

Synchronization of limit cycle oscillations in diffusively-coupled systems

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Abstract—We present analytical and numerical conditions to verify whether limit cycle oscillations synchronize in diffusively coupled systems. We consider both compartmental ODE models, where each compartment represents a spatial domain of components interconnected through diffusion terms with like components in different compartments, and reaction-diffusion PDEs with Neumann boundary conditions. In both the discrete and continuous spatial domains, we assume the uncoupled dynamics are determined by a nonlinear system which admits an asymptotically stable limit cycle. The main contribution of the paper is a method to certify when the stable oscillatory trajectories of a diffusively coupled system are robust to diffusion, and to highlight cases where diffusion in fact leads to loss of spatial synchrony. We illustrate our results with a relaxation oscillator example.

I. INTRODUCTION

Diffusively coupled models are crucial to understanding the dynamical behavior of a range of engineering and biological systems. Synchronization of diffusively coupled models has been an active research area [1]. Conversely, developing conditions that rule out synchrony are also important, as they can facilitate study of the opposite problem of patterning. One of the major ideas behind pattern formation in cells and organisms is based on diffusion-driven instability [2], [3], which occurs when higher-order spatial modes in a reaction-diffusion PDE are destabilized by diffusion [4]–[8].

The majority of synchronization studies address phase coupled oscillators [9]–[12], which rely on the assumption of weak coupling to be able to represent the subsystems with a single phase variable. Full state models have been studied in [13]–[16]; however, these references derive global results that may be conservative when synchronization of trajectories close to a specific attractor, such as a limit cycle, is of interest.

In this paper, we study diffusively coupled nonlinear systems that exhibit limit cycles in the absence of diffusion. We develop analytical and numerical tools to determine whether diffusion stabilizes the spatially homogeneous limit cycle trajectories, thereby synchronizing the oscillations across the spatial domain. Our methods apply to compartmental ODEs and reaction-diffusion PDEs with Neumann boundary conditions. In the former case, each compartment has identical dy-

namics and represents a well-mixed spatial domain wherein like components in adjacent compartments are coupled by diffusion.

We first linearize the system about an asymptotically stable limit cycle trajectory, and then study the resulting periodic linear time varying system. In the case of sufficiently small or large diffusion, we use Floquet theory to decompose the linearized system into fast and slow time scales, and present results using *two-time scale averaging theory* [17], [18] that guarantee synchrony. In the case of diffusion coefficients of intermediate strength, we turn to a numerical approach, in which we use *harmonic balance* [19], [20] to represent the linearized system as an infinite-dimensional linear time invariant system. We make use of concepts from robust control, in particular the structured singular value [21], to determine stability of the linearized system in the presence of diffusion coefficients spanning a specified finite interval. We find that diffusion can indeed lead to loss of spatial synchrony and divergence of trajectories.

The paper is organized as follows. In Section 2, we formulate the problem, and present an example of a system with an asymptotically stable limit cycle that loses spatial synchrony in the presence of diffusion. In Section 3, we outline analytical tests for synchrony in the case of sufficiently small or sufficiently large perturbations. In Section 4, we give a numerical test to study synchrony on the remaining interval of intermediate diffusion coefficients. We present a relaxation oscillator example in Section 5, and give the conclusions in Section 6.

II. PROBLEM FORMULATION

In this section, we formulate the problem of synchronization of limit cycle oscillations in diffusively-coupled systems. For both reaction-diffusion systems of PDEs with Neumann boundary conditions and compartmental systems of ODEs, we show that synchrony amounts to examining stability of a linear system with time-periodic coefficients. To motivate our developments, we also provide an example of a system with an asymptotically stable limit cycle that loses spatial synchrony in the presence of large enough diffusion.

We first discuss systems governed by reaction-diffusion PDEs, and, next, systems of compartmental ODEs. For both cases, we investigate whether diffusive coupling synchronizes limit cycle oscillations.

