MODELING AND ANALYSIS OF PARALLEL AND SPATIALLY-EVOLVING WALL-BOUNDDED SHEAR FLOWS

by

Wei Ran

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Dedication

To my dear parents.
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Abstract

Wall-bounded shear flows such as boundary layer and channel flows are ubiquitous in engineering applications. One of the most concerning aspects in these wall-bounded shear flows is the skin-friction drag, which related to, for example, fuel-efficiency in air/water vehicles and efficiency and longevity of wind turbines—knowing that the processes and mechanism of laminar-turbulent transition are crucial to suppress it since turbulence can significantly increase skin-friction drag. Since transition will eventually take place in these flows, examining techniques that can reduce drag in the turbulent regime is critical as well.

In the beginning, we utilize the externally forced linearized Navier-Stokes equations to study the receptivity of pre-transitional boundary layers to persistent sources of stochastic excitation. Stochastic forcing is used to model the effect of free-stream turbulence that enters at various wall-normal locations, and the fluctuation dynamics are studied via linearized models that arise from locally parallel and global perspectives. In contrast to the widely used resolvent analysis that quantifies the amplification of deterministic disturbances at a given temporal frequency, our approach examines the steady-state response to stochastic excitation that is uncorrelated in time. In addition to stochastic forcing with identity covariance, we utilize the spatial spectrum of homogeneous isotropic turbulence to model the effect of free-stream turbulence. Even though locally parallel analysis does not account for the effect of the spatially evolving base flow, we demonstrate that it captures the essential mechanisms and the prevailing length-scales in stochastically forced boundary layer flows. On the other hand, global analysis, which accounts for the spatially evolving nature of the boundary layer flow, predicts the amplification of a cascade of streamwise scales throughout the streamwise domain. We show that the flow structures that are extracted from a modal decomposition
of the resulting velocity covariance matrix can be closely captured by conducting locally-parallel analysis at various streamwise locations and over different wall-parallel wavenumber pairs. Our approach does not rely on costly stochastic simulations, and it provides insight into mechanisms for perturbation growth, including the interaction of the slowly varying base flow with streaks and Tollmien-Schlichting waves.

The effort is then devoted to developing a successively linearized model to study interactions between different modes in a spatially evolving boundary layer flow. Our method consists of two steps. First, we augment the boundary layer profiles with a disturbance field resulting from the linear Parabolized Stability Equations (PSE) to obtain the modified base flow; and, second, we draw on Floquet decomposition to capture the effect of mode interactions on the spatial evolution of flow fluctuations via a linear progression. The resulting Parabolized Floquet Equations (PFE) can be conveniently advanced downstream to examine the interaction between different modes in slowly varying shear flows. We apply our framework to two canonical settings of transition in boundary layers; the H-type transition scenario initiated by exponential instabilities, and streamwise elongated laminar streaks triggered by the lift-up mechanism. We demonstrate that the PFE captures the growth of various harmonics and provides excellent agreement with the results obtained in direct numerical simulations and experiments.

Finally, we develop a model-based framework to quantify the effect of streamwise-aligned spanwise-periodic riblets on kinetic energy and skin-friction drag in turbulent channel flow. We model the effect of riblets as a volume penalization in the Navier-Stokes equations and use the statistical response of the eddy-viscosity-enhanced linearized equations to quantify the effect of background turbulence on the mean velocity and skin-friction drag. For triangular riblets, our simulation-free approach reliably predicts drag-reducing trends as well as mechanisms that lead to performance deterioration for large riblets. We investigate the effect of height and spacing on drag reduction and demonstrate a correlation between energy suppression and drag-reduction for appropriately sized riblets. We also analyze the effect of riblets on drag reduction mechanisms and turbulent flow structures, including very large
scale motions. Our results demonstrate the predictive power of our framework, which paves
the way for the optimal design of periodic surfaces for turbulent drag reduction using low
complexity models.
Chapter 1

Introduction

One of the most crucial support for modern civilization is the power supply and corresponding energy resources. In daily life, oil is required to produce gasoline for vehicles; natural gas is required for heating and cooking; electricity is required for the running of computers and TV. For transporting oil and natural gas, thousands of kilometers of pipes are built; to generate electricity, water turbines in hydropower stations, steam turbines in fossil-fuel generating and nuclear plants, and wind turbines in wind power plants are rotating at full speed. In all of these situations, fluids are flowing over solid surfaces, forming so-called wall-bounded shear flows.

In the engineering application like transportation mentioned above, the most concerning aspect in these wall-bounded shear flows is the skin-friction drag (cf Figure 1.1), which is closely related to the fuel-efficiency in air/water vehicles and efficiency and longevity of wind turbines. According to to Joslin (1998); Gad-el Hak (2000), the skin-friction drag introduces 50% and 90% of fuel cost in aircraft and submarines, respectively. Thus, it is essential to discover techniques to reduce the skin-friction drag so that the fuel economy can be improved.

In wall-bounded shear flows, streamwise velocity is 0 at the wall and increases as moving away from the wall due to the viscosity. Furthermore, the skin-friction drag is the shear stress at the wall, which is proportional to the gradient of the streamwise velocity of the fluid. In a smooth and ordered (laminar) flow, the velocity gradually increases from the wall. However, as it develops, the flow will lose the laminar state and become chaotic and disordered (turbulence). In the turbulent flow, vortices break the laminar velocity
Figure 1.1: Skin-friction drag is the major source of drag for (a) helicopters, (b) passenger airplanes, (c) cargo ships, and (d) submarines, which is responsible for approximately 50% and 90% fuel consumption in the air- and water- vehicles, respectively.

distribution and transport momentum from the high-speed region (away from the wall) to the near-wall region, which dramatically enhances the velocity gradient and resulting skin-friction drag.

Thus, the processes and mechanisms of laminar-turbulent transition are urged to be understood so that we can control the flow to delay the transition and reduce the overall drag. On the other hand, since transition will eventually arrive in these flows, it is also crucial to investigate ways to control and suppress turbulence for drag reduction. In contrast to expensive experiments and direct numerical simulations, in this dissertation, we utilize systems theory and fluid mechanics to construct low-complexity models that capture the complex dynamics of transiting and turbulent flows, paving ways for analysis, optimization, and control design.

This chapter of the introduction is organized as follows. In Section 1.1, a preliminary discussion on modeling and receptivity analysis is provided for transiting boundary layers. In Section 1.2, we briefly overview various aspects of modeling the nonlinear interactions in
transitional boundary layers. The background of turbulent drag reduction via surface corruga-
gations is discussed in Section 1.3; in Section 1.4, we provide an outline of the dissertation. 
Finally, a preview of the main results and contributions is presented in Section 1.5.

1.1 Receptivity analysis in boundary layer

Laminar-turbulent transition of fluid flows is important in many engineering applications. 
Predicting the point of transition requires an accurate understanding of the mechanisms 
that govern the physics of transitional flows. Since the 1990’s, numerical simulations with 
various levels of fidelity have been used to uncover many essential features of the transition 
phenomenon. In spite of this progress, the complicated sequence of events that leads to 
transition and the inherent complexity of the Navier-Stokes (NS) equations have hindered 
the development of practical control strategies for delaying transition in boundary layer 
flows (Morkovin et al., 1994; Saric et al., 2002; Kim and Bewley, 2007).

It is generally accepted that the transition process can be divided into three stages; recep-
tivity, instability growth, and breakdown (Morkovin et al., 1994). In the laminar boundary 
layer flow, disturbances that lead to transition are amplified either through modal, i.e., 
exponential, instability mechanisms or non-modal amplification, e.g., via transient growth 
mechanisms such as lift-up (Landahl, 1975, 1980) and Orr mechanisms (Orr, 1907; Butler 
and Farrell, 1992; Hack and Moin, 2017). An important aspect in both scenarios is the 
receptivity of the boundary layer flow to external excitation sources, e.g., free-stream tur-
bulence and surface roughness. Such sources of excitation perturb the velocity field and 
give rise to initial disturbances within the shear that can grow to critical levels. Depending 
on the amplitude and frequency of excitation, initial disturbances can take different routes 
to transition. For example, low-amplitude excitation of the boundary layer flow can cause 
the growth of two-dimensional Tollmien-Schlichting (TS) waves, which can trigger natural 
transition to turbulence (Klebanoff et al., 1962; Kachanov and Levchenko, 1984; Mack, 1984; 
Herbert, 1988; Sayadi et al., 2013). On the other hand, sufficiently high levels of broad-band 
excitation can induce the growth of streamwise elongated streaks that play an important
role in bypass transition (Saric et al., 2002). The effect of free-stream turbulence on the
growth of boundary layer streaks has been the subject of various experimental (Matsubara and Alfredsson, 2001; Fransson et al., 2005; Ricco et al., 2016), numerical (Jacobs and Durbin, 2001; Brandt et al., 2004), and theoretical (Goldstein, 2014; Hack and Zaki, 2014) studies. In particular, it has been shown that free-stream disturbances that penetrate into the boundary layer are elongated in the streamwise direction (Jacobs and Durbin, 1998).

While nonlinear dynamical models that are based on the NS equations provide insight into receptivity mechanisms, their implementation typically involves a large number of degrees of freedom and it ultimately requires direct simulations. This motivates the development of low-complexity models that are better suited for comprehensive quantitative studies.

In recent years, increasingly accurate descriptions of coherent structures in wall-bounded shear flows, e.g. (Smits et al., 2011; Hack and Moin, 2018), have inspired the development of reduced-order models. Such models are computationally tractable and can be trained to replicate statistical features that are estimated from experimentally or numerically generated data measurements. However, their data-driven nature is accompanied by a lack of robustness. Specifically, control actuation and sensing may significantly alter the identified modes which introduces nontrivial challenges for model-based control design (Noack et al., 2011). In contrast, models that are based on the linearized NS equations are less prone to such uncertainties and are, at the same time, well-suited for analysis and synthesis using tools of modern robust control.

While the nonlinear terms in the NS equations play an important role in transition to turbulence and in sustaining the turbulent state, they are conservative and, as such, they do not contribute to the transfer of energy between the mean flow and velocity fluctuations but only transfer energy between different Fourier modes (McComb, 1991; Durbin and Reif, 2011). This feature has inspired modeling the effect of nonlinearity using additive stochastic forcing with early efforts focused on homogeneous isotropic turbulence (HIT) (Orszag, 1970; Kraichnan, 1971; Monin and Yaglom, 1975). In the presence of stochastic excitation, the
linearized NS equations have been used to model heat and momentum fluxes and spatio-temporal spectra in quasi-geostrophic turbulence (Farrell and Ioannou, 1993a, 1994; DelSole and Farrell, 1995). Moreover, they have been used to characterize the most detrimental stochastic forcing and determine scaling laws for energy amplification at subcritical Reynolds numbers (Farrell and Ioannou, 1993b; Bamieh and Dahleh, 2001; Jovanovic and Bamieh, 2005), and to replicate structural Hwang and Cossu (2010a,b) and statistical (Moarref and Jovanovic, 2012; Zare et al., 2017b) features of wall-bounded turbulent flows. In these studies, stochastic forcing has been commonly used to model the impact of exogenous excitation sources and initial conditions, or to capture the effect of nonlinearity in the NS equations.

