# Lyapunov-Based Distributed Control of Systems on Lattices

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*Abstract*—**We investigate the properties of systems on lattices with spatially distributed sensors and actuators. These systems arise in a variety of applications such as the control of vehicular platoons, formation of unmanned aerial vehicles, arrays of microcantilevers, and constellations of satellites. We use a Lyapunov-based framework as a tool for stabilization/regulation/asymptotic tracking of both linear and nonlinear systems. We first present results for nominal design and then describe the design of adaptive controllers in the presence of parametric uncertainties. These uncertainties are assumed to be temporally constant, but are allowed to be spatially varying. We show that, in most cases, the distributed design yields controllers with architecture similar to that of the original plant.**

*Index Terms*—**Backstepping, controller architecture, distributed control, systems on lattices.**

## I. Introduction

SYSTEMS on lattices are encountered in a wide range of modern technical applications. Typical examples of such systems include: platoons of vehicles [1]–[4], arrays of microcantilevers [5], unmanned aerial vehicles in formation [6], and satellites in synchronous orbit [7]–[9]. These systems are characterized by the interactions between different subsystems which often results in complex behavior. A distinctive feature of this class of systems is that every unit is equipped with sensors and actuators. The controller design problem is thus dominated by architectural questions such as the choice of localized versus centralized controller architecture. This problem has attracted a lot of attention in the last 25–30 years. A large body of literature in the area that is usually referred to as "decentralized control of large-scale systems" has been created [10]–[17].

A framework for considering spatially distributed systems is that of a spatio-temporal system [18]. In the specific case of systems on lattices, signals of interest are functions of time and a spatial variable $n \in \mathbb{F}$, where $\mathbb{F}$ is a discrete spatial lattice (any countable set, e.g., $\mathbb{Z}$ or $\mathbb{N}$).

A particular class of spatio-temporal systems termed linear spatially invariant systems was considered in [19], where it was shown that optimal controllers (in a variety of norms) for spatially invariant plants inherit this spatially invariant structure. Furthermore, it was shown that optimal controllers with quadratic performance objectives (such as LQR, $\mathcal{H}_2$, and $\mathcal{H}_\infty$) have an inherent degree of spatial localization.

With *a priori* assumptions on the information passing structure in the distributed controller, sufficient conditions for internal stability and strict contractiveness can be expressed using linear matrix inequality (LMI) conditions [20]. Similarly, control design for linear spatially varying distributed systems was considered in [21]. Conditions for synthesis are expressed in terms of convex operator inequalities, which for finite and periodic spatial domains simplify to LMIs.

Adaptive identification and control are extensively studied for both linear (see, for example, [22]–[25]) and nonlinear [26] finite-dimensional systems. Several researchers have also considered these problems in the infinite-dimensional setting [27]–[30] and, for spatially interconnected systems, [9], [11]–[15], [31], [32]. We refer the reader to these references for a fuller discussion.

In this paper, we study distributed control of nonlinear infinite-dimensional systems on lattices. The motivation for studying this class of systems is twofold. First, we want to develop tools for control of systems with large arrays of sensors and actuators. Several examples of systems with this property are given above. In addition to these examples, our results can be used for control of discretized versions of PDEs with distributed controls and measurements. Second, infinite dimensional systems represent an insightful limit of large-but-finite systems: Problems with, for example, stability of an infinite-dimensional system indicate issues with performance of its large-scale equivalent. The latter point was recently illustrated in [4] where the theory for spatially invariant linear systems was utilized to show that care should be exercised when extending standard results from small to large-scale or infinite vehicular platoons.

We extend the well-known finite dimensional *integrator backstepping* design tool to a more general class of systems considered in this paper. Backstepping approach is utilized to provide stability/regulation/asymptotic tracking of nominal systems and systems with parametric uncertainties. In the latter case, we assume that the unknown parameters are temporally constant, but are allowed to be either spatially constant or spatially varying. In both of these situations, we design adaptive Lyapunov-based estimators and controllers. As a result of our design, boundedness of all signals in the closed-loop in the presence of parametric uncertainties is guaranteed. In addition to that, the adaptive controllers provide convergence of the states of the original system to their desired values. We

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also show that, in most cases, the distributed design results in controllers with architecture similar to that of the original plant. This means, for example, that if the plant has only nearest neighbor interactions, then the distributed controller also has only nearest neighbor interactions. The only situation which results in a centralized controller is for plants with constant parametric uncertainties when we start our design with one estimate per unknown parameter. It should be noted however that this problem can be circumvented by the "over-parameterization" of the unknown parameters. As a result, every control unit has its own estimator of unknown parameters and avoids the centralized architecture.

Backstepping is a well-studied design tool [26], [33] for finite-dimensional systems. In the infinite dimensional setting, a backstepping controller was designed to suppress compression system instabilities for the nonlinear PDE Moore–Greitzer model [34]. Furthermore, a backstepping-like approach can be used to obtain stabilizing boundary feedback control laws for a class of parabolic systems (see [35] and [36] for details). Backstepping boundary control can also be used as a tool for vibration suppression in flexible-link gantry robots [37]. However, backstepping has not been applied to distributed control of infinite-dimensional systems on lattices to the best of our knowledge. We note that backstepping represents a recursive design scheme that can be used for systems in strict-feedback form with nonlinearities not constrained by linear bounds [26], [33]. At every step of backstepping, a new control Lyapunov function (CLF) is constructed by augmentation of the CLF from the previous step by a term which penalizes the error between "virtual control" and its desired value (so-called "*stabilizing function*"). A major advantage of backstepping is the construction of a Lyapunov function whose derivative can be made negative definite by a variety of control laws rather than by a specific control law [26]. Furthermore, backstepping can be used for adaptive control of "parametric pure-feedback systems" in which unknown parameters enter into equations in an affine manner [26].

Our presentation is organized as follows. In Section II, we introduce the notation used throughout this paper, give an example of systems on lattices, describe the classes of systems for which we design feedback controllers in Section III (nominal state-feedback design), Section IV (adaptive state-feedback design), and Section V (output-feedback design), discuss well-posedness of the open-loop systems, and describe different strategies that can be used for control of systems on lattices. In Section III-D, we design fully decentralized nominal controllers for both linear and nonlinear mass-spring systems. In Section VI, we analyze architecture of Lyapunov-based distributed controllers. We conclude by summarizing major contributions and future research directions in Section VII.

## II. Systems on Lattices

In this section, an example of systems on lattices is given: We consider a mass-spring system on a line. This system is chosen because it represents a simple nontrivial example of an unstable system where the interactions between different plant units are caused by the physical connections between them. Another example of systems with this property is given by an array of microcantilevers. The interactions between different plant units may also arise because of a specific control objective that the designer wants to meet. Examples of systems on lattices with this property include: A system of cars in an infinite string, aerial vehicles and spacecrafts in formation flights. We also introduce the notation that we use, discuss well-posedness of the open-loop systems, describe the classes of systems for which we design state and output-feedback controllers in Sections III–V, and briefly outline different approaches to control of systems on lattices.

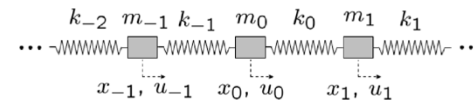


Fig. 1. Mass-spring system.

### A. Notation

The sets of integers and natural numbers are denoted by $\mathbb{Z}$ and $\mathbb{N}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, and $\mathbb{Z}_N := \{-N, \ldots, N\}$, $N \in \mathbb{N}$. Discrete spatial lattice is denoted by $\mathbb{F}$ (e.g., $\mathbb{Z}$ or $\mathbb{N}$). The space of square summable sequences is denoted by $l_2$, and the space of bounded sequences is denoted by $l_\infty$. Similarly, the spaces of square integrable and bounded functions are, respectively, denoted by $L_2$ and $L_\infty$. The $k$th unit vector in $\mathbb{R}^m$ is denoted by $e_k$. Symbol "$*$" is used to denote transpose of a vector (matrix), and adjoint of an operator. The state and control of the $n$th subsystem (cell, unit) are, respectively, represented by $[\psi_{1n} \; \ldots \; \psi_{mn}]^*$ and $u_n$, $m \in \mathbb{N}$, $n \in \mathbb{F}$. The capital letters denote infinite vectors defined, for example, as $\Psi_k := [\cdots \; \psi_{k,n-1} \; \psi_{k,n} \; \psi_{k,n+1} \; \cdots]^* =: \{\psi_{kn}\}_{n \in \mathbb{F}}$, $k \in \{1, \ldots, m\}$. Operators are represented by $\mathcal{A}$, $\mathcal{B}$, etc. The $n$th plant cell is denoted by $G_n$, and the $n$th controller cell is denoted by $K_n$.

### B. Example: Mass-Spring System

A system consisting of an infinite number of masses and springs on a line is shown in Fig. 1. The dynamics of the $n$th mass are given by

$$m_n \ddot{x}_n = F_{n-1} + F_n + u_n, \qquad n \in \mathbb{Z} \tag{1}$$

where $x_n$ represents the displacement from a reference position of the $n$th mass, $F_n$ represent the restoring force of the $n$th spring, and $u_n$ is the control applied on the $n$th mass. For relatively small displacements, restoring forces can be considered as linear functions of displacements $F_n = k_n(x_{n+1} - x_n)$, $F_{n-1} = k_{n-1}(x_{n-1} - x_n)$, where $k_n$ is the $n$th spring constant. We also consider a situation in which the spring restoring forces depend nonlinearly on displacement. One such model is given by the so-called *hardening spring* (see, for example, [33]) where, beyond a certain displacement, large force increments are obtained for small displacement increments

$$\begin{aligned} F_n &= k_n((x_{n+1} - x_n) + c_n^2(x_{n+1} - x_n)^3) \\ &=: k_n(x_{n+1} - x_n) + q_n(x_{n+1} - x_n)^3 \\ F_{n-1} &= k_{n-1}((x_{n-1} - x_n) + c_{n-1}^2(x_{n-1} - x_n)^3) \\ &=: k_{n-1}(x_{n-1} - x_n) + q_{n-1}(x_{n-1} - x_n)^3. \end{aligned}$$

For both cases (1) can be rewritten in terms of its state–space representation for every $n \in \mathbb{Z}$ as

$$\begin{aligned}\dot{\psi}_{1n} &= \psi_{2n}\\ \dot{\psi}_{2n} &= f_n(\psi_{1,n-1}, \psi_{1n}, \psi_{1,n+1}) + b_n u_n\end{aligned} \quad (2)$$

where $\psi_{1n} := x_n$ and $\psi_{2n} := \dot{x}_n$.