We consider and define the spatial domain $\Omega \in \mathbb{R}^r$ with

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smooth boundary $\partial\Omega$, spatial variable $\xi \in \Omega$, and outward normal vector $n(\xi)$ for $\xi \in \partial\Omega$. We study the PDE model:

$$\frac{\partial x}{\partial t} = f(x) + D\nabla^2 x, \quad (1)$$

subject to Neumann boundary conditions $\nabla x_i(t, \xi) \cdot n(\xi) = 0$ for all $\xi \in \partial\Omega$, where $x(t, \xi) \in \mathbb{R}^n$, $D \in \mathbb{R}^{n \times n}$, and

$$\nabla^2 x = [\nabla^2 x_1 \dots \nabla^2 x_n]^T \quad (2)$$

is a vector of Laplacian operators with respect to the spatial variable ξ applied to each entry of x . In a reaction-diffusion system, x represents a vector of concentrations for the reactants and D is a diagonal matrix of diffusion coefficients. However, for generality of our derivations, we will not assume D to be diagonal unless we state otherwise.

We assume that the lumped system $\dot{x} = f(x)$ has an asymptotically stable limit cycle and that $\bar{x}(t)$ is a solution of $\dot{x} = f(x)$ along the limit cycle. Then $\bar{x}(t, \xi) = \bar{x}(t)$ for all $\xi \in \Omega$ is a solution of (1). In the absence of diffusion ($D = 0$), the system (1) admits out-of-phase oscillations, that is, solutions of the form $x(t, \xi) = \bar{x}(t + \phi(\xi))$, where $\phi(\xi)$ is a phase that depends on the location ξ . To determine whether diffusion eliminates such spatial phase differences, we examine the Jacobian linearization about the limit cycle trajectory $\bar{x}(t)$:

$$\frac{\partial \tilde{x}}{\partial t} = (A(t) + D\nabla^2)\tilde{x} \quad (3)$$

where $\tilde{x}(t, \xi) = x(t, \xi) - \bar{x}(t)$ and

$$A(t) = J(\bar{x}(t)) = \left. \frac{\partial f}{\partial x} \right|_{\bar{x}(t)}, \quad (4)$$

with $A(t)$ periodic with period T . Let $0 = \lambda_1 \leq \lambda_2 \leq \dots$ denote the eigenvalues and $\phi_1(\xi), \phi_2(\xi), \dots$ denote the corresponding orthogonal eigenfunctions of the operator $L = -\nabla^2$ on Ω with Neumann boundary conditions:

$$L\phi_i(\xi) = \lambda_i\phi_i(\xi), \quad \nabla\phi_i(\xi) \cdot n(\xi) = 0 \text{ for all } \xi \in \partial\Omega. \quad (5)$$

The solution to (3) can be expressed as:

$$\tilde{x}(t, \xi) = \sum_{i=1}^{\infty} \sigma_i(t)\phi_i(\xi), \quad (6)$$

where $\sigma_i(t) \in \mathbb{R}^n$ satisfy the decoupled ODEs:

$$\dot{\sigma}_i = (A(t) - \lambda_i D)\sigma_i. \quad (7)$$

Since the eigenfunction $\phi_1(\xi)$ for $\lambda_1 = 0$ is constant, the term corresponding to $i = 1$ represents a spatially homogeneous mode σ_1 governed by $\dot{\sigma}_1 = A(t)\sigma_1$. When the subsystems (7) are asymptotically stable for $i = 2, 3, \dots$, the contributions of the inhomogeneous modes $\phi_2(\xi), \phi_3(\xi), \dots$ to the solution $x(t, \xi)$ decay in time, which in turn implies that the oscillations of (1) synchronize.

We also study a *compartmental ODE* model, where each compartment represents a well-mixed spatial domain interconnected with the other compartments over an undirected graph:

$$\dot{x}_i = f(x_i) + D \sum_{j \in N_i} (x_j - x_i), \quad i = 1, \dots, N. \quad (8)$$

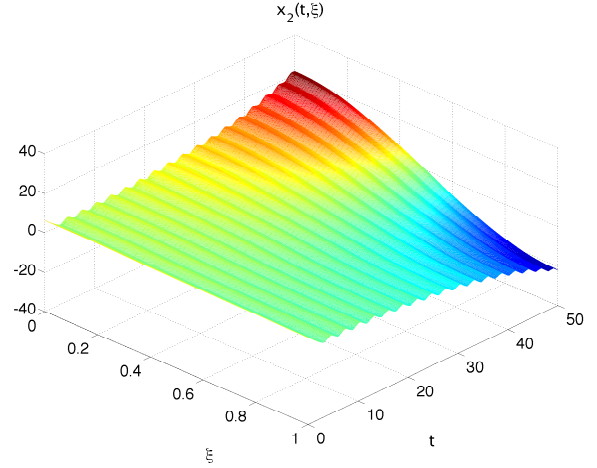


Fig. 1. Spatio-temporal evolution of x_2 for system (1,13) with $d_1 = 100$, $d_2 = 0$, and $\mu = 0.1$ on the one-dimensional spatial domain $\Omega = [0, 1]$ with initial condition $x_2(0, \xi) = 5 + \cos(\pi\xi)$ and Neumann boundary conditions. The oscillations do not synchronize, and in fact growth of the spatial mode $\phi_2(\xi)$ is observed.