The linearized NS equations have been widely used for modal and non-modal stability analysis of both parallel and non-parallel flows (Huerre and Monkewitz, 1990; Schmid and Henningson, 2001; Schmid, 2007). In parallel flows, homogeneity in the streamwise and spanwise dimensions allows for the decoupling of the governing equations across streamwise and spanwise wavenumbers via Fourier transform, which results in significant computational advantages for analysis, optimization, and control. On the other hand, in the flat-plate boundary layer, streamwise and wall-normal inhomogeneity require discretization over two spatial directions and lead to models of significantly larger sizes. Conducting modal and non-modal analyses is thus more challenging than for locally parallel flows. However, due to the slowly varying nature of the boundary layer flow, parallel flow assumptions can still provide meaningful results. For example, primary disturbances can be identified using the eigenvalue analysis of the Orr-Sommerfeld and Squire equations (Schmid and Henningson, 2001) and the secondary instabilities can be obtained via Floquet analysis (Herbert, 1984, 1988). Moreover, the NS equations can be parabolized to account for the downstream propagating nature of waves in slowly varying flows via spatial marching. This technique has enabled the analysis of transitional boundary layers and turbulent jet flows using various forms of the unsteady boundary-region equations (Leib et al., 1999a; Ricco et al., 2011), parabolized stability equations (Herbert, 1997; Lozano-Durán et al., 2018), and the more recent one-way Euler equations (Towne and Colonius, 2015). Furthermore, drawing on Floquet theory, the
linear parabolized stability equations have also been extended to study interactions between different modes in slowly growing boundary layer flow (Ran et al., 2019a).

While the parallel flow assumption offers significant computational advantages, it does not account for the effect of the spatially evolving base flow on the stability of the boundary layer. Global stability analysis addresses this issue by accounting for the spatially varying nature of the base flow and discretizing all inhomogeneous spatial directions. Previously, tools from sparse linear algebra in conjunction with iterative schemes have been employed to analyze the eigenspectrum of the governing equations and provide insight into the dynamics of transitional flows (Ehrenstein and Gallaire, 2005; Alizard and Robinet, 2007; Nichols and Lele, 2011; Paredes et al., 2016; Schmidt et al., 2017). Efforts have also been made to conduct non-modal analysis of spatially evolving flows including transient growth (Barkley et al., 2008; Monokrousos et al., 2010) and resolvent (Brandt et al., 2011; Sipp and Marquet, 2013; Jeun et al., 2016; Schmidt et al., 2018; Dwivedi et al., 2019) analyses. In particular, for the flat-plate boundary layer flow, the sensitivity of singular values of the resolvent operator to base-flow modifications and subsequent effects on the TS instability mechanism and streak amplification was investigated in (Brandt et al., 2011). The growth of flow structures in flat-plate boundary layer flow was also studied in (Sipp and Marquet, 2013) and a connection between the results from local $e^N$ method and global resolvent analysis was established. However, previous studies did not incorporate information regarding the spatio-temporal spectrum and spatial localization of excitation sources. The widely used resolvent analysis (Trefethen et al., 1993; Jovanovic, 2004; McKeon and Sharma, 2010) is limited to monochromatic forcing, and as such, may not fully capture naturally occurring sources of excitation. Furthermore, the evolution of exact optimal perturbations that are identified using resolvent analyses is seldom encountered in practical configurations (Fransson et al., 2004).

The approach advanced in the present work enables the study of receptivity mechanisms in boundary layer flows subject to stochastic sources of excitation. We model the effect of free-stream turbulence as a persistent white-in-time stochastic forcing that enters
at various wall-normal locations and analyze the dynamics of velocity fluctuations around locally parallel and spatially evolving base flows using the solution to the algebraic Lyapunov equation. Our simulation-free approach enables computationally efficient assessment of the energy spectrum of spatially evolving flows, without relying on a particular form of the inflow conditions or computation of the full spectrum of the linearized dynamical generator. Moreover, the broad-band nature of our forcing model captures the aggregate effect of all time-scales without the need to integrate the frequency response over all energetically relevant frequencies.

We compare and contrast results obtained under locally parallel flow assumption with those of global analysis. Coherent structures that emerge as the response to free-stream turbulence are extracted using the modal decomposition of the steady-state velocity covariance matrix. We demonstrate how parallel and global flow analyses can be used to quantify the amplification of streamwise elongated streaks and Tollmien-Schlichting (TS) waves, which are important in the laminar-turbulent transition of boundary layer flows. Our analysis shows that subordinate eigenmodes of the steady-state velocity covariance matrices that result from global flow analyses have nearly equal energetic contributions to that of the principal modes. This observation demonstrates that global covariance matrices cannot be well-approximated by low-rank representations. On the other hand, we show how locally parallel analysis, which breaks up the receptivity process of the boundary layer flow over various streamwise length-scales, can uncover certain flow structures that are difficult to observe in global analysis. We also demonstrate that modeling the effect of free-stream turbulence using the spectrum of HIT yields similar results as the analysis based on white-in-time stochastic excitation with identity covariance matrix. For the considered range of moderate Reynolds numbers, our results support the assumption of parallel flow in the low-complexity modeling and analysis of boundary layer flows.
1.2 Mode interactions in the transitional boundary layer

In the past thirty years, remarkable progress has been made on simulating the physics of transitional flows using models with various levels of fidelity. In spite of this, the multi-layer nature of transition and the inherent complexity of the Navier-Stokes (NS) equations have hindered the development of practical control strategies for delaying transition in boundary layer flows (Kim and Bewley, 2007). Direct Numerical Simulations (DNS) have opened the way to accurate investigations of the underlying physics of transitional flows (Rai and Moin, 1993; Rist and Fasel, 1995; Wu and Moin, 2009). However, due to their high complexity and large number of degrees of freedom, nonlinear dynamical models that are based on the NS equations are not suitable for analysis, optimization, and control. On the other hand, nontrivial challenges, including lack of robustness, may arise in the model-based control of reduced-order models that are obtained using data-driven techniques (Noack et al., 2011).

Linearization of the NS equations around the mean-velocity profile results in models that are well-suited for analysis and control synthesis using tools from modern robust control (Kim and Bewley, 2007). In particular, stochastically forced linearized NS equations have been used to capture structural and statistical features of transitional (Farrell and Ioannou, 1993b; Bamieh and Dahleh, 2001; Jovanovic, 2004; Jovanovic and Bamieh, 2005) and turbulent (Jovanovic and Bamieh, 2001; Hwang and Cossu, 2010b; Moarref and Jovanovic, 2012; Zare et al., 2017b) channel flows. In these models, stochastic forcing may be utilized to model the impact of exogenous excitation sources or to capture the effect of nonlinear terms in the NS equations. Moreover, in conjunction with the parallel-flow assumption, the linearized NS equations are convenient for modal and non-modal stability analysis of spatially evolving flows (Schmid and Henningson, 2001; Schmid, 2007). However, this approach does not account for the effect of the spatially evolving base flow on the stability of the boundary layer. Global stability analysis addresses this issue by accounting for the spatially varying nature of the base flow and discretizing all inhomogeneous spatial directions (Ehrenstein and Gallaire, 2005; Åkervik et al., 2008a; Paredes et al., 2016). Although accurate, this approach leads to problem sizes that may be prohibitively large for flow control and optimization.
In the boundary layer flow, primary disturbances are instigated via receptivity processes that involve internal or external perturbations (Kachanov, 1994), e.g., acoustic noise, freestream turbulence, and surface roughness. Depending on the amplitude of these disturbances, the transition process may take various paths to breakdown (Morkovin et al., 1994; Saric et al., 2002). In particular, primary disturbances can be amplified through modal instability mechanisms or they may experience non-modal amplification, e.g., via transient growth, the lift-up (Landahl, 1975, 1980) and Orr mechanisms (Orr, 1907; Butler and Farrell, 1992). Both pathways can intensify disturbances beyond the critical threshold, trigger secondary instabilities, and induce a strong energy transfer from the mean flow into secondary modes (Herbert, 1988). The H-type (Kachanov and Levchenko, 1984; Herbert, 1988; Sayadi et al., 2013) and K-type (Klebanoff et al., 1962; Sayadi et al., 2013) transition scenarios are typical cases that are triggered by secondary instability mechanisms. Such mechanisms have also been shown to play an important role in the breakdown of laminar streaks at the later stages of transition (Andersson et al., 2001; Asai et al., 2002; Brandt and Henningson, 2002; Fransson et al., 2004; Hack and Zaki, 2016). All of these are initiated after the significant growth of the primary disturbances which intensify the role of nonlinear interactions. The modulation of the base flow by the primary perturbations precludes the usual normal-mode assumptions made in the derivation of the Orr-Sommerfeld equation. Instead, the physics of these secondary growth mechanisms have been commonly studied using Floquet analysis (Herbert, 1988; Andersson et al., 2001; Brandt, 2003) and the parabolized stability equations (PSE) (Joslin et al., 1993; Herbert, 1994, 1997).

The PSE were introduced to account for non-parallel and nonlinear effects and thereby overcome challenges associated with analyses based on a parallel-flow assumption. In particular, the PSE were developed as a means to refine predictions of parallel flow analysis in slowly varying flows (Bertolotti et al., 1992; Herbert, 1994, 1997), e.g., in the laminar boundary layer. The PSE have also been adapted to account for the dynamics of three-dimensional flows that depend strongly on two spatial directions (Galionis and Hall, 2005; De Tullio et al., 2013; Paredes et al., 2015), and more recently, they have been used to
model the amplification of disturbances in DNS and wall-modeled large-eddy simulation of transitional boundary layers (Lozano-Durán et al., 2018). In spite of these successes, the nonlinear nature of this framework has hindered their utility in systematic optimal flow control design. In general, the linear PSE provide reasonable predictions for the evolution of individual primary modes such as Tollmien-Schlichting (TS) waves (Herbert, 1994). Moreover, the predictive capability of the linear PSE has been further refined by modeling the effect of nonlinear terms as a stochastic source of excitation (Ran et al., 2017a). However, secondary growth mechanisms that lead to laminar-turbulent transition of the boundary layer flow originate from interactions between different modes and these interactions cannot be explicitly accounted for using such techniques.

In the transitional boundary layer, primary instability mechanisms can cause disturbances to grow to finite amplitudes and saturate at steady or quasi-steady states. Floquet stability analysis identifies secondary instability modes as the eigen-modes of the NS equations linearized around the modified base flow profile that contains spatially periodic primary velocity fluctuations. In the corresponding eigenvalue problem, the operators inherit a periodic structure from the underlying periodicity of the base flow and, as a result, capture primary-secondary mode interactions. Such representations that account for mode interactions also appear in the analysis of distributed systems with spatially or temporally periodic coefficients (Fardad et al., 2008; Jovanovic and Fardad, 2008) as well as in the model-based design of sensor-free periodic strategies for controlling transitional and turbulent wall-bounded shear flows (Jovanovic, 2008; Moarref and Jovanovic, 2010; Lieu et al., 2010; Moarref and Jovanovic, 2012).

1.3 Drag reduction by structured surface corrugation

Surface roughness typically increases skin-friction drag and degrades performance of engineering systems that involve the motion of rigid bodies in turbulent flows, e.g., ships and submarines with biofouled hulls (Schultz et al., 2011). Using both experiments and numerical simulations, Yusim and Utama (2017) reported an increase in skin-friction drag by
about 41% per year because of marine fouling growth. In contrast, carefully designed surface corrugations can reduce skin-friction drag by as much as 10% (Bechert et al., 1997; Gad-el Hak, 2000). Patterned surface modifications have been used to reduce drag in a number of engineering applications (Coustols and Savill, 1989). Success stories include the 2% drag reduction by spanwise-periodic riblets in commercial aircrafts (Szodruch, 1991), and the 7% drag reduction by shark-skin-inspired design of swimsuits for olympic swimmers (Benjannut-vatra et al., 2002; Mollendorf et al., 2004).