If the restoring forces are linear functions of displacements and all masses and springs are homogeneous, that is, $m_n = m = \text{const.}$, $k_n = k = \text{const.}$, $\forall n \in \mathbb{Z}$, (2) represents a linear *spatially invariant infinite dimensional* system. This implies that it can be analyzed using the tools of [19], [20]. The other mathematical representations of a mass-spring system are either nonlinear or spatially varying. One of the main purposes of the present study is to design feedback controllers and analyze their architecture for this broader class of systems.

### C. Classes of Systems

In this section, we summarize the classes of systems for which we design feedback controllers in Sections III–V. We consider continuous time $m$th-order subsystems over discrete spatial lattice $\mathbb{F}$ with *at most* $2N$ interactions per plant's cell (see Assumption 1). We assume that all subsystems satisfy the *matching condition* [26]. This condition is satisfied for most mechanical systems: for example, the models presented in Section II-B, the models obtained by discretization of heat or wave equations with distributed controls and measurements, and the model of an array of microcantilevers [5] belong to this class of systems. In Section III-C, we show how the matching condition assumption can be removed. Furthermore, for systems with parametric uncertainties we assume that unknown parameters enter into equations in an affine manner.

We consider systems without parametric uncertainties of the form

$$\dot{\psi}_{1n} = \psi_{2n}, \qquad n \in \mathbb{F} \quad (3a)$$
$$\dot{\psi}_{2n} = \psi_{3n}, \qquad n \in \mathbb{F} \quad (3b)$$
$$\vdots$$
$$\dot{\psi}_{mn} = f_n(\Psi_1, \dots, \Psi_m) + b_n u_n, \quad n \in \mathbb{F} \quad (3c)$$

where nonzero numbers $b_n$ denote the so-called *control coefficients* [26].

If, on the other hand, all parameters have constant but unknown values $\forall t \in \mathbb{R}_+$ and $\forall n \in \mathbb{F}$, it is convenient to rewrite $f_n$ and $b_n$ in (3) as

$$\begin{aligned}f_n &= \nu_n(\Psi_1, \dots, \Psi_m) + g_n^*(\Psi_1, \dots, \Psi_m)\theta\\ b_n &= b\end{aligned} \quad (4)$$

where $\theta$ represents a vectors of unknown parameters.

The most general case that we discuss is the one in which all parameters have unknown spatially varying values that do not depend on time. In other words, the parameters are allowed to be a function of $n \in \mathbb{F}$ but not of $t \in \mathbb{R}_+$. In this case, it is convenient to rewrite $f_n$ in (3) as

$$f_n = \tau_n(\Psi_1, \dots, \Psi_m) + h_n^*(\Psi_1, \dots, \Psi_m)\theta_n. \quad (5)$$

A major difference between models [(3), (4)] and [(3), (5)] is in the number of unknown parameters. In the former, the number of unknown parameters is finite, and in the latter the number of unknown parameters can be infinite.

We also consider output-feedback design for nominal systems

$$\dot{\psi}_{1n} = \psi_{2n}, \qquad n \in \mathbb{F} \quad (6a)$$
$$\dot{\psi}_{2n} = \psi_{3n}, \qquad n \in \mathbb{F} \quad (6b)$$
$$\vdots$$
$$\dot{\psi}_{mn} = f_n(\mathrm{Y}) + b_n u_n, \qquad n \in \mathbb{F} \quad (6c)$$
$$y_n = \psi_{1n}, \qquad n \in \mathbb{F} \quad (6d)$$

and systems with parametric uncertainties, for which we represent $f_n$ in (6) as

$$f_n(\mathrm{Y}) = \tau_n(\mathrm{Y}) + h_n^*(\mathrm{Y})\theta_n, \qquad n \in \mathbb{F} \quad (7)$$

where $\mathrm{Y} := \{y_n\}_{n\in\mathbb{F}} = \{\psi_{1n}\}_{n\in\mathbb{F}}$ denotes the distributed output of (6) and [(6), (7)]. Clearly, for these two systems nonlinearities are allowed to depend only on the measured output.

We introduce the following assumptions.

*Assumption 1:* There are at most $2N$ interactions per plant cell: $n$th plant cell $G_n$ interacts only with $\{G_{n-N}, \dots, G_{n+N}\}$. In other words, functions $f_n$, $g_n$, $h_n$, $\nu_n$, and $\tau_n$ depend on at most $2N + 1$ elements of $\Psi_1, \dots, \Psi_m$, for every $n \in \mathbb{F}$. For example, for (3) with $m = 2$, $f_n(\Psi_1, \Psi_2) = f_n(\{\psi_{1,n+j}\}_{j\in\mathbb{Z}_N}, \{\psi_{2,n+j}\}_{j\in\mathbb{Z}_N})$.

*Assumption 2:* Functions $f_n$, $g_n$, $h_n$, $\nu_n$, and $\tau_n$ are known, continuously differentiable functions of their arguments, equal to zero at the origins of their respective systems. In addition to that, for each of these functions, infinite vectors defined as $\mathrm{F} := \{f_n\}_{n\in\mathbb{F}}$ satisfy:$\{\Psi_1 \in l_\infty, \dots, \Psi_m \in l_\infty\} \Rightarrow \mathrm{F}(\Psi_1, \dots, \Psi_m) \in l_\infty$.

*Assumption 3:* Functions $g_n$ and $\nu_n$ are bounded by polynomial functions of their arguments. Furthermore, these polynomials are equal to zero at the origin of [(3), (4)].

*Assumption 4:* The sign of $b$ in [(3), (4)] is known.

*Assumption 5:* The signs of $b_n$, $\forall n \in \mathbb{F}$, in [(3), (5)] and [(6), (7)] are known.

These assumptions are used in the sections devoted to the distributed control design and the well-posedness of both open and closed-loop systems.

*Remark 1:* For notational convenience, both the well-posedness and the control design problems are solved for second-order subsystems over discrete spatial lattice $\mathbb{F}$, that is for $m = 2$.

### D. Well-Posedness of Open-Loop Systems

We next briefly analyze the well-posedness of the open-loop systems of Section II-C for $m = 2$ by considering them as the abstract evolution equations either on a Hilbert space $\mathbb{H} := l_2 \times l_2$ or on a Banach space $\mathbb{B} := l_\infty \times l_\infty$. Either representation is convenient for addressing the questions of existence and uniqueness of solutions. With this in mind, we prove the well-posedness of the open-loop systems on $\mathbb{H}$ and remark that similar argument can be used if the underlying state–space is $\mathbb{B}$ rather than $\mathbb{H}$.

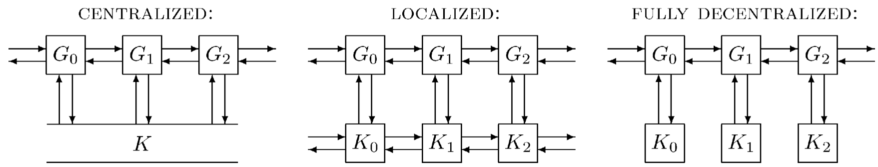


Fig. 2. Distributed controller architectures for centralized, localized (with nearest neighbor interactions), and fully decentralized strategies.

The unforced systems of Section II-C can be rewritten as the abstract evolution equations of the form

$$\dot{\Psi} = \mathcal{A}\Psi + \mathrm{P}(\Psi) \quad \Psi := \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{8}$$

A linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$, and a (possibly) nonlinear mapping $\mathrm{P} : \mathbb{H} \to \mathbb{H}$ are defined as

$$\mathcal{A} := \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad \mathrm{P}(\Psi) := \begin{bmatrix} 0 \\ \mathrm{P}_2(\Psi_1, \Psi_2) \end{bmatrix}$$

where $D(\mathcal{A})$ represents the *domain* of $\mathcal{A}$, and $\mathrm{P}_2$ denotes $\{f_n\}_{n\in\mathbb{F}}$, $\{\nu_n + g_n^*\theta\}_{n\in\mathbb{F}}$, $\{\tau_n + h_n^*\theta_n\}_{n\in\mathbb{F}}$, $\{f_n\}_{n\in\mathbb{F}}$, or $\{\tau_n + h_n^*\theta_n\}_{n\in\mathbb{F}}$ depending on whether systems (3), [(3), (4)], [(3), (5)], (6), or [(6), (7)] are considered.

One can show that $\mathcal{A}$ is a *bounded operator* and, therefore, it generates a *uniformly continuous semigroup*. This automatically implies that the semigroup generated by $\mathcal{A}$ is *strongly continuous*. If, in addition, P is locally Lipschitz then system (8) has a unique *mild solution* on $[0, t_{\max})$ (see [38, Th. 2.73]). Moreover, if P is continuously (Fréchet) differentiable then (8) has a unique *classical solution* on $[0, t_{\max})$ (see [38, Th. 2.74]). If $t_{\max} < \infty$ then $\lim_{t\to t_{\max}} \|\Psi(t)\| = \infty$.

It can be readily shown that for the systems described in Section II-C that satisfy Assumptions 1 and 2, P is a *continuously Fréchet differentiable* function of its arguments which guarantees existence and uniqueness of classical solutions for these systems on $[0, t_{\max})$.

### E. Distributed Controller Architectures

Fig. 2 illustrates control architectures that can be used for distributed control of systems on lattices: centralized, localized, and fully decentralized. Centralized controllers require information from all plant cells for achieving the desired control objective. This approach usually results in best performance, but it requires excessive communication. On the other hand, in fully decentralized strategies control cell $K_n$ uses only information from the $n$th plant cell $G_n$ on which it acts. This approach does not require any communication which often limits performance or even leads to instability. An example of a localized control architecture with nearest neighbor interactions is shown in Fig. 2. The hope is that this approach can provide a good performance with moderate communication. For systems described in Section II-C, we show that backstepping design yields distributed controllers that are intrinsically decentralized with a strong similarity between plant and controller architectures.