The vector $x_i \in \mathbb{R}^n$ represents each compartment's state, \mathcal{N}_i denotes the neighbors of compartment i , and $D \in \mathbb{R}^{n \times n}$. We take the Jacobian linearization about a limit cycle trajectory $\bar{x}(t)$, and aggregate the dynamics of the subsystems using the state variable $\tilde{x} = [\tilde{x}_1^T \dots \tilde{x}_N^T]^T$, $\tilde{x}_i(t) = x_i(t) - \bar{x}(t)$. We represent the interaction between state variables by a graph Laplacian matrix $L = L^T \in \mathbb{R}^{N \times N}$, defined as

$$L = AA^T, \quad (9)$$

where A is an incidence matrix whose rows represent vertices (state variables in this context) and columns represent edges (couplings between state variables). The dynamics of the aggregated system may be written:

$$\dot{\tilde{x}} = (I \otimes A(t) - L \otimes D)\tilde{x}, \quad (10)$$

where $A(t)$ is as in (4) and \otimes denotes the Kronecker product. Let $U \in \mathbb{R}^{N \times N}$ be a similarity transformation that brings L into the diagonal matrix of its eigenvalues $\Sigma \in \mathbb{R}^{N \times N}$: $L = U\Sigma U^{-1}$. Choosing $\tilde{y} = (U^{-1} \otimes I)\tilde{x}$, we rewrite (10) as:

$$\dot{\tilde{y}} = (I \otimes A(t) - \Sigma \otimes D)\tilde{y}, \quad (11)$$

which is decoupled into the subsystems:

$$\dot{\tilde{y}}_l = (A(t) - \lambda_l D)\tilde{y}_l, \quad l = 1, \dots, N, \quad (12)$$

where $\tilde{y}_l \in \mathbb{R}^n$ and λ_l is the l th eigenvalue of the Laplacian, respectively. In particular, $\lambda_1 = 0$ and $\lambda_l > 0$, $l = 2, 3, \dots, N$ when the graph is connected. Note that (12) is analogous to (7) except that it consists of finitely many modes $l = 1, \dots, N$. If the subsystems (12), $l = 2, \dots, N$, are asymptotically stable, then for any pair x_j and x_k , we have $x_j - x_k \rightarrow 0$ exponentially as $t \rightarrow \infty$.

To see that a diagonal $D \succeq 0$ does not necessarily guarantee synchronization, consider the system (1) with the dynamics:

$$f(x) = \begin{bmatrix} \frac{1}{\mu}(x_1 - \frac{1}{3}x_1^3 - x_2) \\ x_1 + \mu x_2 \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (13)$$

with $d_1 > 0$. When $\mu > 0$ is sufficiently large, the vector field $f(x)$ has the behavior of a *relaxation oscillator* [22] and admits a stable limit cycle. The Jacobian linearization about the limit cycle trajectory $\bar{x}(t)$ is:

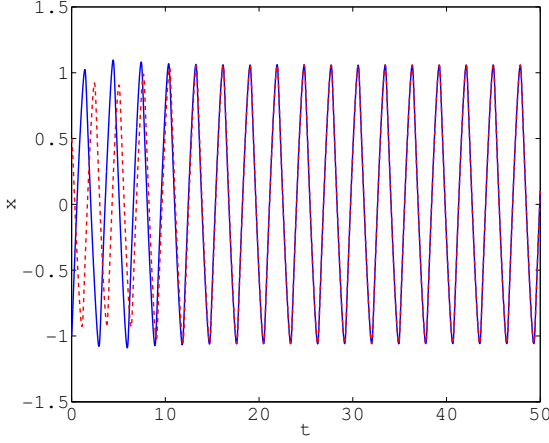


Fig. 2. Trajectories of x_{12} (blue, solid) and x_{22} (red, dashed) of (8) and (13) for two compartments and synchronize under small diffusion coefficient $d_1 = .5$ and initial conditions $(x_{12}(0), x_{22}(0)) = (-1, .5)$.

