.

The realizations of $\mathcal{F}_{rs}(\omega, k_z) : d_s \mapsto r$ can be now easily determined for every $\{r = u, v, w; s = x, y, z\}$. For example, TPBVSR of $\mathcal{F}_{uy}(\omega, k_z) : d_y \mapsto u$ is given by

$$\mathcal{F}_{uy} : \begin{cases} x'(y) = A_0(y)x(y) + B_{y0}d_y(y), \\ u(y) = C_{u0}x(y), \\ 0 = N_1x(-1) + N_2x(1), \quad y \in [-1, 1]. \end{cases}$$

As previously mentioned, $\mathcal{F}_{ux}(\omega, k_z)$ and $\mathcal{F}_{rs}(\omega, k_z)$, for every $\{r = v, w; s = y, z\}$, are, respectively, the second and the fourth-order operators. We note that their minimal realizations can be obtained by combining the respective realizations of (5.4b) and (5.4a) with the corresponding rows of (5.2b). For example, a minimal realization of $\mathcal{F}_{ux}(\omega, k_z)$ is given by

$$\mathcal{F}_{ux} : \begin{cases} \eta'(y) = \begin{bmatrix} 0 & 1 \\ k_z^2 + j\omega & 0 \end{bmatrix} \eta(y) + \begin{bmatrix} 0 \\ -jk_z \end{bmatrix} d_x(y), \\ u(y) = \begin{bmatrix} -\frac{j}{k_z} & 0 \end{bmatrix} \eta(y), \\ 0 = T_1\eta(-1) + T_2\eta(1), \quad y \in [-1, 1]. \end{cases}$$

Alternatively, $\|\mathcal{F}_{ux}(\omega, k_z)\|_{\text{HS}}^2$ can be determined from Theorem 4.1 using the following non-minimal realization of $\mathcal{F}_{ux}(\omega, k_z)$

$$\mathcal{F}_{ux} : \begin{cases} x'(y) = A_0(y)x(y) + B_{x0}d_x(y), \\ u(y) = C_{u0}x(y), \\ 0 = N_1x(-1) + N_2x(1), \quad y \in [-1, 1]. \end{cases}$$

Note that the underlying matrices in (5.5) can be determined from the above matrices using Theorem 4.1.

Appendix B. Formulae for Hilbert–Schmidt norms at $\omega = 0$

We next derive analytical expressions for functions $\|\mathcal{F}_{rs}(0, k_z)\|_{\text{HS}}^2$ (the HS subscript is dropped from the notation below). The mean-flow-dependent quantities $\|\mathcal{F}_{uy}(0, k_z)\|^2$ and $\|\mathcal{F}_{uz}(0, k_z)\|^2$ are obtained for Couette flow. These expressions are determined symbolically in MATHEMATICA using (4.12), and they are given by

$$\|\mathcal{F}_{ux}(0, k_z)\|^2 = \frac{k_z(\coth(2k_z) + 2k_z \operatorname{csch}^2(2k_z)) - 1}{2k_z^4},$$

$$\|\mathcal{F}_{uy}(0, k_z)\|^2 = \frac{h_{uy}(k_z)}{322560k_z^8(\cosh(4k_z) - 8k_z^2 - 1)^2},$$

$$\|\mathcal{F}_{uz}(0, k_z)\|^2 = \frac{h_{uz}(k_z)}{5160960k_z^8(\cosh(4k_z) - 8k_z^2 - 1)^2},$$

$$\|\mathcal{F}_{vy}(0, k_z)\|^2 = \frac{h_{vy}(k_z)}{1440k_z^4(\cosh(4k_z) - 8k_z^2 - 1)^2},$$

$$\|\mathcal{F}_{vz}(0, k_z)\|^2 = \frac{h_{vz}(k_z)}{1440k_z^4(\cosh(4k_z) - 8k_z^2 - 1)^2},$$

$$\|\mathcal{F}_{wy}(0, k_z)\|^2 = \|\mathcal{F}_{vz}(0, k_z)\|^2,$$

$$\|\mathcal{F}_{wz}(0, k_z)\|^2 = \frac{h_{wz}(k_z)}{1440k_z^4(\cosh(4k_z) - 8k_z^2 - 1)^2},$$

where

$$h_{uy}(k_z) := -297675 + 16k_z^2(-187425 + 4k_z^2) \\ \times (-168525 + 8k_z^2(-8337 + 2474k_z^2 + 80k_z^4)) \\ + 12(33075 + 249900k_z^2 + 42000k_z^4) \\ + 28672k_z^6 + 768k_z^8) \cosh(4k_z) - 99225 \cosh(8k_z) \\ + \frac{7}{8}k_z \operatorname{csch}^2(k_z) \operatorname{sech}^2(k_z)(3072k_z^5(45 + 16k_z^4) \\ + (14175 + 64k_z^2(2295 + 660k_z^2 - 1176k_z^4) \\ + 1216k_z^6)) \sinh(4k_z) \\ + 12(-945 + 8k_z^2(-765 + 440k_z^2 + 72k_z^4)) \\ \times \sinh(8k_z) + 2835 \sinh(12k_z),$$