## III. Nominal State-Feedback Design

In this section, we extend the finite dimensional *integrator backstepping* design tool to a class of nonlinear infinite dimensional systems on lattices. We first design state-feedback distributed backstepping controllers for nominal systems (3). For notational convenience, this problem is solved for second-order subsystems over discrete spatial lattice $\mathbb{F}$, that is for $m = 2$. In this case, the dynamics of the $n$th cell (3) simplifies to

$$\dot{\psi}_{1n} = \psi_{2n}, \qquad n \in \mathbb{F} \tag{9a}$$
$$\dot{\psi}_{2n} = f_n(\Psi_1, \Psi_2) + b_n u_n, \qquad n \in \mathbb{F}. \tag{9b}$$

We rewrite the dynamics of the entire system as

$$\dot{\Psi}_1 = \Psi_2 \tag{10a}$$
$$\dot{\Psi}_2 = \mathrm{F}(\Psi_1, \Psi_2) + \mathcal{B}\mathrm{U} \tag{10b}$$

where capital letters denote infinite vectors defined in Section II-A, and $\mathcal{B} := \mathrm{diag}\{b_n\}_{n\in\mathbb{F}}$. System (10) represents an abstract evolution equation in the *strict-feedback form* [26] defined on either a Hilbert space $\mathbb{H} := l_2 \times l_2$ or a Banach space $\mathbb{B} := l_\infty \times l_\infty$. Because of that, it is amenable to be analyzed by the backstepping. Even though a stabilizing controller can be designed using various tools (because of the matching condition), we choose backstepping because it gives both a stabilizing feedback law and a CLF for a system under study. Once CLF is constructed its derivative can be made negative definite using a variety of control laws.

In Section III-A, we study a situation in which the desired properties of system (10) are accomplished by performing a global design. Unfortunately, this is not always possible. Because of this, in Section III-B, we also perform design on individual cells (9) to guarantee the desired behavior of system (10). In Section III-C, we show how integrator backstepping can be employed as a constructive design tool for stabilization of nominal nonlinear systems on lattices.

### A. Global Backstepping Design

Before we illustrate the global distributed backstepping design, we introduce the following assumption.

*Assumption 6:* The initial distributed state is such that both $\Psi_1(0) \in l_2$ and $\Psi_2(0) \in l_2$.

The objective is to provide global asymptotic stability of the origin of (10). This is accomplished using the distributed backstepping design. In the first step of backstepping, (10a) is stabilized by considering $\Psi_2$ as its control. Since $\Psi_2$ is not actually a control, but rather, a state variable, the error between $\Psi_2$ and the value which stabilizes (10a) must be penalized in the augmented Lyapunov function at the next step. In this way, a stabilizing control law is designed for the overall infinite-dimensional system.

*Step 1:* The global recursive design starts with (10a) by considering $\Psi_2$ as control and proposing a radially unbounded CLF $V_1 : l_2 \to \mathbb{R}$, $V_1(\Psi_1) = (1/2)\langle \Psi_1, \Psi_1 \rangle := (1/2)\sum_{n\in\mathbb{F}} \psi_{1n}^2$.

The derivative of $V_1(\Psi_1)$ along the solutions of (10a) is given by $\dot{V}_1 = \langle \Psi_1, \dot{\Psi}_1 \rangle = \langle \Psi_1, \Psi_2 \rangle$.

*Assumption 7:* There exists a continuously differentiable "stabilizing function" $\Psi_{2d} := \Lambda_1(\Psi_1)$, $\Lambda_1(0) = 0$, such that $\Psi_1 \in l_2 \Rightarrow \Lambda_1(\Psi_1) \in l_2$, and $W_1(\Psi_1) := -\langle \Psi_1, \Lambda_1(\Psi_1) \rangle > 0$, for every $\Psi_1 \in l_2 \setminus \{0\}$.

Assumption 7 is always satisfied: for example, $\Lambda_1(\Psi_1) = -k_1 \Psi_1$, $k_1 > 0$, provides negative definiteness of $\dot{V}_1(\Psi_1)$. Since $\Psi_2$ is not actually a control, but rather, a state variable, we introduce the change of variables $Z_2 := \Psi_2 - \Psi_{2d} = \Psi_2 - \Lambda_1(\Psi_1)$, which yields $\dot{V}_1 = -W_1(\Psi_1) + \langle Z_2, \Psi_1 \rangle$. The sign indefinite term in $\dot{V}_1$ will be taken care of at the second step of backstepping.

*Step 2:* Augmentation of the CLF from Step 1 by a term which penalizes the error between $\Psi_2$ and $\Psi_{2d}$ yields a function

$$V_2(\Psi_1, Z_2) := V_1(\Psi_1) + \frac{1}{2}\langle Z_2, Z_2 \rangle \tag{11}$$

whose derivative along the solutions of

$$\begin{aligned}\dot{\Psi}_1 &= \Lambda_1(\Psi_1) + Z_2 \\ \dot{Z}_2 &= F(\Psi_1, \Psi_2) - \frac{\partial \Lambda_1(\Psi_1)}{\partial \Psi_1}\Psi_2 + \mathcal{B}U\end{aligned}$$

is determined by

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + \langle Z_2, \dot{Z}_2 \rangle \\ &= -W_1(\Psi_1) + \left\langle Z_2, \Psi_1 + F - \frac{\partial \Lambda_1(\Psi_1)}{\partial \Psi_1}\Psi_2 + \mathcal{B}U \right\rangle. \end{aligned} \tag{12}$$

In particular, the following choice of control law $U = -\mathcal{B}^{-1}(\Psi_1 + F - (\partial \Lambda_1(\Psi_1)/\partial \Psi_1)\Psi_2 - \Lambda_2(Z_2))$, with $\Lambda_2(0) = 0$, $W_2(Z_2) := -\langle Z_2, \Lambda_2(Z_2) \rangle > 0$, yields $\dot{V}_2(\Psi_1, Z_2) = -W_1(\Psi_1) - W_2(Z_2) < 0$, for every $\Psi_1, Z_2 \in l_2 \backslash \{0\}$. Thus, U guarantees global asymptotic stability of the origin of (10).

Results of this section are summarized in the following theorem.

*Theorem 1:* Suppose that system (10) satisfies Assumptions 1, 2, and 6. Then, there exists a state-feedback control law $U = \Upsilon(\Psi_1, \Psi_2)$ which guarantees global asymptotic stability of the origin of system (10). One such control law is given by: $U = -\mathcal{B}^{-1}\big((1 + k_1 k_2)\Psi_1 + (k_1 + k_2)\Psi_2 + F(\Psi_1, \Psi_2)\big)$, where $k_1$ and $k_2$ are positive design parameters. These properties can be established with the Lyapunov function: $V(\Psi_1, \Psi_2) = (1/2)\langle \Psi_1, \Psi_1 \rangle + (1/2)\langle \Psi_2 + k_1 \Psi_1, \Psi_2 + k_1 \Psi_1 \rangle$.

### B. Individual Cell Backstepping Design

The distributed backstepping design on the space of square summable sequences cannot always be performed. For example, if Assumption 6 is not satisfied the construction of a CLF for (10) is not possible. Some additional conditions need to be met to be able to construct a CLF for systems that do not satisfy matching condition (see Section III-C). In this section, we show that global asymptotic stability of the origin of (10) can be achieved by performing design on each individual cell (9) rather than on the entire system (10).

*Step 1:* The individual cell backstepping design starts with subsystem (9a) by considering $\psi_{2n}$ as control and proposing a quadratic radially unbounded CLF $V_{1n} : \mathbb{R} \rightarrow \mathbb{R}$, $V_{1n}(\psi_{1n}) = \psi_{1n}^2/2$. The derivative of $V_{1n}(\psi_{1n})$ along the solutions of (9a) is determined by $\dot{V}_{1n} = \psi_{1n}\dot{\psi}_{1n} = \psi_{1n}\psi_{2n}$. If $\psi_{2n}$ were a control, subsystem (9a) could be stabilized by $\psi_{2nd} = -k_{1n}\psi_{1n}$, $k_{1n} > 0$. A change of coordinates $\zeta_{2n} := \psi_{2n} - \psi_{2nd} = \psi_{2n} + k_{1n}\psi_{1n}$, yields $\dot{V}_{1n} = -k_{1n}\psi_{1n}^2 + \psi_{1n}\zeta_{2n}$. The sign indefinite term in the last equation will be accounted for at the second step of backstepping.

*Step 2:* Augmentation of $V_{1n}(\psi_{1n})$ by a term which penalizes $\zeta_{2n}$ yields a quadratic CLF: $V_{2n}(\psi_{1n}, \zeta_{2n}) := V_{1n}(\psi_{1n}) + (1/2)\zeta_{2n}^2$. The simplest choice of controller that provides negative definiteness of $\dot{V}_{2n}$ is given by $u_n = -(1/b_n)(\psi_{1n} + f_n + k_{1n}\psi_{2n} + k_{2n}\zeta_{2n})$, $k_{2n} > 0$. This choice of control gives $\dot{V}_{2n}(\psi_{1n}, \zeta_{2n}) = -k_{1n}\psi_{1n}^2 - k_{2n}\zeta_{2n}^2 < 0$, for every $(\psi_{1n}, \zeta_{2n}) \in \mathbb{R}^2 \backslash \{0\}$, and every $n \in \mathbb{F}$. Thus, $u_n$ warrants global asymptotic stability of the origin of (9) for every $n \in \mathbb{F}$, which implies global asymptotic stability of the origin of system (10).

Results of this section are summarized in the following theorem.

*Theorem 2:* Suppose that system (10) satisfies Assumptions 1 and 2. Then, for every $n \in \mathbb{F}$, there exists a state-feedback control law $u_n = \gamma_n(\Psi_1, \Psi_2)$ which guarantees global asymptotic stability of the origin of (10). One such control law is given by: $u_n = -(1/b_n)\big((1 + k_1 k_2)\psi_{1n} + (k_1 + k_2)\psi_{2n} + f_n(\Psi_1, \Psi_2)\big)$, where $k_1$ and $k_2$ are positive design parameters.