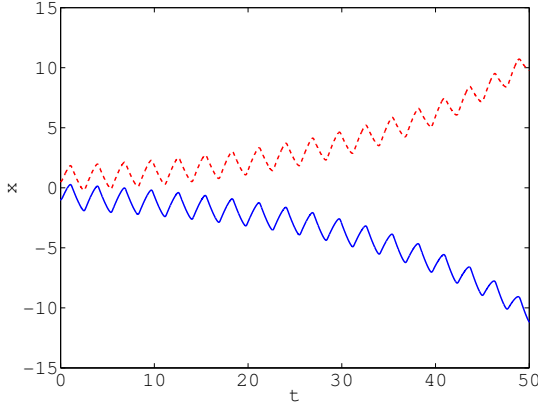


Fig. 3. Trajectories of x_{12} (blue, solid) and x_{22} (red, dashed) of (8) and (13) for two compartments do not synchronize under larger diffusion coefficient $d_1 = 100$ and initial conditions $(x_{12}(0), x_{22}(0)) = (-1, .5)$.

$$A(t) = \begin{bmatrix} \frac{1}{\mu}(1 - \bar{x}_1^2(t)) & \frac{1}{\mu} \\ 1 & \mu \end{bmatrix}. \quad (14)$$

When $\lambda_i d_1 \gg 1/\mu$, system (12) exhibits two-time scale behavior, with the slow dynamics unstable:

$$\dot{\sigma}_{i2} = \mu \sigma_{i2}. \quad (15)$$

Thus, we expect (13) to be unstable when $\lambda_i d_1$ is sufficiently large. Indeed, for $d_1 = 100$ and $\mu = 0.1$, the simulations over the spatial domain $[0, 1]$ demonstrate the growth of the spatial mode $\phi_2(\xi) = \cos(\pi\xi)$; see Figure 1. Unlike standard examples of diffusion-driven instability of a homogeneous steady-state [2]–[4], this example demonstrates destabilization of a spatially homogeneous periodic orbit by diffusion.

Similar behavior can be observed for the compartmental model (8) with two compartments, and $f(x)$ and D given by (13). The two-node graph representing the interconnection of the two compartments has Laplacian eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$. When d_1 is small, we find that oscillations synchronize spatially, as shown in Figure 2. When d_1 is large, the trajectories corresponding to compartments one and two diverge from each other, as shown in Figure 3.

Thus, for both the PDE (1) and the compartmental ODE (8), synchrony is determined by the stability of the time-varying system

$$\dot{x} = (A(t) - D)x, \quad (16)$$

where, for simplicity, we have dropped λ_i since D can be appropriately scaled to account for λ_i .

III. SYNCHRONIZATION UNDER WEAK OR STRONG COUPLING

We study the case of sufficiently small or large diffusion. We use Floquet theory to decompose the linearized system into fast and slow time scales, and develop conditions using two-time scale averaging theory that guarantee synchrony.

Consider system (16), where $A(t) = \frac{\partial f}{\partial x} \Big|_{\bar{x}(t)}$ is the linearization of $f(x)$ about a limit cycle trajectory $\bar{x}(t)$, and let T denote the period of oscillations: $A(t+T) = A(t)$ for all t . We first consider the case with $D = 0$, that is:

$$\dot{x} = A(t)x, \quad (17)$$

and note that it admits the periodic solution $x(t) = \hat{x}(t)$. To see this, observe the following:

$$\dot{\hat{x}}(t) = f(\bar{x}(t)) \implies \ddot{\hat{x}}(t) = \frac{\partial f}{\partial x} \Big|_{\bar{x}(t)} \dot{\hat{x}}(t) = A(t)\dot{\hat{x}}(t). \quad (18)$$

Floquet's Theorem (Thm. 2.2.5, [23]) implies that the state transition matrix $\Phi(t, t_0)$ of (17) is periodic and can be written as

$$\Phi(t, t_0) = U(t) \exp(F(t - t_0))V(t_0), \quad (19)$$

where $U(t+T) = U(t)$ and $U(t) = V^{-1}(t)$, with the columns of $U(t)$ given by $u_i(t)$ and the rows of $V(t)$ given by $v_j^T(t)$. Since (17) results from linearization about a stable limit cycle, F can be written as

$$F = \begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix}, \quad (20)$$

where F_2 is an $(n-1) \times (n-1)$ Hurwitz matrix and $u_1(t) = \hat{x}(t)$. The eigenvalues of F are called *Floquet exponents*, and

the evaluation of the state transition matrix over one period with initial condition t_0 , $\Phi(t_0 + T, t_0) = \exp(FT)$, is called the *monodromy matrix*.