Given the potential economic benefits of riblets, many experimental and numerical studies have been dedicated to examining the dependence of skin-friction drag on design parameters (Walsh, 1982; Walsh and Lindemann, 1984; Choi et al., 1993; Bechert et al., 1997, 2000; García-Mayoral and Jiménez, 2011). These efforts provided a broad range of guidelines for characterizing the drag-reducing trends of riblets based on their size and shape (blade-like, triangular, T-shaped, etc.). In particular, turbulent drag-reduction as a function of various metrics of size (e.g., rib spacing or groove area) appears to follow a consistent trend over a host of riblet shapes, e.g., see Bechert et al. (1997, figure 15). For example, for small riblets (i.e., in the so-called “viscous” regime), the drag reduction is proportional to the riblet size. This linear trend gradually saturates at an optimal size before eventually degrading and leading to a drag increase for large riblets. Furthermore, for various riblet shapes, the optimal riblet spacing (in inner units) satisfies $s^+ \in [10, 20]$ (Bechert et al., 1997). García-Mayoral and Jiménez (2011) discovered that the cross-sectional area $A_g$ of the grooves provides the best predictor of drag-reducing trends over various shapes and identified $l_g^+ = \sqrt{A_g^+} \approx 10.7 \pm 1$ as the optimal size of riblets.

Beyond parametric studies, a considerable effort was made to uncover mechanisms responsible for drag reduction. The presence of small-size riblets results in the suppression of the cross-flows introduced by near-wall turbulence, which weakens the near-wall quasi-streamwise vortices and pushes them away from the wall. This limits the transfer of mean momentum toward the wall and creates a zone of suppressed turbulence within the grooves,
thereby reducing skin-friction drag (Choi et al., 1993; Sirovich and Karlsson, 1997; Lee and Lee, 2001).

Various notions of protrusion height have been proposed to quantify the effect of riblets on near-wall turbulence. Bechert and Bartenwerfer (1989) defined the protrusion height as the offset between the virtual origin for the mean flow and a measure of the average wall location. In contrast, Luchini et al. (1991) proposed to use the difference between the virtual origin for the streamwise and spanwise flows. For blade-like and scalloped riblets, the latter approach provides a good indicator of the shift in mean velocity and it offers a better surrogate for predicting drag reduction in the viscous regime. García-Mayoral et al. (2019); Ibrahim et al. (2019) proposed to quantify the shift in turbulence arising from quasi-streamwise vortices as a function of the wall-normal and spanwise slip lengths. This method can be used to predict the shift in the mean velocity, which is typically difficult to quantify in flows over complex surfaces. On the other hand, by examining 2D/3D roughness, Orlandi and Leonardi (2006) demonstrated a linear relation between the roughness function, i.e., shift of mean velocity in the logarithmic region and the rms of wall-normal velocity at the tip of roughness elements.

Both experiments and simulations have been used to demonstrate that the drag-reducing performance of riblets eventually saturates and degrades with increase in their size. Goldstein and Tuan (1998) suggested that the creation of small secondary streamwise vortices around the tips of riblets by the unsteady cross-flow degrades performance. Choi et al. (1993); Suzuki and Kasagi (1994); Lee and Lee (2001) related this phenomenon to the lodging of streamwise vortices into the grooves, which breaks down the viscous regime near the wall. More recently, the numerical study of García-Mayoral and Jiménez (2011) suggested that the breakdown of the viscous regime is accompanied by the emergence of spanwise rollers of typical streamwise length $\lambda_{x}^{+} \sim 150$ that develop from a two-dimensional Kelvin-Helmholtz (K-H) instability. The emergence of these coherent flow structures was also connected to an increase in the Reynolds shear stress in the vicinity of the corrugated surface.

While these studies offer valuable insights into drag reduction mechanisms, their reliance on costly experiments and simulations has hindered the model-based design of riblet-mounted
surfaces. This motivates the development of low-complexity models that capture the essential physics of turbulent flows over riblets and are well-suited for analysis, design, and optimization. Previously proposed notions of protrusion height (Bechert and Bartenwerfer, 1989; Luchini et al., 1991), spanwise slip length (García-Mayoral et al., 2019; Ibrahim et al., 2019), and the roughness function (Orlandi and Leonardi, 2006) provide surrogate measures for the performance of specific riblet geometries, but are typically constrained to the viscous regime. The self-regular model for wall turbulence regeneration proposed by Bandyopadhyay and Hellum (2014) accounts for the spatio-temporal evolution of flow structures over patterned surfaces and matches experimental results for transitional and turbulent flows at low Reynolds numbers. More recently, the receptivity of channel flow over riblets was studied using the $H_2$ norm of the linearized dynamics (Kasliwal et al., 2012) and the resolvent analysis (Chavarin and Luhar, 2019). While the former study used a change of coordinates to translate spatially-periodic geometry into spatially-periodic differential operators, the latter utilized a volume penalization technique to represent the effect of riblets as a feedback term in the dynamics. Moreover, Chavarin and Luhar (2019) showed that the dependence of the resolvent gain on the spacing of riblets closely follows previously reported drag-reducing trends in turbulent flows.

Prior model-based efforts have shown promise in predicting the energetics of turbulent flows in the presence of riblets. However, apart from Chavarin and Luhar (2019), such studies do not account for the interactions among harmonics of flow fluctuations that are induced by spatially-periodic geometry. Furthermore, in the absence of a systematic framework to quantify the influence of background turbulence on the mean velocity, prior studies cannot provide accurate predictions of skin-friction drag in the presence of riblets. In this work, we account for dynamical interactions and utilize turbulence modeling in conjunction with the eddy-viscosity-enhanced linearized NS equations to quantify the effect of background turbulence on skin-friction drag in turbulent channel flow over riblets.

The linearized NS equations have been used to capture structural and statistical features of transitional (Butler and Farrell, 1992; Trefethen et al., 1993; Farrell and Ioannou, 1993b;
Bamieh and Dahleh, 2001; Jovanovic, 2004; Jovanovic and Bamieh, 2005; Ran et al., 2019b) and turbulent (McKeon and Sharma, 2010; Hwang and Cossu, 2010b; Zare et al., 2017b, 2020) wall-bounded shear flows. In these studies, the effect of disturbances was modeled as an additive source of deterministic or stochastic excitation in the NS equations. Such an input-output approach (Jovanovic, 2020) has also been used for the model-based design of sensor-free control strategies for suppressing turbulence via streamwise traveling waves (Moarref and Jovanovic, 2010; Lieu et al., 2010) and transverse wall oscillations (Jovanovic, 2008; Moarref and Jovanovic, 2012), as well as feedback control strategies (Kim and Bewley, 2007) including opposition control (Luhar et al., 2014; Toedtli et al., 2019).

A challenging aspect of control design for turbulent flows is to quantify the effect of background turbulence on the mean velocity around which we study the dynamics of fluctuations. This effect is often captured by turbulent eddy viscosity models that are prescribed for specific flow configurations and do not account for the influence of control. To capture the influence of background turbulence, Moarref and Jovanovic (2012) developed a framework to determine the turbulent viscosity of channel flow in the presence of control from the statistics of the eddy-viscosity-enhanced linearized NS equations. This study showed that, by accounting for the influence of fluctuation dynamics on the turbulence model, reliable predictions of the mean velocity and the skin-friction drag can be obtained.

In this work, we extend the framework developed in Moarref and Jovanovic (2012) to quantify the effect of riblets on a turbulent channel flow. Following Chavarin and Luhar (2019) we use a volume penalization technique (Khadra et al., 2000) to approximate the effect of spatially-periodic surface on turbulent flow. This method introduces a static feedback term that captures the shape of riblets via a resistive function in the momentum equation. Additionally, we augment kinematic viscosity with turbulent eddy viscosity and examine the dynamics of flow fluctuations around the steady-state solution of the modified governing equations. The spatially-periodic nature of the mean flow introduces interaction between different harmonics of the mean and fluctuating velocity fields, which complicates frequency response analysis relative to the flow over smooth walls. We utilize the second-order statistics
of velocity fluctuations to determine the turbulent viscosity for the flow over riblets and compute their effect on the skin-friction drag.

We use our simulation-free approach to examine the effect of triangular riblets in turbulent channel flow. For various shapes and sizes of riblets, our results are in close agreement with experimental and numerical studies (Bechert et al., 1997; García-Mayoral and Jiménez, 2011). We also study the kinetic energy of velocity fluctuations and observe a strong correlation between energy suppression and drag-reduction trends for certain sizes of riblets. In addition, we use our model to examine dominant flow structures and mechanisms for drag reduction. The close agreement between our predictions and prior experimental and DNS results demonstrates that our model-based approach can be used for systematic design of periodic drag-reducing surfaces.

1.4 Dissertation structure

Our presentation is organized as follows:

In Part I, we discuss the receptivity analysis of boundary layer flows with stochastic forcing. Specifically, in Chapter 2, the problem formulation is given, and the stochastic model is discussed. In Chapter 3, receptivity analyses for boundary layer flows using our stochastic model are implemented with both local parallel and spatially developing base flows. Then in Chapter 4, the relation between receptivity analyses under local and global setups is discovered. Finally in Chapter 5 we summarize main contribution of Part I and discuss the future research directions.

Part II considers the modeling of mode interactions in the spatially-evolving boundary layer flows. In Chapter 6, we formulate the linear Parabolized Floquet Equations framework that can do the weakly nonlinear analysis of the boundary layers using its slowly evolving property. Moreover, two typical cases, i.e., H-type transition and nonlinear evolution of streaks of the boundary layer flows, are studied via our linear model in Chapter 7. In the end, we conclude Part II and discuss broader implications of our framework in Chapter 8.
Part III is devoted to the model-based framework for the analysis of drag reduction via riblets in turbulent channel flows. We first introduce the framework and our model’s procedures based on stochastically-forced Navier-Stokes equations in Chapter 9. Then in Chapter 10, we consider using triangular riblets to study the drag reduction, receptivity, and typical flow structures with our framework. In Chapter 11, we summarize Part III and outlines the possible extensions in the future work.

In Appendix. A, we discuss the sponge layers techniques that used in the global analysis of Chapter 3. Appendix. B provides the operators and formulas that used in Part I. And Appendix. C lists the operators of the Parabolized Floquet Equations. The operators and formulations of our model for drag reduction study in the presence of riblets are given in Appendix. D.

1.5 Preview of main results and contributions

In this section, the dissertation structure is provided along with the main contributions of each part.

Part I

**Receptivity analysis of boundary layer with stochastic input.** We utilize the externally forced linearized Navier-Stokes equations to study the receptivity of pre-transitional boundary layers to persistent stochastic excitation sources. Stochastic forcing is used to model the effect of free-stream turbulence that enters at various wall-normal locations, and the fluctuation dynamics are studied via linearized models that arise from locally parallel and global perspectives. In contrast to the widely used resolvent analysis that quantifies amplification mechanisms surrounding input-output pairs of identical temporal frequency, our approach pursues the steady-state response to white-in-time stochastic excitation. In addition to stochastic forcing with trivial (identity) covariance, we utilize the spatial spectrum of homogeneous isotropic turbulence to model the effect of free-stream turbulence.
Relating the local and global analysis of boundary layer flows. Although models based on the parallel flow assumption do not account for the effect of the spatially evolving base flow, we demonstrate that they capture essential trends and prevailing length-scales in stochastically forced boundary layer flows. On the other hand, the global flow analysis, which accounts for the spatially evolving nature of the boundary layer flow, predicts the amplification of a cascade of streamwise scales throughout the streamwise domain. We show that the flow structures that can be extracted from a modal decomposition of the resulting velocity covariance matrix, can be closely captured by a carefully conducted parallel flow analysis at various streamwise locations and over individual length-scales.

Part II

Linear model that account mode interactions in slowly-evolving boundary layers. We developed a successively linearized model to study interactions between different modes in boundary layer flows. Our method consists of two steps. First, we augment the boundary layer profiles with a disturbance field resulting from the linear Parabolized Stability Equations (PSE) to obtain the modified base flow; and, second, we draw on Floquet decomposition to capture the effect of mode interactions on the spatial evolution of flow fluctuations via a linear progression. The resulting Parabolized Floquet Equations (PFE) can be conveniently advanced downstream to examine the interaction between different modes in slowly varying shear flows. We apply our framework to two canonical settings of transition in boundary layers; the H-type transition scenario initiated by exponential instabilities, and streamwise elongated laminar streaks triggered by the lift-up mechanism. We demonstrate that the PFE captures the growth of various harmonics and provides excellent agreement with the results obtained in direct numerical simulations and experiments.