*Remark 2:* If a control objective is to asymptotically track a reference output $r_n(t)$, with output of system (10) being defined as $y_n := \psi_{1n}$, $\forall n \in \mathbb{F}$, then the following control law:

$$\begin{aligned}u_n = -\frac{1}{b_n}&((1 + k_1 k_2)(\psi_{1n} - r_n(t)) \\ &+ (k_1 + k_2)(\psi_{2n} - \dot{r}_n(t)) + f_n(\psi_1, \psi_2) - \ddot{r}_n(t))\end{aligned}$$

fulfills this objective. We assume that, for every $n \in \mathbb{F}$, $r_n$, $\dot{r}_n$, and $\ddot{r}_n$ are known and uniformly bounded, and that $\ddot{r}_n$ is piecewise continuous.

### C. Distributed Integrator Backstepping

We next present results that allow for constructive design of stabilizing controllers for systems on lattices without parametric uncertainties. We show how to extend previously described nominal state-feedback controllers to a more general situation in which a system

$$\dot{\psi}_n = q_n(\Psi) + r_n(\Psi)u_n, \qquad q_n(0) = 0, \quad n \in \mathbb{F} \tag{13}$$

is augmented by an infinite number of integrators. We assume that $q_n$ and $r_n$ satisfy Assumptions 1 and 2 [with the exception that $r_n(\Psi)$ does not have to be equal to zero at the origin of (13)], and denote the system's state and control vectors by $\Psi := [\ldots \ \psi_{n-1}^* \ \ \psi_n^* \ \ \psi_{n+1}^* \ \ldots]^*$, $U := [\ldots \ u_{n-1} \ \ u_n \ \ u_{n+1} \ \ldots]^*$, with $\psi_n \in \mathbb{R}^m$ and $u_n \in \mathbb{R}$ for all $n \in \mathbb{F}$. For the purpose of global design, it is convenient to rewrite (13) as

$$\dot{\Psi} = Q(\Psi) + \mathcal{R}(\Psi)U \tag{14}$$

where $\mathcal{R}(\Psi) := \operatorname{diag}\{r_n(\Psi)\}_{n\in\mathbb{F}}$ and $\mathrm{Q}(\Psi) := [\ldots\ q_{n-1}^*(\Psi)\ \ q_n^*(\Psi)\ \ q_{n+1}^*(\Psi)\ \ \ldots]^*$. Note that $\mathcal{R}(\Psi)$ represents an operator from $l_\infty$ ($l_2$) to $l_\infty^m$ ($l_2^m$).

The results summarized in Lemmas 3 and 4 represent extensions of the well-known finite dimensional *integrator backstepping* design tool [26], [33], and they are, respectively, based on the following assumptions.

*Assumption 8:* There exists a continuously differentiable state-feedback control law $u_n = \lambda_n(\Psi)$, $\lambda_n(0) = 0$, and a smooth positive–definite radially unbounded function $V_n(\psi_n)$, $V_n : \mathbb{R}^m \to \mathbb{R}$, for the $n$th subsystem of (13) such that $\partial V_n(\psi_n)/\partial\psi_n(q_n(\Psi) + r_n(\Psi)\lambda_n(\Psi)) \le -W_n(\psi_n) < 0$, for every $\psi_n \in \mathbb{R}^m\backslash\{0\}$, where $W_n : \mathbb{R}^m \to \mathbb{R}$ is a positive–definite function for every $n \in \mathbb{F}$.

*Assumption 9:* There exists a continuously differentiable state-feedback control law $\mathrm{U} = \Lambda(\Psi)$, $\Lambda(0) = 0$, such that $\{\Psi \in l_2^m \Rightarrow \Lambda(\Psi) \in l_2\}$, and a smooth positive–definite radially unbounded functional $V(\Psi)$, $V : l_2^m \to \mathbb{R}$, for (14) such that $\partial V(\Psi)/\partial\Psi(\mathrm{Q}(\Psi) + \mathcal{R}(\Psi)\Lambda(\Psi)) \le -W(\Psi) < 0$, for every $\Psi \in l_2^m\backslash\{0\}$, where $W : l_2^m \to \mathbb{R}$ is a positive–definite functional.

Assumption 8 guarantees global asymptotic stability of the origin of the $n$th subsystem of (13) for every $n \in \mathbb{F}$, which implies global asymptotic stability of the origin of (14). Similarly, if the conditions of Assumption 9 are satisfied, then global asymptotic stability of the origin of (14) can be concluded as well. Moreover, Assumption 9 guarantees existence of a CLF for infinite-dimensional system (14). This can be used to obtain controllers with less interactions, as illustrated in Section III-D.

In the remainder of this section, we consider (13) augmented by an infinite number of integrators

$$\dot{\psi}_n = q_n(\Psi) + r_n(\Psi)\phi_n, \qquad n \in \mathbb{F} \tag{15a}$$
$$\dot{\phi}_n = u_n, \qquad n \in \mathbb{F} \tag{15b}$$

or, equivalently

$$\dot{\Psi} = \mathrm{Q}(\Psi) + \mathcal{R}(\Psi)\Phi \tag{16a}$$
$$\dot{\Phi} = \mathrm{U}. \tag{16b}$$

*Individual Cell Distributed Integrator Backstepping:*

*Lemma 3:* Suppose that, for every $n \in \mathbb{F}$, the $n$th subsystem of (15a) satisfies Assumption 8 with $\phi_n \in \mathbb{R}$ as its control. Then, the augmented function $V_{an}(\Psi, \phi_n) := V_n(\psi_n) + (1/2)(\phi_n - \lambda_n(\Psi))^2$, represents a CLF for the $n$th subsystem of (15). Thus, there exists a state-feedback control law $\mathrm{U} := \{u_n(\Psi, \Phi)\}_{n\in\mathbb{F}} = \{\lambda_{an}(\Psi, \Phi)\}_{n\in\mathbb{F}}$ which guarantees global asymptotic stability of the origin of (16). One such control law is given by

$$\lambda_{an}(\Psi, \Phi) = -k_n(\phi_n - \lambda_n(\Psi)) - \frac{\partial V_n(\psi_n)}{\partial \psi_n} r_n(\Psi) + \sum_{j\in\mathbb{F}} \frac{\partial \lambda_n}{\partial \psi_j}\left(q_j(\Psi) + r_j(\Psi)\phi_j\right), \qquad k_n > 0.$$

*Global Distributed Integrator Backstepping:*

*Lemma 4:* Suppose that (16a) satisfies Assumption 9 with $\Phi \in l_2$ as its control. Then, the augmented function $V_a(\Psi, \Phi) := V(\Psi) + (1/2)\langle \Phi - \Lambda(\Psi), \Phi - \Lambda(\Psi)\rangle$, represents a CLF for system (16). Thus, there exists a state-feedback control law $\mathrm{U} := \Lambda_a(\Psi, \Phi)$ which guarantees global asymptotic stability of the origin of system (16). One such control law is given by

$$\Lambda_a(\Psi, \Phi) = -k(\Phi - \Lambda(\Psi)) - \mathcal{R}^*(\Psi)\left(\frac{\partial V(\Psi)}{\partial \Psi}\right)^* + \frac{\partial \Lambda(\Psi)}{\partial \Psi}\left(\mathrm{Q}(\Psi) + \mathcal{R}(\Psi)\Phi\right), \qquad k > 0.$$

*Remark 3:* Results of Lemma 3 (Lemma 4) can be also applied to a more general class of systems

$$\dot{\psi}_n = q_n(\Psi) + r_n(\Psi)\phi_n, \qquad n \in \mathbb{F} \tag{17a}$$
$$\dot{\phi}_n = q_{an}(\Psi, \Phi) + r_{an}(\Psi, \Phi)u_{an}, \qquad n \in \mathbb{F} \tag{17b}$$

where we assume that $r_{an}(\Psi, \Phi) \neq 0$, for all $n \in \mathbb{F}$. In this case, the input transformation: $u_{an} := (u_n - q_{an}(\Psi, \Phi))/r_{an}(\Psi, \Phi)$, renders (17) into (15), which allows for the application of Lemma 3 (Lemma 4).

### *D. Fully Decentralized Controllers for Mass-Spring System*

We next demonstrate how global backstepping design can be utilized to obtain fully decentralized controllers for mass-spring system. This is accomplished by a careful analysis of the interactions in the underlying system, and feedback domination rather than cancellation of harmful interactions. For notational convenience we consider a situation in which all masses and springs are homogeneous. We note that similar procedure can employed in the nonhomogeneous case, as long as all parameters have known values.

*1) Linear Mass-Spring System:* We first consider a linear spatially invariant mass-spring system described by (2) with $n \in \mathbb{Z}$ and

$$f_n = \frac{k}{m}(\psi_{1,n-1} - 2\psi_{1n} + \psi_{1,n+1}) \quad b_n = \frac{1}{m}. \tag{18}$$

The backstepping design closely follows the procedure described in Section III-A: by choosing $\Lambda_1(\Psi_1) := \{-\kappa_{1n}\psi_{1n}\}_{n\in\mathbb{Z}}$, $\kappa_{1n} > 0$, $\forall n \in \mathbb{Z}$, expression (12) simplifies to the equation shown at the bottom of the page, where $s_n := (1 - 2k/m)\psi_{1n} + \kappa_{1n}\psi_{2n}$. We now invoke Young's Inequality (see [26, (2.254)]): $\zeta_{2n}\psi_{1i} \le p\zeta_{2n}^2 + (1/4p)\psi_{1i}^2$, $p > 0$, $n \in \mathbb{Z}$, $i := \{n-1, n+1\}$, and choose

$$\begin{aligned} u_n &= -\, m(s_n + \kappa_{2n}\zeta_{2n}) \\ &= -\, m\left(\left(1 - \frac{2k}{m} + \kappa_{1n}\kappa_{2n}\right)\psi_{1n} + (\kappa_{1n} + \kappa_{2n})\psi_{2n}\right) \end{aligned}$$

$$\dot{V}_2 = -\sum_{n\in\mathbb{Z}}\left(\kappa_{1n}\psi_{1n}^2 - \zeta_{2n}\left(s_n + \frac{1}{m}u_n\right) - \frac{k}{m}(\zeta_{2n}\psi_{1,n-1} + \zeta_{2n}\psi_{1,n+1})\right)$$

to obtain: $\dot{V}_2 \leq -\sum_{n\in\mathbb{Z}}((\kappa_{1n} - k/2mp)\psi_{1n}^2 + (\kappa_{2n} - 2kp/m)\zeta_{2n}^2)$. Thus, the aforementioned fully decentralized controller guarantees global asymptotic stability of the origin of the closed-loop system if $\kappa_{1n}$ and $\kappa_{2n}$ satisfy $\{\kappa_{1n} > k/2mp,\ \kappa_{2n} > 2kp/m,\ \forall n \in \mathbb{Z}\}$. Since [(2), (18)] is a spatially invariant system it may be of interest to preserve this property under feedback. This can be achieved by assigning constant values $\kappa_1$ and $\kappa_2$ to all design parameters $\kappa_{1n}$ and $\kappa_{2n}$, that is $\{\kappa_{1n} := \kappa_1 > k/2mp,\ \kappa_{2n} := \kappa_2 > 2kp/m,\ \forall n \in \mathbb{Z}\}$.