In what follows, we derive a condition that relates the stability of (16) with sufficiently small D to $u_1(t)$ and $v_1(t)$. First, we review properties of $u_1(t)$ and $v_1(t)$ that follow from Floquet theory. The definition of $U(t)$ and $V(t)$ implies that $v_j^T(t)u_i(t) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. In particular, $v_1(t)$ is a periodic solution of the adjoint system:

$$\dot{\rho} = -A^T(t)\rho. \quad (21)$$

To compute $u_1(t)$ and $v_1(t)$, we follow [24] and numerically integrate

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0) \quad (22)$$

over one period with the initial condition $\Phi(t_0, t_0) = I$. We then compute the eigenvector of the monodromy matrix corresponding to its eigenvalue at one:

$$u_1(t_0) = \Phi(t_0 + T, t_0)u_1(t_0). \quad (23)$$

Using the numerically-computed state transition matrix, we then calculate the trajectory $u_1(t) = \Phi(t, t_0)u_1(t_0)$. To compute $v_1(t)$, we begin by computing the left eigenvector of the monodromy matrix corresponding to its eigenvalue at one:

$$v_1^T(t_0)\Phi(t_0 + T, t_0) = v_1^T(t_0). \quad (24)$$

We scale $v_1(t_0)$ such that $v_1^T(t_0)u_1(t_0) = 1$. Finally, to obtain $v_1(t)$, we numerically integrate the adjoint system (21) backwards in time with the terminal condition $\rho(t_0 + T) = v_1(t_0)$.

Having reviewed the case $D = 0$, we now prove a result about the stability of (17) with sufficiently small D .

Proposition 3.1: Let $v_1^T(t)$ be the first row of $V(t)$ and $u_1(t)$ be the first column of $U(t)$, where $\Phi(t, t_0) = U(t)\exp(F(t - t_0))V(t_0)$ as described above. Given a matrix $D \in \mathbb{R}^{n \times n}$, if the inequality

$$\int_{t_0}^{t_0+T} v_1^T(t)Du_1(t) dt > 0 \quad (25)$$

holds, then the origin of the system

$$\dot{x} = (A(t) - \epsilon D)x \quad (26)$$

is exponentially stable for sufficiently small $\epsilon > 0$.

Proof: Floquet theory implies that the time-varying change of coordinates $y = V(t)x$ transforms (17) into a linear time invariant system:

$$\dot{y} = Fy, \quad (27)$$

where F is as in (20). Introducing the decomposition $y = [w^T \ z^T]^T$, we rewrite (27) as:

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}. \quad (28)$$

When applied to system (26), the preceding change of coordinates yields:

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix} - \epsilon V(t)DU(t) \right) \begin{bmatrix} w \\ z \end{bmatrix}. \quad (29)$$

For small ϵ , this time-varying periodic system exhibits two-time scale behavior, which allows us to exploit the theory of two-time scale averaging [17], [18]. The *averaged slow system* corresponding to (29) is given by

$$\begin{aligned} \dot{w} &= -\epsilon aw, \\ a &= \frac{1}{T} \int_{t_0}^{t_0+T} v_1^T(t)Du_1(t)dt. \end{aligned} \quad (30)$$

Since F_2 is Hurwitz, an application of Lemma A.1 in the Appendix shows that if $a > 0$, then the equilibrium $y = 0$ is exponentially stable for sufficiently small ϵ . ■

Note that Proposition 3.1 does not require D to be diagonal. When D is diagonal, the test (25) can be simplified as follows:

Corollary 3.2: Let u_{1i} and v_{1j}^T be the i th and j th components of u_1 and v_1^T , respectively. If the inequalities

$$\int_{t_0}^{t_0+T} v_{1i}^T(t)u_{1i}(t) dt > 0, \quad i = 1, \dots, n \quad (31)$$

hold, then given any diagonal matrix $D \succeq 0$, $D \neq 0$, the periodic solution of the linearized system (26) is stable for sufficiently small $\epsilon > 0$.