Part III

Model-based framework that is able to predict drag reduction induced by riblets. We impose a model-based approach to predict the drag reduction of turbulent channel flows
over corrugated surfaces using turbulent-eddy-viscosity enhanced Navier-Stokes equations. A volume penalization to the Navier-Stokes equations is introduced to account for the influence of corrugated surface to both the mean flow and velocity fluctuations. The dynamics of velocity fluctuations around the resulting base velocity profile are studied using the linearized Navier-Stokes equations that couples different harmonics due to the presence of periodic surface corrugation. The resulting second-order statistics of velocity fluctuations are adopted to determine a correction to the turbulent viscosity from the $k$-$\epsilon$ model. This correction in turn influences the turbulent mean velocity and modifies skin-friction drag. For triangular riblets, our simulation-free approach reliably predicts drag-reducing trends, typical flow structures, as well as mechanisms that lead to performance deterioration for large riblets. Meanwhile, we investigate the effect of height and spacing on drag reduction and demonstrate a correlation between energy suppression and drag-reduction for appropriately sized riblets. Our study paves the way for the optimal design of periodic surfaces using low complexity models to bypass the need for costly numerical simulations and experiments.
Part I

Stochastic receptivity analysis of boundary layer flow
Chapter 2

Stochastically forced linearized NS equations

The analysis, optimization, and control of dynamical models based on the Navier-Stokes (NS) equations is often hindered by their complexity and a large number of degrees of freedom. While the existence of coherent structures in wall-bounded shear flows (Smits et al., 2011) has inspired the development of reduced-order models using data-driven techniques, the essential features of such models can be crucially altered by control actuation and sensing. This gives rise to nontrivial challenges for model-based control design (Noack et al., 2011).

In contrast, the linearization of the NS equations around mean-velocity is well-suited for analysis and synthesis using modern robust control tools. Linearized models subject to stochastic excitation have been employed to replicate structural and statistical features of transitional (Farrell and Ioannou, 1993b; Bamieh and Dahleh, 2001; Jovanovic and Bamieh, 2005) and turbulent (Hwang and Cossu, 2010b; Moarref and Jovanovic, 2012; Zare et al., 2017b) wall-bounded shear flows. In these models, the nonlinear terms in the NS equations are replaced by stochastic forcing. Most studies have focused on parallel flow configurations in which translational invariance allows for decoupling the dynamical equations across streamwise and spanwise wavenumbers. This offers significant computational advantages for analysis, optimization, and control.

In the flat-plate boundary layer streamwise and normal inhomogeneity leads to the temporal eigenvalue problem for PDEs with two spatial variables. This problem is computationally more difficult to solve than for parallel flows. Previously, tools from sparse linear algebra and iterative schemes have been employed to analyze the spectra of the governing
equations and provide insight into the dynamics of transitional flows (Ehrenstein and Gal- 
laire, 2005; Åkervik et al., 2008b; Paredes, 2014). Efforts have also been made to conduct 
a non-modal analysis of spatially-evolving flows, including transient growth (Åkervik et al., 
2008b; Monokrousos et al., 2010), and resolvent (Jeun et al., 2016) analyses. Despite these 
successes, many challenges remain.

In this chapter, we formulate the linearized NS equations and employ tools from control 
theory to study receptivity mechanisms in boundary layer flows subject to stochastic sources 
of excitation. The effect of free-stream turbulence is modeled as a persistent white-in-time 
stochastic forcing that enters at various wall-normal locations.

### 2.1 Problem formulation

In a flat-plate boundary layer, the linearized incompressible NS equations around the Blasius 
base flow profile \( \bar{u} = [U(x, y) \ V(x, y) \ 0]^T \) are given by

\[
\begin{align*}
\mathbf{v}_t &= - (\nabla \cdot \bar{u}) \mathbf{v} - (\nabla \cdot \mathbf{v}) \bar{u} - \nabla p + \frac{1}{Re_0} \Delta \mathbf{v} + \mathbf{d}, \\
0 &= \nabla \cdot \mathbf{v},
\end{align*}
\]  

(2.1)

where \( \mathbf{v} = [u \ v \ w]^T \) is the vector of velocity fluctuations, \( p \) denotes pressure fluctuations, \( u \), 
\( v \), and \( w \) represent components of the fluctuating velocity field in the streamwise \( (x) \), wall-
normal \( (y) \), and spanwise \( (z) \) directions, and \( \mathbf{d} \) denotes an additive zero-mean stochastic 
body forcing. The stochastic perturbation \( \mathbf{d} \) is used to model the effect of exogenous sources 
of excitation on the boundary layer flow and, as illustrated in figure 2.1, it can be introduced 
in various wall-normal regions. In Eqs. (2.1), \( Re_0 = U_\infty \delta_0/\nu \) is the Reynolds number based 
on the Blasius length-scale \( \delta_0 = \sqrt{\nu x_0/U_\infty} \), where the initial streamwise location \( x_0 \) denotes 
the distance from the leading edge, \( U_\infty \) is the free-stream velocity, and \( \nu \) is the kinematic 
viscosity. The local Reynolds number at distance \( x \) to the starting position \( x_0 \) is thus given 
by \( Re = Re_0 \sqrt{1 + x/x_0} \). The velocities are non-dimensionalized by \( U_\infty \), time by \( \delta_0/U_\infty \), and 
pressure by \( \rho U_\infty^2 \), where \( \rho \) is the fluid density.
2.1.1 Evolution model

Elimination of the pressure yields an evolution form of the linearized equations with the state variable \( \varphi = [v \ \eta]^T \), which contains the wall-normal velocity \( v \) and vorticity \( \eta = \partial_z u - \partial_x w \) (Schmid and Henningson, 2001). In addition, homogeneity of the Blasius base flow in the spanwise direction allows a normal-mode representation with respect to \( z \), yielding the evolution model

\[
\begin{align*}
\frac{\partial \varphi(x, y, k_z, t)}{\partial t} & = [A(k_z) \varphi(\cdot, k_z, t)](x, y) + [B(k_z) d(\cdot, k_z, t)](x, y), \\
v(x, y, k_z, t) & = [C(k_z) \varphi(\cdot, k_z, t)](x, y),
\end{align*}
\]

(2.2)

which is parameterized by the spanwise wavenumber \( k_z \). Definitions of the operators \( A, B, \) and \( C \) are provided in Appendix B.1. We note that an additional wall-parallel base flow assumption that entails \( \bar{u} = [U(y) \ 0 \ 0]^T \) renders the coefficients in Eqs. (2.1) independent of \( x \) and thus enables a normal-mode representation in that dimension as well.

We obtain finite-dimensional approximations of the operators in Eqs. (2.2) using a pseudospectral discretization scheme (Weideman and Reddy, 2000) in the spatially inhomogeneous directions. For streamwise-varying base flows we consider \( N_x \) and \( N_y \) Chebyshev collocation points in \( x \) and \( y \), and for streamwise invariant base flows we use \( N_y \) points in \( y \). Furthermore, a change of variables is employed to obtain a state-space representation.
in which the kinetic energy is determined by the Euclidean norm of the state vector; see Appendix B.2. We thus arrive at the state-space model

\[
\dot{\psi}(t) = A\psi(t) + B d(t), \\
v(t) = C\psi(t),
\]

where \(\psi(t)\) and \(v(t)\) are vectors with \(2N_x N_y\) and \(3N_x N_y\) complex-valued components, respectively (\(2N_y\) and \(3N_y\) components, respectively, for parallel flows), and state-space matrices \(A\), \(B\), and \(C\) incorporate the aforementioned change of variables and wavenumber parameterization over \(k_z\) (over \((k_x, k_z)\) for parallel flows).

### 2.1.2 Second-order statistics of velocity fluctuations

We next characterize the structural dependence between the second-order statistics of the state and forcing term in the linearized dynamics. We also describe how the energy amplification arising from persistent stochastic excitation and the energetically dominant flow structures can be computed from these flow statistics. All mathematical statements in the remainder of this section are parameterized over homogeneous directions.

In statistical steady-state, the covariance matrices \(\Phi = \lim_{t \to \infty} \langle v(t) v^*(t) \rangle\) of the velocity fluctuation vector and \(X = \lim_{t \to \infty} \langle \psi(t) \psi^*(t) \rangle\) of the state vector in Eq. (2.3) are related by

\[
\Phi = C X C^*,
\]

where \(\langle \cdot \rangle\) denotes the expectation and superscript * denotes complex conjugate transpose. The matrix \(\Phi\) contains information about all second-order statistics of the fluctuating velocity field in statistical steady-state, including the Reynolds stresses. We assume that the persistent source of excitation \(d(t)\) in Eq. (2.3) is zero-mean and white-in-time with spatial covariance matrix \(W = W^*\),

\[
\langle d(t_1) d^*(t_2) \rangle = W \delta(t_1 - t_2),
\]

where \(\delta\) is the Dirac delta function.
where $\delta$ is the Dirac delta function. When the linearized dynamics (2.3) are stable, the steady-state covariance $X$ of the state $\psi(t)$ can be determined as the solution to the algebraic Lyapunov equation

$$AX +XA^* = -BB^*.$$ (2.6)

The Lyapunov equation (2.6) relates the statistics of white-in-time forcing, represented by $W$, to the infinite-horizon state covariance $X$ via system matrices $A$ and $B$. It can also be used to compute the energy spectrum of velocity fluctuations $v$,

$$E = \text{trace} (\Phi) = \text{trace} (CXC^*) .$$ (2.7)

We note that the steady-state velocity covariance matrix $\Phi$ can be alternatively obtained from the spectral density matrix of velocity fluctuations $S_v(\omega)$ as (Kwakernaak and Sivan, 1972),

$$\Phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_v(\omega) \, d\omega .$$

For the linearized NS equations, we have

$$S_v(\omega) := T_{vd}(\omega) W T_{vd}^*(\omega)$$ (2.8)

where the frequency response matrix

$$T_{vd}(\omega) = C (i\omega I - A)^{-1} B,$$ (2.9)

is obtained by applying the temporal Fourier transform on system (2.3). We note that the solution $X$ to the algebraic Lyapunov equation (2.6) allows us to avoid integration over temporal frequencies and compute the energy spectrum $E$ using (2.7); see Section 4.3 for additional details.
Following the proper orthogonal decomposition of (Bakewell and Lumley, 1967; Moin and Moser, 1989), the velocity field can be decomposed into characteristic eddies by determining the spatial structure of fluctuations that contribute most to the energy amplification. For turbulent channel flow, it has been shown that the dominant characteristic eddy structures extracted from second-order statistics of the stochastically forced linearized model qualitatively agree with results obtained using eigenvalue decomposition of DNS-generated autocorrelation matrices; see figures 15 in (Moarref and Jovanovic, 2012) and (Moin and Moser, 1989). In addition to examining the energy spectrum of velocity fluctuations, we will use the eigenvectors of the covariance matrix $\Phi$ (defined in Eq. (2.4)) to study dominant flow structures that are triggered by stochastic excitation.

Remark 1 Since linearized dynamics (2.3) are globally stable even when the flow is convectively unstable (Huerre and Monkewitz, 1990), the Lyapunov-based approach can be used to conduct the steady-state analysis of the velocity fluctuations statistics for many flow configurations that are not stable from the perspective of local analysis.