We can also analyze properties of system [(2), (18)] using the tools of [19]. Application of the "bilateral $\mathcal{Z}$–transform" evaluated on the unit circle $z = \mathrm{e}^{j\theta}$ transforms [(2), (18)] into the $\theta$–parameterized family of second-order systems

$$\begin{aligned}\dot{\hat{\psi}}_\theta &= \hat{\mathcal{A}}_\theta \hat{\psi}_\theta + \hat{\mathcal{B}}_\theta \hat{u}_\theta,\ \ \theta \in [0,\, 2\pi)\\ \hat{\mathcal{A}}_\theta &:= \begin{bmatrix} 0 & 1 \\ \frac{2k}{m}(\cos\theta - 1) & 0 \end{bmatrix} \quad \hat{\mathcal{B}}_\theta := \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}\end{aligned} \tag{19}$$

where, for example, $\hat{u}_\theta := \sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{-j\theta n}$. Properties of system [(2), (18)] are completely determined by properties of (19). Namely, [(2), (18)] is exponentially stable (stabilizable) if and only if (19) is exponentially stable (stabilizable) for every $\theta \in [0,\, 2\pi)$ (see [19, Cor. 3]). Since the transformed system is exponentially stabilizable, the original system can be stabilized by a fully decentralized state-feedback if and only if there exist a constant matrix $\mathcal{K} := [\,k_1 \quad k_2\,]$ such that $\hat{\mathcal{A}}_\theta - \hat{\mathcal{B}}_\theta \mathcal{K}$ is Hurwitz for every $\theta \in [0,\, 2\pi)$. Clearly, this is the case when both $k_1 > 0$ and $k_2 > 0$. Thus, the Lyapunov-based design yields somewhat conservative estimates on feedback gains to guarantee stability. This is not surprising and we certainly do not recommend the use of backstepping for control of linear spatially invariant systems since, for example, quadratically optimal controllers can be easily obtained for these systems [19]. However, we remark that backstepping does not give one fixed controller, but rather a CLF that can be used to generate a variety of different control laws. Furthermore, backstepping can be applied as a constructive design tool for a broad class of systems, including spatially varying and nonlinear systems. We illustrate in Section III-D.2 that it can be used to obtain a fully decentralized stabilizing controller for a nonlinear mass-spring system. This further demonstrates the power of backstepping and its flexibility as a design tool.

*2) Nonlinear Mass-Spring System:* We next employ global backstepping to obtain a fully decentralized stabilizing controller for a nonlinear mass-spring system described by (2) with $n \in \mathbb{Z}$ and

$$\begin{aligned} f_n &= \frac{k}{m}(\psi_{1,n-1} - 2\psi_{1n} + \psi_{1,n+1}) \\ &\quad + \frac{q}{m}((\psi_{1,n-1} - \psi_{1n})^3 + (\psi_{1,n+1} - \psi_{1n})^3) \\ b_n &= \frac{1}{m}. \end{aligned} \tag{20}$$

The backstepping design closely follows the procedure described in Section III-A. In particular, we select $\Lambda_1(\Psi_1) := \{-\kappa_{1n}\psi_{1n} - \kappa_{2n}\psi_{1n}^3\}_{n\in\mathbb{Z}}$, $\{\kappa_{1n}, \kappa_{2n} > 0, \forall n \in \mathbb{Z}\}$, and use Young's Inequality (see [26, (2.253) and (2.254)]) to bound the interactions between $G_n$ and its immediate neighbors $G_{n-1}$ and $G_{n+1}$ : $\zeta_{2n}\psi_{1i} \leq p\zeta_{2n}^2 + (1/4p)\psi_{1i}^2$, $\zeta_{2n}\psi_{1i}^3 \leq (r^4/4)\zeta_{2n}^4 + (3/4r^{4/3})\psi_{1i}^4$, $-\zeta_{2n}\psi_{1n}\psi_{1i}^2 \leq l\zeta_{2n}^2\psi_{1n}^2 + (1/4l)\psi_{1i}^4$, $\zeta_{2n}\psi_{1n}^2\psi_{1i} \leq s\zeta_{2n}^2\psi_{1n}^4 + (1/4s)\psi_{1i}^2$

$$\{p, r, l, s > 0,\ \zeta_{2n} := \psi_{2n} + \kappa_{1n}\psi_{1n} + \kappa_{2n}\psi_{1n}^3 \quad \forall n \in \mathbb{Z} \qquad \forall i := \{n-1,\, n+1\}\}.$$

Based on this, it can be shown that the following fully decentralized controller:

$$\begin{aligned} u_n = -m\Bigg(&\left(1 - \frac{2k}{m} - \frac{2q}{m}\psi_{1n}^2\right)\psi_{1n} + (\kappa_{1n} + 3\kappa_{2n}\psi_{1n}^2)\psi_{2n} \\ &+ \kappa_{3n}\zeta_{2n} + \kappa_{4n}\zeta_{2n}^3 + \kappa_{5n}\zeta_{2n}\psi_{1n}^2 + \kappa_{6n}\zeta_{2n}\psi_{1n}^4\Bigg), \qquad n \in \mathbb{Z} \end{aligned}$$

provides global asymptotic stability of the origin of a nonlinear mass-spring system [(2), (20)] if the design parameters satisfy: $\kappa_{1n} > k/2mp + 3q/2ms$, $\kappa_{2n} > 3q/2mr^{4/3} + 3q/2ml$, $\kappa_{3n} > 2kp/m$, $\kappa_{4n} > qr^4/2m$, $\kappa_{5n} > 6ql/m$, $\kappa_{6n} > 6qs/m$, and for every $n \in \mathbb{Z}$.

*Remark 4:* Fully decentralized controllers for mass-spring system cannot be obtained using the individual cell backstepping procedure of Section III-B. This is because the harmful interactions that are dominated by feedback in the global design are treated as the exogenous signals in the individual cell design. Thus, the cancellation controller for a mass-spring system in which $K_n$ interacts with $K_{n-1}$ and $K_{n+1}$ is pretty much the only controller that can come out of the individual cell backstepping design.

## IV. Adaptive State-Feedback Design

In this section, we study adaptive distributed design for systems with time independent unknown parameters that are either spatially constant or spatially varying. The dynamic controllers that guarantee boundedness of all signals in the closed-loop and achieve "regulation" of the plant's state are obtained using backstepping.

We show that systems with constant unknown parameters are amenable to a global design. This implies that an adaptive CLF (ACLF) can be constructed for the entire infinite dimensional system. Based on our experience with nominal backstepping this may seem advantageous, but we demonstrate that this approach yields centralized dynamical controllers, which is undesirable. On the other hand, for systems with spatially varying unknown parameters global design is not possible without some *a priori* information about values of these parameters. For this class of systems we perform an individual cell adaptive backstepping design which yields localized distributed dynamical controllers. As in the nominal case, the architecture of these controllers has strong similarity with the plant architecture. Furthermore, the dynamical order of $K_n$ is determined by the number of unknown parameters in $G_n$. We also note that this approach can be utilized to obtain localized distributed controllers for systems with constant unknown parameters.

### A. Constant Unknown Parameters

We first study adaptive state-feedback design for systems with constant unknown parameters [(3), (4)]. We establish that global adaptive backstepping yields centralized controllers.

For $m = 2$, [(3), (4)] simplifies to

$$\dot{\psi}_{1n} = \psi_{2n}, \qquad n \in \mathbb{Z}$$
$$\dot{\psi}_{2n} = \nu_n(\Psi_1, \Psi_2) + g_n^*(\Psi_1, \Psi_2)\theta + b u_n, \qquad n \in \mathbb{Z}$$

where $\theta \in \mathbb{R}^r$ and $b \in \mathbb{R}$ represent constant unknown parameters. We rewrite dynamics of the entire system as

$$\dot{\Psi}_1 = \Psi_2 \tag{22a}$$
$$\dot{\Psi}_2 = \mathrm{N}(\Psi_1, \Psi_2) + \mathcal{G}^*(\Psi_1, \Psi_2)\theta + b\mathrm{U} \tag{22b}$$

where $\mathcal{G}$ is an operator ($\mathcal{G} : l_2 \to \mathbb{R}^{r_k}$, $k = 1, 2$) defined by: $\mathcal{G}(\Psi_1, \Psi_2) := [\cdots \ g_{n-1} \ g_n \ g_{n+1} \ \cdots]$. Using Assumptions 1 and 3, we conclude boundedness of $\mathcal{G}$.

The first step of backstepping is the same as in Section III-A, where we choose $\Lambda_1(\Psi_1) := -k_1\Psi_1$, $k_1 > 0$. At the second step we have to account for the lack of knowledge of parameters in (22b): we need to estimate $\theta$ and reciprocal of $b$, $\varrho := 1/b$, to avoid the division with an estimate of $b$ which can occasionally be zero.