We now turn to the case where D is large. Standard results from perturbation theory [22] guarantee stability of (16) when D is nonsingular and sufficiently large. When D is singular, we again leverage two-time scale arguments to derive a condition that guarantees stability of (16):

Proposition 3.3: Consider the linear time varying system:

$$\dot{x} = (A(t) - \epsilon^{-1}D)x \quad (32)$$

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix}, \quad (33)$$

where $x \in \mathbb{R}^n$, $A(t + T) = A(t)$ for all t , $A_{22}(t)$ and D_2 have the same dimension, $-D_2$ is Hurwitz, and $\epsilon > 0$. If

$$\bar{A}_{11} = \frac{1}{T} \int_{t_0}^{t_0+T} A_{11}(t) dt \quad (34)$$

is Hurwitz, then $x = 0$ is an exponentially stable equilibrium of (32) for sufficiently small ϵ .

The proof follows from Lemma A.1. Note that if D is not block diagonal, but is singular with trivial Jordan blocks corresponding to its eigenvalues at zero and all remaining eigenvalues in the closed right half plane, there exists a similarity transformation that will bring (32) to the form required by (33).

IV. NUMERICAL VERIFICATION OF SYNCHRONIZATION USING SSV

In this section, we develop numerical tools to determine the stability of (16) for a family of matrices D parametrized as:

$$D = M + B\Delta C, \quad (35)$$

where $M \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times n}$ are fixed matrices, and $\Delta \in \mathbb{R}^{m \times m}$ is a diagonal matrix whose entries take values in $[-1, 1]$. For example, suppose that the system (16) has one diffusible component, with

$$D = \text{diag}([d_1 \ 0 \ \dots \ 0]), \quad (36)$$

where $d_1 \in [r, R]$. Then D can be written as in (35) with $M = \frac{R+r}{2} e_1 e_1^T$ where e_i is a standard basis vector, $B = [\frac{R-r}{2} \ 0 \ \dots \ 0]^T$, $C = [1 \ 0 \ \dots \ 0]$, and $\Delta = \delta$ is a scalar. The problem is then to ascertain that the system (16) is stable for all values of δ on the interval $[-1, 1]$.

Structured singular value (SSV) analysis provides a useful test for determining the robustness of a stable linear time-invariant system to structured modeling uncertainty. However, since (37) is time-varying, in order to apply SSV analysis directly we must first bring the system to an equivalent time-invariant form. For such analysis, it is useful to rewrite the system (16) as:

$$\begin{aligned} \dot{x} &= (A(t) - M)x - Bq \\ y &= Cx \\ q &= \Delta y. \end{aligned} \quad (37)$$

Previous efforts to apply SSV analysis to time-varying systems have focused on the *lifting* idea of [25], [26], outlined in [27], [28], where system (37) is discretized and converted to a continuous time-invariant system.

Instead, we pursue an SSV analysis that makes use of the *harmonic balance* approach [19], which avoids the numerical difficulties and sensitivity of computing the state transition matrix and discretizing with an adequate number of samples in the lifting approach. Our computational experiments show that the harmonic balance approach frequently leads to less conservative results in establishing the values of diffusion coefficients that lead to instabilities. We give a brief summary of harmonic balance, and then outline its application to the problem of determining the stability of (16).

We assume that each entry of the matrix $A(t)$ is a continuous function of t that has an absolutely convergent Fourier series, and so $A(t)$ may be expressed as:

$$A(t) = \sum_{m \in \mathbb{Z}} A_m e^{jm\omega_p t}, \quad (38)$$

where ω_p is the fundamental frequency. Define doubly infinite vectors representing the harmonics of the state:

$$X^T = [\dots x_{-1}^T \ x_0^T \ x_1^T \ \dots], \quad (39)$$

and do the same for the input Q and output Y . The doubly infinite block Toeplitz matrix \mathcal{A} is determined by the harmonics of $A(t)$:

$$\mathcal{A} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & A_0 & A_{-1} & A_{-2} & \dots \\ \dots & A_1 & A_0 & A_{-1} & \dots \\ \dots & A_2 & A_1 & A_0 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (40)$$

We define the doubly infinite matrices $\mathcal{I} = \text{blkdiag}(I)$, $\mathcal{B} = \text{blkdiag}(B)$, and $\mathcal{C} = \text{blkdiag}(C)$, and define the modulation frequency matrix as:

$$\mathcal{N} = \text{blkdiag}\{jm\omega_p I\}, \quad \forall m \in \mathbb{Z}. \quad (41)$$