2.1.3 Filtered excitation and receptivity coefficient

Let us specify the spatial region in which the forcing enters, by introducing

$$d(x, y, z, t) := f(y) h(x) d_s(x, y, z, t), \tag{2.10}$$

where $d_s$ represents a white solenoidal forcing, $f(y)$ is a smooth filter function defined as

$$f(y) := \frac{1}{\pi} \left( \tan(a(y - y_1)) - \tan(a(y - y_2)) \right), \tag{2.11}$$

and $h(x)$ is a filter function that determines the streamwise extent of the forcing. Here, $y_1$ and $y_2$ determine the wall-normal extent of $f(y)$ and $a$ specifies the roll-off rate; figure 2.2 shows $f(y)$ with $y_1 = 5$ and $y_2 = 10$, for two cases of $a = 1$ and $a = 10$. In Sections 3.1 and 3.2, we study energy amplification arising from stochastic excitation that enters at various wall-normal locations; see Table 2.1. For the near-wall forcing (with $y_1 = 0$ and
with \( a = 1 \), more than 96% of the energy of the forcing is applied within the \( \delta_{0.99} \) boundary layer thickness; on the other hand, for the outer-layer forcing (with \( y_1 = 15 \) and \( y_2 = 20 \)) with \( a = 1 \), less than 0.1% is applied in that region. Our study mainly focuses on the forcing with \( h(x) = 1 \); the effect of changing the function \( h \) is considered in Section 3.2.2.

**Table 2.1:** Cases of stochastic excitation entering at various wall-normal regions

<table>
<thead>
<tr>
<th>case number</th>
<th>wall-normal region of excitation; ([y_1, y_2]) in Eq. (2.11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (near-wall)</td>
<td>([0, 5])</td>
</tr>
<tr>
<td>2</td>
<td>([5, 10])</td>
</tr>
<tr>
<td>3</td>
<td>([10, 15])</td>
</tr>
<tr>
<td>4 (outer-layer)</td>
<td>([15, 20])</td>
</tr>
</tbody>
</table>

**Figure 2.2:** The shape of the filter function \( f(y) \) for \( y_1 = 5, \ y_2 = 10 \) with \( a = 1 \) (−) and \( a = 10 \) (−−).

We quantify the receptivity of velocity fluctuations to stochastic forcing that enters at various wall-normal regions using the receptivity coefficient

\[
C_R := \frac{\lim_{t \to \infty} \langle (D_g v(t))^* D_g v(t) \rangle}{\lim_{t \to \infty} \langle d^*(t) d(t) \rangle} = \frac{\text{trace} \left( D_g \Phi D_g^* \right)}{\text{trace} \left( W \right)},
\]  

(2.12)
which determines the ratio of the energy of velocity fluctuations within the boundary layer to the energy of the forcing. Here, $D_g := g(x, y)I$, where the function $g(x, y)$ is a top-hat filter that extracts velocity fluctuations within the $\delta_{0.99}$ boundary layer thickness. In parallel flows, the function $g$ is invariant with respect to the streamwise direction.
Chapter 3

Receptivity analysis of boundary layer flows

In this chapter, we employ the linearized Navier-Stokes equations formulated in Chapter 2 to study the receptivity analyses for boundary layer flows. The energy response, as well as the dominant flow structures, are investigated using the fluctuation dynamics linearized around both locally-parallel and spatially developing base flows subject to persistent stochastic forcing enters at various wall-normal locations.

3.1 Receptivity analysis of locally parallel flow

We first examine the dynamics of the stochastically forced Blasius boundary layer under the locally parallel flow assumption. In this case, the base flow only depends on the wall-normal coordinate $y$ and evolution model (2.3) is parameterized by horizontal wavenumbers $(k_x, k_z)$, which significantly reduces the computational complexity. We perform an input-output analysis to quantify the energy amplification of velocity fluctuations subject to free-stream turbulence.

We compute the energy spectrum of stochastically excited parallel Blasius boundary layer flow with $Re_0 = 232$ (the Blasius length-scale is $\delta_0 = 1$). Here, we consider a wall-normal region with $L_y = 35$ and discretize the differential operators in Eqs. (2.2) using $N_y = 100$ Chebyshev collocation points in $y$. In the wall-normal direction, homogenous Dirichlet boundary conditions are imposed on wall-normal vorticity, $\eta(0) = \eta(L_y) = 0$ and Dirichlet/Neumann boundary conditions are imposed on wall-normal velocity, $v(0) = v(L_y) = 0$, $v_y(0) = v_y(L_y) = 0$, where $v_y$ denotes the derivative of $v$ with respect to $y$. In the
horizontal directions, we use $50 \times 51$ logarithmically spaced wavenumbers with $k_x \in [10^{-4}, 1]$ and $k_z \in [5 \times 10^{-3}, 10]$ to parameterize the linearized model (2.3). Thus, for each pair $(k_x, k_z)$, the state $\psi = [v^T \eta^T]^T$ is a complex-valued vector with $2N_y$ components. Grid convergence has been verified by doubling the number of points used in the discretization of the differential operators in the wall-normal coordinate.

### 3.1.1 Energy response

We first consider a streamwise invariant ($h(x) = 1$) solenoidal white-in-time excitation $d$ with covariance $W = I$ in the immediate vicinity of the wall (case 1 in Table 2.1). Figure 3.1(a) shows largest receptivity at low streamwise wavenumbers ($k_x \approx 0$) with a global peak at $k_z \approx 0.25$. This indicates that streamwise elongated streaks are the dominant flow structures that result from persistent stochastic excitation of the boundary layer flow. Such streamwise elongated structures are reminiscent of energetically dominant streaks with spanwise wavenumbers $k_z \approx 0.26$ (in Blasius length-scale) that were identified in analyses of optimal disturbances (Andersson et al., 1999; Luchini, 2000). Slightly smaller spanwise wavenumbers have been recorded from hot-wire signal correlations in the boundary layer subject to free-stream turbulence (Matsubara and Alfredsson, 2001). In addition to streaks, figure 3.1(a) also predicts the emergence of TS waves at $k_x \approx 0.19$. For outer-layer forcing, the amplification of streamwise elongated structures persists while the amplification of the TS waves weakens; see figure 3.1(b). It is also observed that as the region of excitation moves away from the wall, energy amplification becomes weaker and the peak of the receptivity coefficient shifts to lower values of $k_z$. As we demonstrate in Section 3.2, these observations are in agreement with the global receptivity analysis of stochastically excited boundary layer flow.

As noted in Section 2.1.2, the solution $X$ to Lyapunov equation (2.6) represents the steady-state (i.e., long-time average) covariance matrix of the state $\psi$ of stochastically forced linearized evolution model (2.3), which can be used to compute the energy spectrum in Eq. (2.7) or the receptivity coefficient in Eq. (2.12) in a simulation-free manner. To verify the
values of $C_R$ reported in figure 3.1, we conduct stochastic simulations of the forced linearized flow equations at the wavenumber pair $(k_x, k_z) = (0.19, 0.005)$, which is marked by the red dot in figure 3.1(a). This wavenumber pair allows us to examine the amplification of TS waves identified in figure 3.1(a). Since proper comparison with the result of the Lyapunov equation requires ensemble-averaging, rather than comparison at the level of individual stochastic simulations, we have conducted twenty simulations of system (2.3). The total simulation time was set to $1.6 \times 10^4$ dimensionless time units. Figure 3.2 shows the time evolution of $C_R$ for twenty realizations of white-in-time forcing $\mathbf{d}$ to system (2.3). The receptivity coefficient averaged over all simulations is marked by the thick black line. The results indicate that the average of the sample sets asymptotically approaches the correct steady-state value of $C_R$.

The one-dimensional energy spectrum shown in figure 3.3(a) quantifies the energy amplification $E$ over various spanwise wavenumbers when forcing enters at different distances from the wall. This quantity can be computed by integrating the energy spectrum $E(k_x, k_z)$ (cf. Eq. (2.7)) over streamwise wavenumbers. In figure 3.3, the locations at which the energy spectrum peaks correspond to the spanwise scale associated with streamwise elongated
Figure 3.2: Time evolution of the receptivity coefficient $C_R$ for twenty realizations of near-wall stochastic forcing to linearized dynamics (2.3) with $(k_x, k_z) = (0.19, 0.005)$ and $Re_0 = 232$. The receptivity coefficient averaged over all simulations is marked by the thick black line.

Figure 3.3: (a) The one-dimensional energy spectrum, and (b) the receptivity coefficient for the parallel Blasius boundary layer flow with $Re_0 = 232$ subject to white stochastic excitation entering in the wall-normal regions covered in Table 2.1; case 1 (black), case 2 (blue), case 3 (red), and case 4 (green). The forcing region moves away from the wall in the direction of the arrows.

streaks. When the forcing region shifts away from the wall, the energy amplification decreases, indicating that the flow region in the immediate vicinity of the wall is more susceptible to external excitation. As mentioned earlier, we also observe that, when the forcing region shifts upward, the boundary layer streaks become wider in the spanwise direction. Figure 3.3(b) shows similar trends in the receptivity coefficient as a function of spanwise
wavenumber $k_z$, which is computed by integrating $C_R$ presented in figure 3.1 over streamwise wavenumbers.

3.1.2 Most energetic flow structures

The eigenvalue decomposition of the velocity covariance matrix $\Phi$ can be used to identify the energetically dominant flow structures resulting from stochastic excitation. In particular, symmetries in the wall-parallel directions can be used to express velocity components as

$$u_j(x, z, t) = 4 \cos(k_z z) \Re \left( \tilde{u}_j(k_x, k_z) e^{ik_x x} \right),$$

$$v_j(x, z, t) = 4 \cos(k_z z) \Re \left( \tilde{v}_j(k_x, k_z) e^{ik_x x} \right),$$

$$w_j(x, z, t) = -4 \sin(k_z z) \Im \left( \tilde{w}_j(k_x, k_z) e^{ik_x x} \right),$$

(3.1)

Here, $\Re$ and $\Im$ denote real and imaginary parts, and $\tilde{u}_j$, $\tilde{v}_j$, and $\tilde{w}_j$ correspond to the streamwise, wall-normal, and spanwise components of the $j$th eigenvector of the matrix $\Phi$ in Eq. (2.4). While all amplitudes have been normalized, the phase of these components have been modulated to ensure the compactness of $v_j(x, y, z)$ around $z = 0$ (Moin and Moser, 1989); see (Moarref and Jovanovic, 2012, Appendix F) for additional details.

![Figure 3.4](image)

**Figure 3.4:** Contribution of the first 8 eigenvalues of the velocity covariance matrix $\Phi$ of the Blasius boundary layer flow with $Re_0 = 232$ subject to (a) near-wall, and (b) outer-layer white-in-time stochastic forcing.
Figure 3.5: Principal modes with \((k_x, k_z) = (7 \times 10^{-3}, 0.32)\), resulting from excitation of the boundary layer flow with \(Re_0 = 232\) in the vicinity of the wall. (a) Streamwise velocity component where red and blue colors denote regions of high and low velocity. (b) Streamwise velocity at \(z = 0\). (c) \(y-z\) slice of streamwise velocity (color plots) and vorticity (contour lines) at \(x = 500\), which corresponds to the cross-plane slice indicated by the black dashed lines in (b).

Figure 3.6: Principal modes with \((k_x, k_z) = (7 \times 10^{-3}, 0.15)\), resulting from outer-layer excitation of the boundary layer flow with \(Re_0 = 232\). (a) Streamwise velocity component where red and blue colors denote regions of high and low velocity. (b) Streamwise velocity at \(z = 0\). (c) \(y-z\) slice of streamwise velocity (color plots) and vorticity (contour lines) at \(x = 500\), which corresponds to the cross-plane slice indicated by the black dashed lines in (b).