*Step 2:* We augment CLF (11) by two terms that account for the errors between $\theta$ and $\varrho$ and their estimates $\hat{\theta}$ and $\hat{\varrho}$

$$V_a(\Psi_1, \mathrm{Z}_2, \tilde{\theta}, \tilde{\varrho}) := V_2(\Psi_1, \mathrm{Z}_2) + \frac{1}{2}\tilde{\theta}^*\Gamma^{-1}\tilde{\theta} + \frac{|b|}{2\beta}\tilde{\varrho}^2 \tag{23}$$

where $\tilde{\theta}(t) := \theta - \hat{\theta}(t)$, $\tilde{\varrho}(t) := \varrho - \hat{\varrho}(t)$, $\Gamma$ is a positive definite matrix, and $\beta$ is a positive constant. The derivative of $V_a$ along the solutions of

$$\dot{\Psi}_1 = -k_1\Psi_1 + \mathrm{Z}_2$$
$$\dot{\mathrm{Z}}_2 = k_1\Psi_2 + \mathrm{N}(\Psi_1, \Psi_2) + \mathcal{G}^*(\Psi_1, \Psi_2)(\hat{\theta} + \tilde{\theta}) + b\mathrm{U}$$

is determined by

$$\dot{V}_a = -k_1\langle\Psi_1, \Psi_1\rangle + \langle \mathrm{Z}_2, \mathrm{S} + b\mathrm{U}\rangle + \tilde{\theta}^*(\Gamma^{-1}\dot{\tilde{\theta}} + \mathcal{G}\mathrm{Z}_2) + \left(\frac{|b|}{\beta}\right)\tilde{\varrho}\dot{\tilde{\varrho}}$$

where $\mathrm{S} := \Psi_1 + k_1\Psi_2 + \mathrm{N}(\Psi_1, \Psi_2) + \mathcal{G}^*(\Psi_1, \Psi_2)\hat{\theta}$. We can eliminate $\tilde{\theta}$ from $\dot{V}_a$ by selecting: $\dot{\hat{\theta}} = \Gamma\mathcal{G}(\Psi_1, \Psi_2)\mathrm{Z}_2$. A choice of control law of the form: $\mathrm{U} = -\hat{\varrho}(\mathrm{S} + k_2\mathrm{Z}_2)$, $k_2 > 0$, together with parameter $\theta$ update law and the relationship $b\hat{\varrho} = b(\varrho - \tilde{\varrho}) = 1 - b\tilde{\varrho}$, yields

$$\dot{V}_a = -k_1\langle\Psi_1, \Psi_1\rangle - k_2\langle \mathrm{Z}_2, \mathrm{Z}_2\rangle + \frac{|b|}{\beta}\tilde{\varrho}(\dot{\tilde{\varrho}} + \beta\,\mathrm{sign}(b)\langle \mathrm{Z}_2, \mathrm{S} + k_2\mathrm{Z}_2\rangle).$$

With the following choice of update law for the estimate $\hat{\varrho} : \dot{\hat{\varrho}} = \beta\,\mathrm{sign}(b)\langle \mathrm{Z}_2, \mathrm{S} + k_2\mathrm{Z}_2\rangle$, we finally obtain: $\dot{V}_a = -k_1\langle\Psi_1, \Psi_1\rangle - k_2\langle \mathrm{Z}_2, \mathrm{Z}_2\rangle =: -W(\Psi_1, \mathrm{Z}_2) \leq 0$. Boundedness of all signals in the closed-loop and asymptotic convergence of $\Psi_1$ and $\Psi_2$ to zero is established in Appendix.

The developments of this section are summarized in the following theorem.

*Theorem 5:* Suppose that system (22) satisfies Assumptions 1–4 and 6. Then, the following centralized dynamical controller:

$$\begin{aligned}
\mathrm{U} &= -\hat{\varrho}(\mathrm{S} + k_2\mathrm{Z}_2), \quad k_2 > 0\\
\mathrm{Z}_2 &= \Psi_2 + k_1\Psi_1, \quad k_1 > 0\\
\mathrm{S} &= \Psi_1 + k_1\Psi_2 + \mathrm{N}(\Psi_1, \Psi_2) + \mathcal{G}^*(\Psi_1, \Psi_2)\hat{\theta}\\
\dot{\hat{\theta}} &= \Gamma\mathcal{G}(\Psi_1, \Psi_2)\mathrm{Z}_2, \quad \Gamma > 0\\
\dot{\hat{\varrho}} &= \beta\,\mathrm{sign}(b)\langle \mathrm{Z}_2, \mathrm{S} + k_2\mathrm{Z}_2\rangle, \quad \beta > 0
\end{aligned} \tag{24}$$

guarantees boundedness of all signals in the closed-loop system [(22), (24)] and asymptotic convergence of the state of (22) to zero. These properties can be established with the Lyapunov function

$$V_a(\Psi_1, \mathrm{Z}_2, \tilde{\theta}, \tilde{\varrho}) = \frac{1}{2}\langle\Psi_1, \Psi_1\rangle + \frac{1}{2}\langle \mathrm{Z}_2, \mathrm{Z}_2\rangle + \frac{1}{2}\tilde{\theta}^*\Gamma^{-1}\tilde{\theta} + \frac{|b|}{2\beta}\tilde{\varrho}^2.$$

### B. Spatially Varying Unknown Parameters

Here, we consider state-feedback design for systems with time independent spatially varying unknown parameters. An example of such a system is given by [(3), (5)], which for $m = 2$ becomes

$$\dot{\psi}_{1n} = \psi_{2n} \tag{25a}$$
$$\dot{\psi}_{2n} = \tau_n(\Psi_1, \Psi_2) + h_n^*(\Psi_1, \Psi_2)\theta_n + b_n u_n \tag{25b}$$

where $n \in \mathbb{F}$.

*Remark 5:* Even if Assumption 6 holds, the infinite number of unknown parameters in (25) rules out a global design. Namely, the finiteness of the global ACLF candidate for (25) at $t = 0$

$$V_a(\Psi_1, \mathrm{Z}_2, \{\tilde{\theta}_n\}_{n\in\mathbb{F}}, \{\tilde{\varrho}_n\}_{n\in\mathbb{F}}) := \frac{1}{2}\langle\Psi_1, \Psi_1\rangle + \frac{1}{2}\langle \mathrm{Z}_2, \mathrm{Z}_2\rangle + \frac{1}{2}\sum_{n\in\mathbb{F}}\left(\tilde{\theta}_n^*\Gamma_n^{-1}\tilde{\theta}_n + \frac{|b_n|}{\beta_n}\tilde{\varrho}_n^2\right)$$

would imply that most of unknown parameters are initially known, which is somewhat artificial. If we have some *a priori* information about values that these parameters can assume we can choose a sequence of positive–definite matrices $\{\Gamma_n\}_{n\in\mathbb{F}}$ and positive parameters $\{\beta_n\}_{n\in\mathbb{F}}$ such that $V_a(\Psi_1(0), \mathrm{Z}_2(0), \{\tilde{\theta}_n(0)\}_{n\in\mathbb{F}}, \{\tilde{\varrho}_n(0)\}_{n\in\mathbb{F}})$ is finite. However, this would lead to parameter update laws with very large gains since elements of these sequences have to increase their values as $n \to \infty$. Clearly, this is not desirable for implementation.

In view of Remark 5, we carry out the individual cell adaptive backstepping design for (25) with spatially varying parametric uncertainties. This approach can be also used for adaptive control of systems with constant unknown parameters, and we will demonstrate that it yields localized distributed controllers. Thus, individual cell adaptive design for infinite dimensional systems on lattices has advantages over global adaptive design of Section IV-A, because the latter leads to centralized dynamical controllers.

The first step of backstepping is the same as in Section III-B. However, at the second step we need to estimate the values of $\theta_n$ and $\varrho_n := 1/b_n$.

*Step 2:* We augment $V_{2n}$ from Section III-B by two terms to account for $\tilde{\theta}_n(t) := \theta_n - \hat{\theta}_n(t)$ and $\tilde{\varrho}_n(t) := \varrho_n - \hat{\varrho}_n(t)$

$$V_{an}(\psi_{1n}, \zeta_{2n}, \tilde{\theta}_n, \tilde{\varrho}_n) := V_{2n}(\psi_{1n}, \zeta_{2n}) + \frac{1}{2}\tilde{\theta}_n^* \Gamma_n^{-1}\tilde{\theta}_n + \frac{|b_n|}{2\beta_n}\tilde{\varrho}_n^2$$

where $\Gamma_n$ is a positive–definite matrix, and $\beta_n$ is a positive constant. It is readily established that the following localized distributed controller: $u_n = -\hat{\varrho}_n s_n$, $\dot{\hat{\theta}}_n = (\psi_{2n} + k_{1n}\psi_{1n})\Gamma_n h_n$, $\dot{\hat{\varrho}}_n = \beta_n \operatorname{sign}(b_n)(\psi_{2n} + k_{1n}\psi_{1n})s_n$, $s_n = (1 + k_{1n}k_{2n})\psi_{1n} + (k_{1n} + k_{2n})\psi_{2n} + \tau_n + h_n^*\hat{\theta}_n$, with $k_{1n}, k_{2n} > 0$, renders $\dot{V}_{an}$ for every $n \in \mathbb{F}$ into $\dot{V}_{an} = -k_{1n}\psi_{1n}^2 - k_{2n}\zeta_{2n}^2 \le 0$. Boundedness of all signals in the closed-loop and asymptotic convergence of both $\psi_{1n}(t)$ and $\psi_{2n}(t)$ to zero, for all $n \in \mathbb{F}$, can be established using similar argument as in Appendix.

The main result of this section is summarized in the following theorem.

*Theorem 6:* Suppose that system (25) satisfies Assumptions 1, 2, and 5. Then, the following distributed dynamical controller:

$$\begin{aligned} u_n &= -\hat{\varrho}_n s_n \\ \dot{\hat{\theta}}_n &= (\psi_{2n} + k_{1n}\psi_{1n})\Gamma_n h_n(\Psi_1, \Psi_2) \\ \dot{\hat{\varrho}}_n &= \beta_n \operatorname{sign}(b_n)(\psi_{2n} + k_{1n}\psi_{1n})s_n \\ s_n &= (1 + k_{1n}k_{2n})\psi_{1n} + (k_{1n} + k_{2n})\psi_{2n} \\ &\quad + \tau_n(\Psi_1, \Psi_2) + h_n^*(\Psi_1, \Psi_2)\hat{\theta}_n \end{aligned} \tag{26}$$

with $\{k_{1n} > 0,\ k_{2n} > 0,\ \Gamma_n > 0,\ \beta_n > 0,\ \forall n \in \mathbb{F}\}$, guarantees boundedness of all signals in the closed-loop system [(25), (26)] and asymptotic convergence of the state of (25) to zero.