We define the matrix $\tilde{\Delta} = \text{blkdiag}(\Delta)$ to be block diagonal with copies of the diagonal matrix Δ in each block, and the matrix $\mathcal{M} = \text{blkdiag}(M)$ to be a block diagonal scaling matrix with copies of the matrix M in each block. We now introduce the harmonic state space model, where $s = j\omega$:

$$\begin{aligned} sX &= (\mathcal{A} - \mathcal{M} - \mathcal{N})X - \mathcal{B}Q \\ Y &= \mathcal{C}X \\ Q &= \tilde{\Delta}Y. \end{aligned} \quad (42)$$

We perform SSV analysis to determine if there exist matrices D such that (16) is unstable. For the precise definition of the structured singular value in the context of periodic linear-time-varying systems represented by a harmonic state space model, we refer the reader to [29]. To obtain a computationally tractable test, we truncate the doubly infinite system. In the examples we consider, there exist fewer than ten significant harmonics, and we represent the doubly infinite system by a finite dimensional system. We then perform SSV analysis on the truncated version of (42) to determine the range of matrices Δ for which (16) remains stable. In particular, we use the MATLAB command *mussv* in the Robust Control Toolbox, which performs SSV analysis to test if there exists a Δ such that (42) is unstable.

V. EXAMPLE

We discuss numerical results for the *relaxation oscillator* example given by (13) in Section II. We set the parameter $\mu = 0.1$, and first study the two-compartment ODE model (8). When D is small, the techniques of Section III apply, and we can easily check that the conditions of Corollary 3.2 are satisfied for nonnegative $\lambda_i d_1 < \epsilon^*$, where ϵ^* is computed from the proof of Lemma A.1. In Figure 2, we show the oscillations of the solution of x_2 synchronizing spatially under small D , as expected.

We next examine the case of larger D for both (8) and (1). To apply the harmonic balance method, we compute the harmonic components of $x_1(t)$ and find that eight harmonics are sufficient to represent the signal. We then use the harmonic expansion to generate a corresponding finite-dimensional approximation of the matrix \mathcal{A} . Because D is diagonal and

nonnegative, we set $M = \frac{r+\epsilon^*}{2}e_1e_1^T$, $B = [\frac{r-\epsilon^*}{2} \ 0]^T$, $C = [1 \ 0]$, and $\Delta = \delta$, and perform SSV analysis to determine values of d_1 that lead to instabilities. We find that at $\lambda_i d_1 \geq 87.6$, stability is lost.

Indeed, when the product $\lambda_i d_1 \geq 87.6$, the two compartment ODE, with $\lambda_2 = 2$, will exhibit trajectories that diverge, and the reaction-diffusion PDE model, with $\lambda_i = (i - 1)^2$, $i = 2, 3, \dots$, will lose spatial uniformity for initial spatial modes with large enough wavenumber i regardless of d_1 . In Figures 1 and 3, we show that the oscillations of the solution of x_2 do not synchronize spatially for large D , and observe increasing spatial inhomogeneity over time.

VI. CONCLUSION

We have studied diffusively coupled systems that admit stable limit cycles, and shown an example demonstrating destabilization of a spatially homogeneous periodic orbit. This intriguing phenomenon underscores the necessity of establishing analytic and numerical methods that may be used to determine whether limit cycle oscillations synchronize. Furthermore, our tests could also aid in determining coupling strengths in diffusively-coupled multiagent systems.

APPENDIX

We state a lemma that follows from standard results in two-time scale averaging (see, e.g., [17], Thm. 4.4.3). Its proof may be found in [29].

Lemma A.1: Let $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$, and consider the linear time varying system:

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} - \epsilon \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix} \right) \begin{bmatrix} w \\ z \end{bmatrix}, \quad (43)$$

where each $H_{ij}(t)$, $i, j \in \{1, 2\}$ is a bounded piecewise continuous matrix-valued function of time such that $H_{ij}(t + T) = H_{ij}(t)$, $G \in \mathbb{R}^{q \times q}$, and $\epsilon > 0$. Define the associated averaged slow system:

$$\dot{w} = -\epsilon \bar{H}_{11} w, \quad \bar{H}_{11} = \frac{1}{T} \int_{t_0}^{t_0+T} H_{11}(t) dt. \quad (44)$$

If $-\bar{H}_{11}$ and G are Hurwitz, then there exists ϵ^* such that $[w^T \ z^T]^T = 0$ is an exponentially stable equilibrium of (43) for $0 < \epsilon < \epsilon^*$.

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