While the sum of all eigenvalues of the matrix \(\Phi\) determines the overall energy amplification reported in figure 3.3(a), it is also useful to examine the spatial structure of modes with dominant contribution to the energy of the flow. Figure 3.4 shows the contribution of the first 8 eigenvalues of \(\Phi\) to the energy amplification, \(\lambda_j / \sum \lambda_i\) when the boundary layer flow is subject to stochastic forcing. For fluctuations with \((k_x, k_z) = (7 \times 10^{-3}, 0.32)\) and near-wall excitation (cross in figure 3.1(a)) the principal mode which corresponds to the largest eigenvalue, contains approximately 93% of the total energy. On the other hand, for fluctuations with \((k_x, k_z) = (7 \times 10^{-3}, 0.15)\) and outer-layer excitation (cross in figure 3.1(b)) the principal mode contains approximately 52% of the total energy. Figures 3.5 and 3.6
show the flow structures associated with the streamwise component of these most significant modes. From figures 3.5(b) and 3.6(b) it is evident that the core of streamwise elongated structures moves away from the wall with the shift of the stochastically excited region. These streamwise elongated structures are situated between counter-rotating vortical motions in the cross-stream plane (cf. figures 3.5(c) and 3.6(c)) and contain alternating regions of fast- and slow-moving fluid, which are slightly inclined (and detached) relative to the wall. Even though these structures do not capture the full complexity of transitional flow, as we show in Section 3.2, they contain information about energetic streamwise elongated flow structures that are amplified by external excitation of the boundary layer flow. In particular, such alignment of counter-rotating vortices and streaks is closely related to the lift-up mechanism and the generation of streamwise elongated streaks (Andersson et al., 1999; Luchini, 2000; Hack and Zaki, 2015).

### 3.2 Global analysis of stochastically forced linearized NS equations

The parallel flow assumption applied in Section 3.1 allows for the efficient parameterization of the governing equations over all wall-parallel wavenumbers $k_x$ and $k_z$. While this significantly reduces computational complexity, it excludes the effect of the spatially evolving base flow on the dynamics of velocity fluctuations. In global stability analysis, the NS equations are linearized around a spatially evolving Blasius boundary layer profile and the finite dimensional approximation is obtained by discretizing all inhomogeneous spatial directions. In this section, we employ global receptivity analysis to quantify the influence of stochastic excitation on the velocity fluctuations around the spatially evolving Blasius boundary layer base flow.

At any spanwise wavenumber $k_z$, the state $\psi = [v^T \eta^T]^T$ of linearized evolution model (2.3) is a complex vector with $2N_xN_y$ components, where $N_x$ and $N_y$ denote the number of collocation points used to discretize the differential operators in the streamwise and wall-normal directions, respectively. While this choice of state variables is not commonly used in conventional global stability analysis of boundary layer flows, in Appendix B.3 we
demonstrate that it yields consistent results with the descriptor form in which the state is determined by all velocity and pressure fluctuations. We consider a Reynolds number $Re_0 = 232$ and a computational domain with $L_x = 900$ and $L_y = 35$, where the differential operators are discretized using $N_x = 101$ and $N_y = 50$ Chebyshev collocation points in $x$ and $y$, respectively. Similar to locally parallel analysis, we verify convergence by doubling the number of grid points.

As in Chapter 3.1, in the wall-normal direction we enforce homogeneous Dirichlet boundary conditions on $\eta$ and homogeneous Dirichlet/Neumann boundary conditions on $v$. At the inflow, we impose homogeneous Dirichlet boundary conditions on $\eta$, i.e., $\eta(0,y) = 0$, and homogeneous Dirichlet/Neumann boundary conditions on $v$, i.e., $v(0,y) = v_y(0,y) = 0$. At the outflow, we apply linear extrapolation conditions on both state variables $(v, \eta)$ and the wall-normal derivative of the first component (Theofilis, 2003a),

$$
v(x(N_x), y) = \alpha v(x(N_x - 1), y) + \beta v(x(N_x - 2), y),
$$
$$
\eta(x(N_x), y) = \alpha \eta(x(N_x - 1), y) + \beta \eta(x(N_x - 2), y),
$$
$$
v_y(x(N_x), y) = \alpha v_y(x(N_x - 1), y) + \beta v_y(x(N_x - 2), y),
$$

\[ \alpha = \frac{x(N_x) - x(N_x - 2)}{x(N_x - 1) - x(N_x - 2)}, \quad \beta = \frac{x(N_x - 1) - x(N_x)}{x(N_x - 1) - x(N_x - 2)}. \]

We also introduce sponge layers at the inflow and outflow to mitigate the influence of boundary conditions on the fluctuation dynamics within the computational domain (Nichols and Lele, 2011; Mani, 2012); see (Ran et al., 2017b) for an in-depth study on the effect of sponge layer strength in the global stability analysis of boundary layer flow. The results presented in this section are obtained after adjusting the sponge layer parameters to match the energy amplification obtained via the descriptor form of the linearized dynamics; see Appendix B.3 for details.
3.2.1 Response to white-in-time stochastic forcing

For boundary layer flows, the global operator in Eqs. (2.3) has no exponentially growing eigenmodes (Huerre and Monkewitz, 1990); see Remark 1. Thus, the steady-state covariance of the fluctuating velocity field can be obtained from the solution to Lyapunov equation (2.6) and the energy amplification can be computed using Eq. (2.7). As in Section 3.1, we examine the influence of streamwise-invariant \((h(x) = 1)\) white-in-time stochastic forcing with covariance \(W = I\) which enters at various wall-normal regions; this is achieved by filtering the forcing using the function \(f(y)\) in Eq. (2.11). Figure 3.7 shows the \(k_z\)-dependence of energy amplification and receptivity coefficient for stochastic excitation entering at various wall-normal regions. Our computations show that the energy amplification increases as the region of influence for the external forcing approaches the wall, which qualitatively matches the result of the locally parallel analysis in Section 3.1. In particular, for \(Re_0 = 232\), the energy amplification reduces from \(2.0 \times 10^6\) (for stochastic excitation that enters in the vicinity of the wall (case 1 in Table 2.1) with \(k_z = 0.32\)) to \(9.6 \times 10^4\) (for stochastic excitation that enters away from the wall (case 4 in Table 2.1) with \(k_z = 0.21\)). Moreover, the structures that correspond to the largest energy amplification or receptivity coefficient become slightly wider in the spanwise direction, but this shift to smaller values of \(k_z\) is not as pronounced as in parallel flows (cf. figure 3.3). The largest energy amplification and receptivity are observed for structures with \(k_z \in [0.21, 0.32]\), which is in close agreement with previous experimental (Matsubara and Alfredsson, 2001) and theoretical studies (Andersson et al., 1999; Luchini, 2000).

For \(k_z = 0.32\), figure 3.8 shows the contribution of the first 50 eigenvalues of the velocity covariance matrix \(\Phi\) resulting from near-wall and outer-layer stochastic excitation. In contrast to locally parallel analysis (cf. figure 3.4), we observe that other eigenvalues play a more prominent role. The implication is that in global analysis the principal eigenmode of \(\Phi\) cannot capture the full complexity of the spatially evolving flow. Nevertheless, we examine the shape of such flow structures to gain insight into the effect of stochastic excitation on the eigenmodes of the covariance matrix \(\Phi\) that comprise the fluctuation field. Figures 3.9(a)
Figure 3.7: (a) Energy amplification and (b) receptivity coefficient resulting from stochastic excitation of the linearized NS equations around a spatially varying Blasius profile with $Re_0 = 232$. Stochastic forcing enters at the wall-normal regions covered in Table 2.1; case 1 (black), case 2 (blue), case 3 (red), and case 4 (green). The forcing region moves away from the wall in the direction of the arrows.

Figure 3.8: Contribution of the first 50 eigenvalues of the velocity covariance matrix $\Phi$ of the Blasius boundary layer flow with $Re_0 = 232$ subject to white-in-time stochastic forcing (a) in the vicinity of the wall with spanwise wavenumber $k_z = 0.32$; and (b) away from the wall with spanwise wavenumber $k_z = 0.21$.

and 3.10(a) show the spatial structure of the streamwise component of the principal response to white-in-time stochastic forcing that enters in the vicinity of the wall and in the outer-layer, respectively. The streamwise growth of the streaks can be observed. Figures 3.9(b) and 3.10(b) display the cross-section of these streamwise elongated structures at $z = 0$. As the forcing region gets detached from the wall, the cores of the streaky structures also
move away from it. As shown in figures 3.9(c) and 3.10(c), these streaky structures are situated between counter-rotating vortical motions in the cross-stream plane and they contain alternating regions of fast- and slow-moving fluid that are slightly inclined to the wall.

We next examine the spatial structure of less energetic eigenmodes of $\Phi$. As illustrated in figure 3.8(a), for near-wall stochastic forcing the first six eigenmodes respectively contribute 8.9%, 7.3%, 6.1%, 5.3%, 4.6%, and 4.0% to the total energy amplification. We again use the streamwise velocity component to study the spatial structure of the corresponding eigenmodes. As shown in figure 3.11(b), while the principal mode consists of a single
Figure 3.11: Streamwise velocity at $z = 0$ corresponding to the first six eigenmodes of the steady-state covariance matrix $\Phi$ resulting from near-wall excitation of the boundary layer flow with $Re_0 = 232$ and at $k_z = 0.32$; (a) $j = 1$, (b) $j = 2$, (c) $j = 3$, (d) $j = 4$, (e) $j = 5$, and (f) $j = 6$ where $j$ corresponds to ordering in figure 3.8(a).

streamwise-elongated streak, the second mode is comprised of two shorter high- and low-speed streaks. Similarly, the third and fourth modes respectively contain three and four streaks. These streaks become shorter in the streamwise direction and their energy content reduces; see figures 3.11(c) and 3.11(d). As the mode number increases, the streamwise extent of these structures further reduces, they appear at an earlier streamwise location, and their peak value moves closer to the leading edge. This breakup into shorter streaks for higher modes can be related to the dominant modes identified in locally parallel analysis for increasingly larger streamwise wavenumbers and at various streamwise locations (or Reynolds numbers).

As shown in figure 3.11, spatial visualization of various eigenmodes of $\Phi$ resulting from global receptivity analysis uncovers approximately periodic flow structures in the streamwise direction. The fundamental spatial frequency extracted from the streamwise variation of the principal eigenmode of $\Phi$ provides information about the streamwise length-scales associated with the dominant flow structures. Figure 3.12(a) shows the dominant TS wave-like spatial
Figure 3.12: Blasius boundary layer flow with initial Reynolds number $Re_0 = 232$ subject to white-in-time stochastic excitation of the near-wall region (case 1 in Table 2.1). (a) The TS wave-like spatial structure of the streamwise velocity component of the principal eigenmode of the matrix $\Phi$ at $k_z = 0.01$; (b) Fourier transform in streamwise dimension; and (c) the distribution of streamwise length-scales obtained from various eigenmodes of the covariance matrix $\Phi$ for various values of the spanwise wavenumber $k_z$. Filled dots represent the dominant streamwise wavenumber associated with the principal eigenmode of $\Phi$, the red dot corresponds to the fundamental wavenumber extracted from (b), and circles are the streamwise wavenumbers resulting from less significant eigenmodes.