*Remark 6:* It is readily shown that the following distributed adaptive controller:

$$\begin{aligned} u_n &= -\hat{\varrho}_n s_n \\ \dot{\hat{\theta}}_n &= (\psi_{2n} - \dot{r}_n(t) + k_{1n}(\psi_{1n} - r_n(t)))\Gamma_n h_n \\ \dot{\hat{\varrho}}_n &= \beta_n \operatorname{sign}(b_n)((\psi_{2n} - \dot{r}_n(t)) + k_{1n}(\psi_{1n} - r_n(t)))s_n \\ s_n &= (1 + k_{1n}k_{2n})(\psi_{1n} - r_n(t)) + (k_{1n} + k_{2n})(\psi_{2n} - \dot{r}_n(t)) \\ &\quad + \tau_n + h_n^*\hat{\theta}_n - \ddot{r}_n(t) \end{aligned} \tag{27}$$

with $\{k_{1n} > 0,\ k_{2n} > 0,\ \Gamma_n > 0,\ \beta_n > 0,\ \forall n \in \mathbb{F}\}$, guarantees boundedness of all signals in the closed-loop system [(25), (27)] and asymptotic convergence of $\psi_{1n}(t)$ to $r_n(t)$, for all $n \in \mathbb{F}$. It is assumed that, for every $n \in \mathbb{F}$, the reference signal $r_n$, and its first two derivatives $\dot{r}_n$, and $\ddot{r}_n$ are known and uniformly bounded, and that $\ddot{r}_n$ is piecewise continuous.

## V. Output-Feedback Design

The controllers of Sections III and IV provide desired properties of the closed-loop systems under the assumption that the full state information is available. In this section, we study a more realistic situation in which only a distributed output is measured. We show that, as for finite-dimensional systems [26], the observer backstepping can be used as a tool for fulfilling the desired objective for systems on lattices in which nonlinearities depend only on the measured signals. The starting point of the nominal output-feedback approach is a design of an observer which guarantees the exponential convergence of the state estimates to their real values. Once this is accomplished, the combination of backstepping and nonlinear damping is used to account for the observation errors and provide closed-loop stability. In the adaptive case, filters which provide "virtual estimates" of unmeasured state variables also need to be designed.

We solve output-feedback problems for systems of local dynamical order two ($m = 2$) that satisfy the matching condition. General case can be handled using similar tools. In this situation, systems (6) and [(6), (7)], respectively, simplify to

$$\dot{\psi}_{1n} = \psi_{2n}, \qquad n \in \mathbb{F} \tag{28a}$$
$$\dot{\psi}_{2n} = f_n(\mathrm{Y}) + b_n u_n, \qquad n \in \mathbb{F} \tag{28b}$$
$$y_n = \psi_{1n}, \qquad n \in \mathbb{F} \tag{28c}$$

and

$$\dot{\psi}_{1n} = \psi_{2n}, \qquad n \in \mathbb{F} \tag{29a}$$
$$\dot{\psi}_{2n} = \tau_n(\mathrm{Y}) + h_n^*(\mathrm{Y})\theta_n + b_n u_n, \qquad n \in \mathbb{F} \tag{29b}$$
$$y_n = \psi_{1n}, \qquad n \in \mathbb{F}. \tag{29c}$$

Both nominal and adaptive output-feedback problems are solved using individual cell backstepping design. We note that nominal output-feedback problem can be also solved using global backstepping if the initial state of (28) satisfies Assumption 6.

### A. Nominal Output-Feedback Design

We rewrite (28) in a form suitable for observer design

$$\dot{\psi}_n = A\psi_n + \varphi_n(\mathrm{Y}) + b_n e_2 u_n, \qquad n \in \mathbb{F} \tag{30a}$$
$$y_n = C\psi_n, \qquad n \in \mathbb{F} \tag{30b}$$

where

$$\psi_n := \begin{bmatrix} \psi_{1n} \\ \psi_{2n} \end{bmatrix} \ \varphi_n(\mathrm{Y}) := \begin{bmatrix} 0 \\ f_n(\mathrm{Y}) \end{bmatrix} \ e_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ C := [1 \quad 0].$$

We proceed by designing an equivalent of *Krener–Isidori observer* (see, for example, [26] and [39]) for (30)

$$\dot{\hat{\psi}}_n = A\hat{\psi}_n + L_n(y_n - \hat{y}_n) + \varphi_n(\mathrm{Y}) + b_n e_2 u_n \tag{31a}$$
$$\hat{y}_n = C\hat{\psi}_n \tag{31b}$$

where $L_n := [\, l_{1n} \quad l_{2n} \,]^*$ is chosen such that $A_{0n} := A - L_n C$ is a Hurwitz matrix for every $n \in \mathbb{F}$. Clearly, this is going to be satisfied if and only if $l_{in} > 0, \forall i = \{1, 2\}, \forall n \in \mathbb{F}$. In this case, an exponentially stable system of the form

$$\dot{\tilde{\psi}}_n = A_{0n}\tilde{\psi}_n, \qquad n \in \mathbb{F} \tag{32}$$

is obtained by subtracting (31) from (30). The properties of $A_{0n}$ imply the exponential convergence of $\tilde{\psi}_n := \psi_n - \hat{\psi}_n$ to zero and the existence of the positive–definite matrix $P_{0n}$ that satisfies

$$A_{0n}^* P_{0n} + P_{0n} A_{0n} = -I \qquad \forall n \in \mathbb{F}. \tag{33}$$

The main result of this section, whose proof can be found in [40], is summarized in the following theorem.

*Theorem 7:* Consider (28) with Assumptions 1–2 and observer (31). The output-feedback distributed controller

$$\begin{aligned} u_n = -\frac{1}{b_n}(\psi_{1n} + (k_{1n}+d_{1n})\hat{\psi}_{2n} + l_{2n}(\psi_{1n}-\hat{\psi}_{1n}) \\ + f_n(\mathrm{Y}) + d_{2n}(k_{1n}+d_{1n})^2\zeta_{2n} + k_{2n}\zeta_{2n}) \end{aligned} \tag{34}$$

with $\{k_{1n} > 0,\ k_{2n} > 0,\ d_{1n} > 0,\ d_{2n} > 0,\ \forall n \in \mathbb{F}\}$, guarantees global asymptotic stability of the origin of closed-loop system [(28), (32), (34)]. These properties can be established with

$$V_{2n}(\psi_{1n}, \zeta_{2n}, \tilde{\psi}_n) := \frac{1}{2}\psi_{1n}^2 + \frac{1}{2}\zeta_{2n}^2 + \left(\frac{1}{d_{1n}} + \frac{1}{d_{2n}}\right)\tilde{\psi}_n^* P_{0n} \tilde{\psi}_n$$

where $\zeta_{2n} := \hat{\psi}_{2n} + (k_{1n}+d_{1n})\psi_{1n}$.

### *B. Adaptive Output-Feedback Design*

We rewrite (29) in a form suitable for adaptive output-feedback design

$$\dot{\psi}_n = A\psi_n + \eta_n(\mathrm{Y}) + \sum_{j=1}^{r}\theta_{jn}\varphi_{jn}(\mathrm{Y}) + b_n e_2 u_n \tag{35a}$$

$$y_n = C\psi_n \tag{35b}$$

where, for every $n \in \mathbb{F}$, $\psi_n := [\,\psi_{1n} \quad \psi_{2n}\,]^*$ and

$$\eta_n(\mathrm{Y}) := \begin{bmatrix} 0 \\ \tau_n(\mathrm{Y}) \end{bmatrix} \quad \varphi_{jn}(\mathrm{Y}) := \begin{bmatrix} 0 \\ h_{jn}(\mathrm{Y}) \end{bmatrix}$$
$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad e_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C := [\,1 \quad 0\,].$$

We proceed by designing filters which provide "virtual estimates" of unmeasured state variables (see [26, § 7.3])

$$\dot{\xi}_n^{(0)} = A_{0n}\xi_n^{(0)} + L_n y_n + \eta_n(\mathrm{Y}) \tag{36a}$$

$$\dot{\xi}_n^{(j)} = A_{0n}\xi_n^{(j)} + \varphi_{jn}(\mathrm{Y}),\ 1 \le j \le r \tag{36b}$$

$$\dot{v}_n = A_{0n}v_n + e_2 u_n \tag{36c}$$

where $L_n := [\,l_{1n} \quad l_{2n}\,]^*$ is chosen such that $A_{0n} := A - L_n C$ is Hurwitz for every $n \in \mathbb{F}$. Clearly, $A_{0n}$ is going to be Hurwitz if and only if $l_{in} > 0,\ \forall i = \{1,2\},\ \forall n \in \mathbb{F}$. In this case, an exponentially stable system: $\dot{\varepsilon}_n = A_{0n}\varepsilon_n$, is obtained by combining (35) and (36) for every $n \in \mathbb{F}$, with $\varepsilon_n := \psi_n - (\xi_n^{(0)} + \sum_{j=1}^{r}\theta_{jn}\xi_n^{(j)} + b_n v_n)$. The properties of $A_{0n}$ imply the exponential convergence of $\varepsilon_n$ to zero and the existence of the positive definite matrix $P_{0n}$ that satisfies (33).

The main result of this section, whose proof can be found in [40], is summarized in the following theorem.

*Theorem 8:* Consider (29) with Assumptions 1, 2, and 5 and filters (36). The output-feedback distributed controller

$$\begin{aligned} u_n = -(\sigma_n + \psi_{1n}e_{r+1}^*\hat{\vartheta}_n^{(2)} + k_{2n}\zeta_{2n} \\ + d_{2n}((k_{1n}+d_{1n})\hat{\vartheta}_{1n}^{(1)})^2\zeta_{2n}) \end{aligned}$$

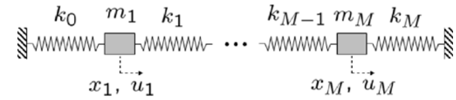


Fig. 3. Finite-dimensional mass-spring system.