$$h_{uz}(k_z) := + \operatorname{csch}^2(k_z) \operatorname{sech}^2(k_z)(34650 + 16k_z^2(31185 + 4k_z^2) \\ \times (2835 + 16k_z^2(630 + 2577k_z^2 - 592k_z^4))) \\ - (51975 + 64k_z^2(10395 + 8k_z^2(735 - 70k_z^2) \\ + 1776k_z^4 + 160k_z^6))) \cosh(4k_z) \\ + 20790 \cosh(8k_z) - 3465 \cosh(12k_z) \\ + 2k_z(8k_z(10395 + 12180k_z^2 - 2240k_z^4 - 576k_z^6) \\ \times \cosh(8k_z) + 224(315 \cosh(2k_z) \sinh(2k_z)^5 \\ + k_z^2(2(855 + 8k_z^2(45 + 197k_z^2 - 136k_z^4)) \sinh(4k_z) \\ + (-855 + 180k_z^2 - 136k_z^4) \sinh(8k_z))))),$$

$$h_{vy}(k_z)$$

$$\begin{aligned} &:= -1485 + 8k_z^2(-1845 + 128k_z^2(-45 \\ &\quad + 2k_z^2(-8 + k_z^2))) - 495 \cosh(8k_z) \\ &\quad + 4(495 + 3690k_z^2 + 720k_z^4 + 256k_z^6) \cosh(4k_z) \\ &\quad + 30k_z(-15 - 216k_z^2 + 128k_z^4 \\ &\quad + 15 \cosh(4k_z)) \sinh(4k_z), \end{aligned}$$

$$h_{vz}(k_z)$$

$$\begin{aligned} &:= -135 - 8k_z^2(225 - 128k_z^4 + 256k_z^6) \\ &\quad + 4(45 + 450k_z^2 + 720k_z^4 - 256k_z^6) \cosh(4k_z) \\ &\quad - 45 \cosh(8k_z) + 90k_z(-1 - 40k_z^2 \\ &\quad + \cosh(4k_z)) \sinh(4k_z), \end{aligned}$$

$$h_{wz}(k_z)$$

$$\begin{aligned} &:= -135 + 8k_z^2(-225 + 16k_z^2(-5 \\ &\quad + 4k_z^2)(3 + 4k_z^2)) - 45 \cosh(8k_z) \\ &\quad + 4(45 + 450k_z^2 + 1200k_z^4 + 256k_z^6) \cosh(4k_z) \\ &\quad - 30k_z(3 + 120k_z^2 + 128k_z^4 - 3 \cosh(4k_z)) \sinh(4k_z). \end{aligned}$$

References

- [1] P.J. Antsaklis, A.N. Michel, Linear Systems, McGraw-Hill, New York, 1997.
- [2] B. Bamieh, M. Dahleh, Energy amplification in channel flows with stochastic excitation, *Phys. Fluids* 13 (2001) 3258–3269.
- [3] B. Bamieh, M. Dahleh, Exact computation of traces and H^2 norms for a class of infinite dimensional problems, *IEEE Trans. Automat. Control* 48 (2003) 646–649.
- [4] C.T. Chen, Linear System Theory and Design, Oxford University Press, Oxford, 1998.
- [5] R.F. Curtain, H.J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer, New York, 1995.
- [6] R.C. DiPrima, G.J. Habetler, A completeness theorem for non-selfadjoint eigenvalue problems in hydrodynamic stability, *Arch. Rat. Mech. Anal.* 34 (1969) 218–227.
- [7] I. Gohberg, M.A. Kaashoek, Time varying linear systems with boundary conditions and integral operators, I. The transfer operator and its properties, *Integral Equations Operator Theory* 7 (1984) 325–391.
- [8] M.R. Jovanović, modeling, analysis, and control of spatially distributed systems, Ph.D. Thesis, University of California, Santa Barbara, 2004. available at <http://www.ece.umn.edu/users/mihailo/>
- [9] M.R. Jovanović, B. Bamieh, Frequency domain analysis of the linearized Navier–Stokes equations, in: Proceedings of the 2003 American Control Conference, Denver, CO, 2003, pp. 3190–3195.
- [10] M.R. Jovanović, B. Bamieh, Componentwise energy amplification in channel flows, *J. Fluid Mech.* 534 (2005) 145–183.
- [11] S.C. Reddy, D.S. Henningson, Energy growth in viscous channel flows, *J. Fluid Mech.* 252 (1993) 209–238.
- [12] P.J. Schmid, D.S. Henningson, Stability and Transition in Shear Flows, Springer, New York, 2001.
- [13] N. Young, An introduction to Hilbert Spaces, Cambridge University Press, New York, 1988.