Structure that results from near-wall stochastic excitation of the boundary layer flow with $Re_0 = 232$ and $k_z = 0.01$. The Fourier transform in the streamwise direction can be used to extract the fundamental value of $k_x$ associated with this spatial structure. As illustrated in figure 3.12(b), the Fourier coefficient peaks at $k_x \approx 0.1$, which corresponds to the most significant streamwise flow structures (cf. figure 3.12(a)). The identified fundamental wavenumber is representative of the streamwise variation of this flow structure and it provides a good approximation of the dominant value of $k_x$ that is excited by the near-wall forcing. For different values of $k_z$, the filled black dots in figure 3.12(c) denote the streamwise wavenumbers extracted from the principal eigenmodes of the covariance matrix $\Phi$, which contribute most to the energy amplification. The circles represent the tail of streamwise wavenumbers extracted from other eigenmodes of the matrix $\Phi$. As shown in figure 3.11, for any $k_z$, less significant eigenmodes are associated with flow structures that are shorter in the streamwise direction. The observed trends are in close agreement with the results obtained using locally parallel analysis (cf. figure 3.1(a)). In particular, streamwise elongated structures are most amplified for $k_z \approx 0.3$. On the other hand, for low spanwise wavenumbers, the TS wave-like structures are most amplified for $k_x \gtrsim 0.1$ (cf. $k_x \approx 0.19$ from locally parallel analysis).
3.2.2 Modeling the effect of homogeneous isotropic turbulence

So far, we have studied the energy amplification of the boundary layer flow subject to persistent white-in-time stochastic excitation with a trivial covariance matrix ($W = I$). It is also of interest to model the effect of free-stream turbulence on the boundary layer flow using Homogeneous Isotropic Turbulence (HIT) (Brandt et al., 2004). The spectrum of HIT has been previously used as an initial condition to study transient growth in boundary layer flows based on the temporal evolution of the solution to the differential Lyapunov equation (Hœpffner and Brandt, 2008). Herein, we consider the persistent stochastic forcing $d$ in system (2.3) to be of the form defined in Eq. (2.10). The filter function $h(x) := 10^{-rx/L_x}$ is used to model the streamwise decay of turbulence intensity (cf. (Brandt et al., 2004, figure 2)) and the spatial covariance matrix $W$ of the forcing term $d_s$ is selected to match the spectrum of HIT; see Appendix B.4 for additional details. We utilize such forcing model as well as the input matrix $B$ in the infinite-horizon Lyapunov equation (2.6) to compute the steady-state covariance matrix $X$ and determine the corresponding energy spectrum via Eq. (2.7).

![Figure 3.13](image)

**Figure 3.13:** (a) Receptivity coefficient resulting from HIT-based stochastic excitation of system (2.3) with $Re_0 = 232$ in the near-wall (black) and outer-layer (blue) regions. The solid lines correspond to streamwise-invariant forcing ($r = 0$ in $h(x)$) and the dashed lines correspond to streamwise decaying forcing with a decay rate of $r = 1.5$ in $h(x)$. (b) Receptivity coefficient corresponding to the streamwise-invariant ($r = 0$ in $h(x)$) HIT-based forcing (solid) and white-in-time forcing with covariance $W = I$ (dotted) entering in the near-wall region.

We first study the receptivity of the linearized NS equations to HIT-based stochastic forcing. The receptivity coefficient as a function of spanwise wavenumber $k_z$ is shown in

41
As shown in this figure, the streamwise decay of forcing using the filter function $h(x) = 10^{-rx/L_x}$ has a minimal damping effect on the receptivity coefficient. Figure 3.13(b) illustrates a similar trend in the receptivity coefficient obtained from both types of white-in-time stochastic forcing, which suggests that stochastic forcing with covariance $W = I$ provides a reasonable approximation of the effect of HIT. However, it is clear that the boundary layer flow is more receptive to the scale-dependent distribution of energy (von Kármán spectrum) realized by the HIT-based forcing.

Figure 3.14 shows the streamwise component of the principal eigenmodes of the velocity covariance matrix $\Phi$ resulting from near-wall HIT-based excitation of the boundary layer flow with $k_z = 0.26$. The flow structures closely resemble the streamwise elongated streaks presented in figure 3.9(b). From figure 3.14(b) we conclude that an exponentially decaying excitation further elongates the streaks in the streamwise direction. We note that the amplification of streaks and their prominence in the downstream regions persists, even if the streamwise-decaying forcing completely vanishes towards the end of the domain. Figure 3.15 shows the dominant flow structure that results from near-wall HIT-based forcing of the boundary layer flow with $k_z = 0.01$. This figure demonstrates that our stochastic analysis is able to predict the amplification of TS wave-like structures arising from persistent excitation that matches the spectrum of HIT, which is in agreement with the global stability analysis of (Alizard and Robinet, 2007). In contrast, similar stochastic analysis of the parallel flow dynamics fails to capture such structures; see (Lin and Jovanovic, 2008) for the predictions resulting from locally parallel analysis.

**Figure 3.14:** The $x$-$y$ slice of the streamwise component of the principal eigenmode from the covariance matrix $\Phi$ at $k_z = 0.26$ resulting from near-wall HIT-based excitation of the boundary layer flow with $Re_0 = 232$. (a) Streamwise-invariant forcing ($r = 0$ in $h(x)$); and (b) streamwise-decaying forcing ($r = 1.5$ in $h(x)$).
Figure 3.15: The TS wave-like spatial structure of the streamwise component of the principal eigenmode of matrix $\Phi$ at $k_z = 0.01$ resulting from global analysis of the boundary layer flow subject to near-wall streamwise-invariant ($r = 0$ in $h(x)$) HIT-based stochastic forcing.

Figure 3.16: The rms amplitude of the streamwise velocity resulting from stochastic excitation that corresponds to the spectrum of HIT entering in the near-wall region (case 1 in Table 2.1). The decay rate for the intensity of stochastic forcing, $r$, increases in the direction of the arrow as $r = 0, 0.5, 1, \text{ and } 1.5$.

Figure 3.16 illustrates the growth of the root-mean-square (rms) amplitude of the streamwise velocity resulting from HIT-based stochastic forcing with various streamwise decay rates; $r = 0, 0.5, 1, \text{ and } 1.5$. This figure is obtained by integrating the steady-state response ($\text{diag}(\Phi)$) over 50 logarithmically spaced spanwise wavenumbers with $0.01 < k_z < 10$. When the forcing is not damped ($r = 0$), the growth is linear and proportional to the Reynolds number for $Re < 400$, which is in agreement with previous studies based on linear stability theory (Andersson et al., 1999; Wundrow and Goldstein, 2001). We observe that this linear trend is no longer present for stochastic forcing with large streamwise decay rates $r$. 
Chapter 4

Discussion on the receptivity analysis of boundary layer flows

We have observed the evidence of the connections between the local and global analysis for both energy response and flow structures in Chapter 3. This chapter compares and contrasts results obtained under a locally-parallel flow assumption with those of global analysis. Coherent structures that emerge as the response to free-stream turbulence are extracted using the modal decomposition of the steady-state velocity covariance matrix. On the one hand, we present how parallel and global flow analyses can quantify the amplification of crucial laminar-turbulent transition of boundary layer flows, e.g., streamwise elongated streaks and Tollmien-Schlichting (TS) waves. On the other hand, we also show how locally parallel analysis, which breaks up the receptivity process of the boundary layer flow over various streamwise length-scales, can uncover specific flow structures that are difficult to observe in global analysis. Finally, we examine frequency responses of the boundary layer flow subject to near-wall stochastic excitation.

4.1 Relations between locally parallel and global analyses

The eigenmodes resulting from locally parallel and global stability analysis are closely related (Huerre and Monkewitz, 1990; Alizard and Robinet, 2007). As shown in the previous sections, both locally parallel and global receptivity analyses predict largest amplification of streamwise elongated structures and the appearance of TS waves. However, the size of flow structures and their wall-normal extent can vary with the streamwise location (Reynolds
Figure 4.1: Streamwise velocity fluctuations resulting from near-wall stochastic excitation of the boundary layer flow. (a) Principal eigenmode of $\Phi$ obtained in locally parallel analysis with $Re = 300$ and $(k_x, k_z) = (0.11, 0.32)$; and (b) 6th eigenmode of $\Phi$ resulting from global analysis with $k_z = 0.32$. In the global computations $L_x = 200$ and the dominant flow structures appear at $Re \approx 300$.

number). For a proper comparison between the streamwise/wall-normal extent of flow structures, herein, we adjust the Reynolds number used in locally parallel analysis to capture the dominant flow structures toward the end of the global streamwise domain. Moreover, a shorter global domain length $L_x$ should be considered to accommodate subcritical Reynolds numbers ($Re \lesssim 360$) beyond which the local dynamics are unstable. To ensure stability of the global dynamics, we extend the streamwise domain in the upstream direction to $Re_0 = 133$, but for consistency, display results for $Re \geq 232$ after appropriate scaling based on the Blasius length-scale at $Re = 232$.

For near-wall stochastic excitation (case 1 in Table 2.1), both locally parallel and global receptivity analyses predict the dominant amplification of streamwise elongated structures with $k_z \approx 0.3$; see figures 3.3 and 3.7. For near-wall excitations with $k_z = 0.32$, figure 4.1 shows that locally parallel analysis of the flow with $Re = 300$ subject to near-wall excitation yields similar flow structures (with $k_x = 0.11$) to those appearing at $Re \approx 300$ in the 6th eigenmode of the covariance matrix $\Phi$ resulting from global analysis. Here, $k_x = 0.11$ is the wavenumber extracted from spatial Fourier transform of the 6th eigenmode of $\Phi$. Moreover, for long spanwise wavelengths, both models predict the amplification of similar TS wave-like
structures in the presence of near-wall excitation (see figure 4.2). These observations can also be explained by evaluating the source terms in the energy balance equation. The intrinsic source terms are dominated by production terms that account for interactions between the fluctuation field and the mean rate of strain, i.e., $\langle u, u \partial_x U \rangle$, $\langle u, v \partial_y U \rangle$, $\langle v, u \partial_x V \rangle$, and $\langle v, v \partial_y V \rangle$; e.g., see (Sipp and Marquet, 2013, Section 4.2). In transitional boundary layer flow, since $\partial_x U$, $\partial_y V \sim O(1/Re)$, and $\partial_x V \sim O(1/Re^2)$, energy production is dominated by the term $\langle u, v \partial_y U \rangle$ and it is well-captured by a locally parallel analysis.

![Figure 4.2](image)

**Figure 4.2:** The TS wave-like spatial structure of the streamwise velocity component of the principal eigenmode of the matrix $\Phi$ resulting from near-wall stochastic excitation of the boundary layer flow. (a) Locally parallel analysis with $Re = 300$ and $(k_x, k_z) = (0.13, 0.01)$; and (b) global flow analysis with $k_z = 0.01$. The wavenumber pair for the locally parallel analysis corresponds to the TS wave branch in the energy spectrum of velocity fluctuations. In the global computations $L_x = 200$ and the dominant flow structures appear at $Re \approx 300$.

In certain scenarios, locally parallel analysis can extract information about streamwise scales that may be hidden in global analysis. This feature of locally parallel analysis can be attributed to the parameterization of the velocity field over streamwise wavenumbers, which enables the separate study of various streamwise length-scales. For example, for wavenumbers at which the global receptivity analysis of the flow subject to outer-layer excitation is dominated by near-wall streaks, locally parallel analysis can uncover the trace of weakly growing outer-layer oscillations at TS frequencies. This is in agreement with experiments (Kendall, 1998) which observe outer-layer oscillations of comparable length to
width \( k_x \approx k_z \) that travel at the phase speed of free-stream velocity with similar temporal frequency as TS waves.

To further investigate this observation, we re-examine the flow structures that can be extracted from locally parallel and global flow analyses of the boundary layer flow at \( Re \approx 300 \) subject to stochastic excitation covering the entire free stream region. In particular, the parameters in Eq. (2.11) are set to \( y_1 = 7, y_2 = 33, \) and \( a = 10 \) for locally parallel analysis, and \( y_1 = \delta_{0.99} + 2, y_2 = 33, \) and \( a = 10 \) for global flow analysis. Note that \( \delta_{0.99} \) in the global analysis is a function of \( x \). By comparing the phase speed of the outer-layer oscillations to that of TS waves \( (c \approx 0.4 U_\infty) \) obtained from local temporal stability analysis with \( Re_0 = 232 \) and \( k_x \approx 0.19 \) we obtain \( \omega \approx 0.076 \). Finally, Taylor’s hypothesis \( (c \approx U_\infty) \) can be used to obtain \( k_x \approx 0.076 \) for outer-layer oscillations.