$\dot{\hat{\vartheta}}_n^{(1)} = \mathrm{sign}(b_n)\psi_{1n}\Gamma_{1n}\omega_n^{(1)}$, $\dot{\hat{\vartheta}}_n^{(2)} = \zeta_{2n}\Gamma_{2n}(\psi_{1n}e_{r+1} + \hat{\vartheta}_{1n}^{(1)}\omega_n^{(2)})$, $\zeta_{2n} := v_{2n} + \omega_n^{(1)^*}\hat{\vartheta}_n^{(1)}$

$$\begin{aligned} \sigma_n := -l_{2n}v_{1n} + \omega_n^{(1)^*}\dot{\hat{\vartheta}}_n^{(1)} + \mu_n^*\hat{\vartheta}_n^{(1)} \\ + \hat{\vartheta}_{1n}^{(1)}((k_{1n}+d_{1n})\xi_{2n}^{(0)} + \omega_n^{(2)^*}\hat{\vartheta}_n^{(2)}) \end{aligned}$$

$\omega_n^{(1)} := [\,\xi_{2n}^{(0)} + (k_{1n}+d_{1n})\psi_{1n} \quad \xi_{2n}^{(1)} \quad \cdots \quad \xi_{2n}^{(r)}\,]^*$, $\omega_n^{(2)} := (k_{1n}+d_{1n})[\,\xi_{2n}^{(1)} \quad \cdots \quad \xi_{2n}^{(r)} \quad v_{2n}\,]^*$, $\vartheta_n^{(1)} := [\,1/b_n \quad \theta_{1n}/b_n \quad \cdots \quad \theta_{rn}/b_n\,]^*$, $\vartheta_n^{(2)} := [\,\theta_{1n} \quad \cdots \quad \theta_{rn} \quad b_n\,]^*$, $\mu_n := [\,\dot{\xi}_{2n}^{(0)} \quad \dot{\xi}_{2n}^{(1)} \quad \cdots \quad \dot{\xi}_{2n}^{(r)}\,]^*$, with

$$\begin{aligned} \{k_{1n} > 0,\ k_{2n} > 0,\ d_{1n} > 0,\ d_{2n} > 0 \\ \Gamma_{1n} > 0,\ \Gamma_{2n} > 0 \qquad \forall n \in \mathbb{F}\} \end{aligned}$$

guarantees boundedness of all signals in the closed-loop adaptive system and asymptotic convergence of $\psi_{1n}$, $\zeta_{2n}$, and $\varepsilon_n$ to zero for every $n \in \mathbb{F}$. These properties can be established with

$$\begin{aligned} V_{an}(\psi_{1n}, \zeta_{2n}, \tilde{\vartheta}_n^{(1)}, \tilde{\vartheta}_n^{(2)}, \varepsilon_n) \\ = \frac{1}{2}\psi_{1n}^2 + \frac{1}{2}\zeta_{2n}^2 + \frac{|b_n|}{2}\tilde{\vartheta}_n^{(1)^*}\Gamma_{1n}^{-1}\tilde{\vartheta}_n^{(1)} \\ + \frac{1}{2}\tilde{\vartheta}_n^{(2)^*}\Gamma_{2n}^{-1}\tilde{\vartheta}_n^{(2)} + \left(\frac{1}{d_{1n}} + \frac{1}{d_{2n}}\right)\varepsilon_n^* P_{0n}\varepsilon_n. \end{aligned}$$

*Remark 7:* The results established in Sections III–V are valid only if the solution to the resulting system of equations exists. The well-posedness of closed-loop systems can be established using similar argument as in Section II-D. Since control design guarantees boundedness of all signals in the closed-loop, we conclude that the backstepping yields systems with unique classical solutions on time interval $[0, \infty)$.

*Remark 8:* In applications, we clearly have to work with systems that contain large-but-finite number of units. All infinite dimensional results are applicable here, but with minor modifications. For example, for the mass-spring system shown in Fig. 3 with $M$ masses ($n = 1, \ldots, M$) both the equations presented in Section II-B and the control laws of Sections III–V are still valid with appropriate "boundary conditions": $x_j = \dot{x}_j = u_j \equiv 0$, $\forall j \in \mathbb{Z} \setminus \{1, \ldots, M\}$. The performance of distributed backstepping controllers is validated using computer simulations of mass-spring system with $M = 100$ units in [40].

## VI. Architecture of Distributed Controllers

In this section, we remark on the architecture of controllers developed in Sections III–V.

The controllers of Theorems 1, 2, and 6–8 inherit the plant architecture. Thus, when matching condition is satisfied the design objective can be always achieved using controller of the same architecture as the original plant. Furthermore, as illustrated in Section III-D, nominal controllers with less interactions can be obtained by performing a global backstepping design to identify beneficial interactions, and/or to dominate harmful interactions.

On the other hand, since information from all plant cells is used to determine estimates of constant unknown parameters $\theta \in \mathbb{R}^r$ and $\varrho \in \mathbb{R}$, dynamical controller (24) of Theorem 5 is centralized. This controller builds one estimate per unknown parameter, and its dynamical order is equal to $r+1$. We remark that results of Theorem 6 can be also applied for the control of systems with constant parametric uncertainties. Controller (26) of Theorem 6 has different parameter update laws (for $\hat{\theta}_n$ and $\hat{\varrho}_n$) in every control unit $K_n$, even when all unknown parameters are constant. The dynamical order of $K_n$, for every $n \in \mathbb{F}$, is equal to the number of unknown parameters: $r+1$. This "over-parameterization" is advantageous in applications because it allows for implementation of *localized distributed adaptive controllers*. This useful property cannot be achieved with controller (24) because it requires information about entire distributed state to estimate unknown parameters. Furthermore, the design procedure described in Section IV-B does not require Assumptions 3 and 6 to hold and, consequently, it can be applied to a broader class of problems.

## VII. Concluding Remarks

This paper studies the distributed control of infinite dimensional systems on lattices. It is illustrated that Lyapunov-based approach can be successfully used to obtain state and output-feedback controllers for both nominal systems and systems with parametric uncertainties. It is also shown that the control problem can always be posed in such a way to yield controllers of the same architecture as the original plant. Therefore, as a result of Lyapunov-based design systems with an intrinsic degree of decentralization are obtained. For a nominal mass-spring system we illustrate that fully decentralized stabilizing controllers can be obtained by a careful analysis of nonlinearities and interactions between different subsystems.

Our current efforts are directed toward development of modular adaptive schemes in which parameter update laws and controllers are designed separately. The major advantage of using this approach rather than the Lyapunov-based design is the versatility that it offers. Namely, adaptive controllers of this paper are limited to Lyapunov-based estimators. From a practical point of view it might be advantageous to use the appropriately modified standard gradient or least-squares type identifiers.

## Appendix
## Completing the Proof of Theorem 5

Negative semidefiniteness of $\dot{V}_a$ implies that $V_a$ is a nonincreasing function of time. Hence, based on the definition of $V_a$ and its boundedness at $t=0$ (see Assumption 6), we conclude that $\Psi_1$, $\Psi_2$, $\hat{\theta}$, and $\hat{\varrho}$ are globally uniformly bounded, that is

$$\left\{ \|\Psi_k(t)\|^2 := \sum_{n \in \mathbb{F}} \psi_{kn}^2(t) < \infty,\ \hat{\theta}^*(t)\hat{\theta}(t) < \infty,\ \hat{\varrho}^2(t) < \infty \quad \forall k = 1,2,\ \forall t \geq 0 \right\}.$$

In view of this, properties of functions $g_n$ and $\nu_n$ (see Assumptions 1–3), and definition of U, it follows that $\{\hat{\theta} \in L_\infty,\ \hat{\varrho} \in L_\infty\}$, and $\{\psi_{1n} \in L_\infty,\ \psi_{2n} \in L_\infty,\ u_n \in L_\infty,\ \forall n \in \mathbb{F}\}$, which in turn implies $\{\dot{\psi}_{1n} \in L_\infty,\ \dot{\psi}_{2n} \in L_\infty,\ \forall n \in \mathbb{F}\}$. Using Assumption 3 and the fact that $\{\Psi_1(t) \in l_2,\ \Psi_2(t) \in l_2,\ \forall t \geq 0\}$, we also conclude that $\{\dot{\hat{\theta}}(t) < \infty,\ \dot{\hat{\varrho}}(t) < \infty,\ \forall t \geq 0\}$. This follows from the properties of functions $\nu_n$ and $g_n$ (see Assumptions 1–3) and a simple observation that $\{\Psi(t) \in l_p \times l_q,\ \forall t \geq 0,\ \forall p,q \in (2,\infty)\}$ whenever $\{\Psi(t) \in l_2 \times l_2,\ \forall t \geq 0\}$. Furthermore, since $V_a(\Psi_1(t), \mathrm{Z}_2(t), \tilde{\theta}(t), \tilde{\varrho}(t))$ is a nonincreasing nonnegative function, it has a limit $V_{a\infty}$ as $t \to \infty$. Thus, integration of $\dot{V}_a = -k_1 \langle \Psi_1, \Psi_1 \rangle - k_2 \langle \mathrm{Z}_2, \mathrm{Z}_2 \rangle =: -W(\Psi_1, \mathrm{Z}_2) \leq 0$ yields

$$\begin{aligned}\int_0^\infty W(\Psi_1(t), \mathrm{Z}_2(t))\mathrm{d}t &= k_1 \int_0^\infty \langle \Psi_1(t), \Psi_1(t) \rangle\, \mathrm{d}t \\ &\quad + k_2 \int_0^\infty \langle \mathrm{Z}_2(t), \mathrm{Z}_2(t) \rangle \mathrm{d}t \\ &\leq V_a(\Psi_1(0), \mathrm{Z}_2(0), \tilde{\theta}(0), \tilde{\varrho}(0)) \\ &\quad - V_{a\infty} < \infty\end{aligned}$$

which together with the definition of $\mathrm{Z}_2$ ($\mathrm{Z}_2 := \Psi_2 + k_1 \Psi_1$) implies $\{\psi_{1n}, \psi_{2n} \in L_2,\ \forall n \in \mathbb{F}\}$. Therefore, we have shown that $\{\psi_{1n},\ \psi_{2n} \in L_2 \cap L_\infty,\ \dot{\psi}_{1n},\ \dot{\psi}_{2n} \in L_\infty,\ \forall n \in \mathbb{F}\}$. Using the Barbălat lemma (see, for example, [25] and [33]), we conclude that both $\psi_{1n}(t)$ and $\psi_{2n}(t)$ go to zero as $t \to \infty$, for all $n \in \mathbb{F}$. Therefore, the dynamical controller of Section IV-A guarantees boundedness of all signals in the closed-loop and asymptotic convergence of the state $\Psi^*(t) := [\, \Psi_1^*(t) \quad \Psi_2^*(t)\,]^*$ of (22) to zero.

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