Figures 4.3(a) and 4.3(b) show the streamwise component of the steady-state response of the boundary layer flow with \( Re = 300 \) and \( k_z = 0.076 \) resulting from locally parallel and global flow analyses, respectively. As aforementioned, locally parallel analysis considers \( k_x = k_z = 0.076 \), which is in concert with the experimentally observed outer-layer oscillations. These flow structures represent the aggregate contribution of all eigenmodes of \( \Phi \) and they have been obtained from \( \text{diag}(C_u X C_u^*) \), where \( C_u \) is the streamwise component of the output matrix \( C \). Note that the spatial structure shown in figure 4.3(a) is obtained by enforcing streamwise periodicity with \( k_x = 0.076 \). While locally parallel analysis of the stochastically forced flow predicts the amplification of structures that reside in the outer-layer, the response obtained in global analysis is dominated by inner-layer streaks and a weaker amplification of outer-layer fluctuations is observed in the presence of stochastic forcing. As shown in figure 4.3(c), such weak outer-layer oscillations can be observed in the 7th mode of the covariance matrix \( \Phi \) resulting from global analysis. Figure 4.3(d) shows the streamwise variation of these flow structures at \( y = 20 \), which corresponds to the wall-normal location where the largest amplitude occurs. The streamwise wavelength of this signal is approximately the same as the parallel flow estimate \( (\lambda_x = 81 \text{ vs } \lambda_x = 82.7) \). Such flow structures may be dominated by higher amplitude streaks as their contribution to the total energy...
Figure 4.3: The rms amplitude of the streamwise velocity component of the response (\(\text{diag} (C_u X C_u^*)\)) obtained from receptivity analysis of boundary layer flow with \(Re = 300\) subject to full outer-layer stochastic excitation: (a) locally parallel analysis with \(k_x = k_z = 0.076\); (b) global flow analysis with \(k_z = 0.076\). (c) Contribution of the first 7 eigenvalues of the velocity covariance matrix \(\Phi\) obtained via global receptivity analysis and the flow structures corresponding to the first and 7th eigenmodes. (d) The streamwise velocity profile at \(y = 20\) from the 7th eigenmode of \(\Phi\) illustrated in (c). In the global computations, we set \(L_x = 200\) and the outer-layer oscillating structures appear at \(Re \approx 300\).

amplification is much smaller than the contribution of the principal mode (0.15\% vs 1.9\%). Nonetheless, similar to the cascade shown in figure 3.11(f), their presence in the eigenmodes of the covariance matrix points to the physical relevance of flow structures that are identified via locally parallel analysis.

4.2 Explanation to the energy distribution of flow structures

In chapter 3, we have observed the similarities of the typical flow structures (cf. figures 3.5 and 3.9) as well as the distinct energy distribution of these structures (cf figures 3.4 and 3.8)
in the locally parallel and global receptivity analyses. Now, we seek an explanation for the connections and differences between the results of these two analyses. We first investigate the influence of a spatially evolving base flow on the steady-state response obtained from solving Eq. (2.6). For this purpose, we consider linearization around a parallel (streamwise invariant) boundary layer flow, but do not exploit the homogeneity of the dynamics in $x$ and discretize the operators in the streamwise direction in the same way as in the wall-normal direction. Figure 4.4 illustrates the arrangement of eigenvalues of the covariance matrix $\Phi$
resulting from linearization around parallel and non-parallel base flow profiles. As shown in figure 4.4(a), when the base flow is assumed to be parallel, the primary eigenvalues retains a more significant portion of the total energy. Furthermore, figure 4.4(b) demonstrates how spatial evolution of the base flow can lead to increased energy amplification across all eigenmodes of the covariance matrix. On the other hand, as shown in figure 4.5, spatial visualization of various eigenmodes of $\Phi$ uncovers similar periodic flow structures to that of linearizing around a spatially evolving base flow (cf. figure 3.11). While the spatial evolution of the base flow plays an important role in distributing energy across various eigenmodes, it appears to have little effect on the shape of the amplified flow structures. Finally, we note that even in the presence of a spatially invariant base flow, spatial discretization of the streamwise dimension results in a full-rank covariance matrix $\Phi$. This is in contrast to the result of parallel flow analysis (cf. figure 3.4). We next provide an illustrative example to gain insight into the significance of subordinate eigenmodes of the covariance matrix resulting from spatial discretization of non-homogeneous directions.

Consider a stochastically forced 2D diffusion equation

$$\xi_t = (\partial_{xx} + \partial_{yy})\xi + f$$

in which Dirichlet boundary conditions are imposed on the state $\xi$ in both $x$ and $y$ directions and the size of the domain is given by $L_x$ and $L_y$. The eigenvalues of the dynamical generator in Eq. (4.1) are

$$\lambda_{m,n} = -\left((m\pi/L_x)^2 + (n\pi/L_y)^2\right); \quad m, n \in \mathbb{Z}^+$$

and the corresponding eigenfunctions are sinusoids that include $m/2$ and $n/2$ periods in $x$ and $y$ directions, respectively. When the stochastic forcing is white-in-time with spatial covariance $W = I$, the steady-state covariance matrix $\Xi = \lim_{t \to \infty} E(\xi(t)\xi^*(t))$ can be expressed via the integral representation,

$$\Xi = \int_0^\infty e^{At} We^{A^*t} dt$$
where $e^{At}$ is the strongly-continuous semi-group generated by $A := \partial_{xx} + \partial_{yy}$ with proper boundary conditions. Since $A$ is normal, the expression for $\Xi$ can be simplified to

$$\Xi = G \text{diag}\left\{-\frac{1}{2\lambda_{m,n}}\right\} G^*$$  \hspace{1cm} (4.2)

where $G$ contains orthonormal eigenfunctions of $A$ that are weighted according to the eigenvalues $-1/(2\lambda_{m,n})$.

We emulate a boundary layer configuration, by considering a stretched spatial domain with $L_x = 20$ and $L_y = 2$ and use 50 Chebyshev collocation points to discretize in $y$. Figure 4.6 shows the energy distribution of various eigenmodes of the covariance matrix $\Xi$ for two instances that differ in the application of Fourier transform in the $x$. When the equations are parameterized over wavenumbers $k_x$, the covariance matrix $\Xi$ retains a dominant principal mode (figure 4.6(a)). For $k_x = \pi/20$, the eigenvalues of $\Xi$ are given by $\frac{1}{2}((\pi/20)^2 + (n\pi/2)^2)^{-1}$, and they decrease at an approximate rate of $1/(n\pi)^2$, thereby resulting in the dominance of the principal eigenmode. In contrast, when the homogeneity in $x$ is not exploited, a global approach, which discretizes the $x$ dimension in the same way as $y$, results in a covariance matrix $\Xi$ that is not low-rank (figure 4.6(b)). This approach yields eigenvalues of the form $\frac{1}{2}((m\pi/20)^2 + (n\pi/2)^2)^{-1}$. Because of the stretched domain ($L_x \gg L_y$), the eigenvalues initially decrease more gradually as their ordering in $m$ precedes $n$. To illustrate the ordering of eigenfunctions, figure 4.7 shows the spatial structure of the 17th mode ($m = 17, n = 1$), which precedes the 18th mode ($m = 1, n = 2$). The eigenvalues corresponding to these modes are marked by red circles in figure 4.6(b). The first 17 modes are sinusoids that include $1/2$ period in $y$ and respectively $1/2, \cdots, 17/2$ periods in $x$.

**Remark 2** In boundary layer flow, the dynamical generator of the linearized NS equations (2.3) is non-normal, and as a result, the covariance matrix $\Phi$ cannot be expressed in a form similar to Eq. (4.2). Nevertheless, the eigenmodes of $\Phi$ inherit a periodic structure from a (non-trivially) weighted combination of spatially periodic eigenmodes of dynamical
Figure 4.6: Contribution of the first 50 eigenvalues of the covariance matrix \( \Xi \) resulting from (a) parameterization in the \( x \) direction \( (k_x = \pi/20) \), and (b) without parameterization in the \( x \) direction (global approach). Red circles mark the 17th and 18th eigenvalues.

Figure 4.7: The spatial structure of the 17th (a) and 18th (b) eigenfunctions of \( \Xi \) from a global approach that does not account for the homogeneity in \( x \) to parameterize the dynamics across Fourier modes.

generator \( A \). Furthermore, similar to the diffusion equation, the spatial domain that contains the shear of the boundary layer flow is stretched in the streamwise direction \( (L_x \gg L_y) \) resulting in a gradually degrading eigenspectrum in \( \Phi \). A rigorous mathematical explanation requires further scrutiny and it is a topic of our ongoing research.
4.3 Frequency response analysis

The receptivity analysis conducted in this part quantifies the energy amplification of stochastically-forced linearized NS equations and identifies the dominant flow structures in statistical steady-state. We utilize the solution $X$ to the algebraic Lyapunov equation (2.6) to avoid the need for performing either costly stochastic simulations or integration over all temporal frequencies. This approach facilitates efficient computations by aggregating the impact of different frequencies on energy amplification. In what follows, we illustrate how additional insight into temporal aspects of the linearized dynamics can be obtained by examining the spectral density associated with velocity fluctuations (2.8).

Application of the temporal Fourier transform on system (2.3) in combination with a coordinate transformation

$$d(t) = W^{1/2} \tilde{d}(t),$$

where $d(t)$ and $\tilde{d}(t)$ are white-in-time forcings with the spatial covariance matrices $W$ and $I$, respectively, yields

$$v(k, \omega) = T_{vd}(k, \omega) d(k, \omega) = T_{vd}(k, \omega) \tilde{d}(k, \omega).$$  \hspace{1cm} (4.3)

Here, $k$ denotes the spatial wavenumbers, $\omega$ is the temporal frequency, $T_{vd}(k, \omega)$ is the frequency response of system (2.3) given in Eq. (2.9), and

$$T_{vd}(k, \omega) := T_{vd}(k, \omega) W^{1/2} = C (i \omega I - A)^{-1} B W^{1/2}. \hspace{1cm} (4.4)$$

Singular value decomposition of $T_{vd}(k, \omega)$ brings the input-output representation (4.3) into the following form:

$$v(k, \omega) = T_{vd}(k, \omega) \tilde{d}(k, \omega) = \sum_i \sigma_i(k, \omega) u_i(k, \omega) \langle w_i(k, \omega), \tilde{d}(k, \omega) \rangle,$$
where \( \sigma_i \) is the \( i \)th singular values of \( T_{vd}(k, \omega) \), \( u_i(k, \omega) \) is the associated left singular vector, and \( w_i(k, \omega) \) is the corresponding right singular vector. The power spectral density (PSD) quantifies the energy of velocity fluctuations \( v(k, \omega) \) across temporal frequencies \( \omega \) and spatial wavenumbers \( k \),

\[
\Pi_v(k, \omega) = \text{trace} \left( T_{vd}(k, \omega) T_{vd}^*(k, \omega) \right) = \text{trace} \left( T_{vd}(k, \omega) W T_{vd}^*(k, \omega) \right) = \text{trace} \left( S_v(k, \omega) \right),
\]

and is determined by the sum of squares of the singular values of the frequency response \( T_{vd}(k, \omega) \),

\[
\Pi_v(k, \omega) = \sum_i \sigma_i^2(k, \omega).
\]

As described in Section 2.1.2, the energy spectrum \( E \) in Eq. (2.7) can be obtained by the integration of \( \Pi_v(k, \omega) \) over temporal frequency (Jovanovic and Bamieh, 2005),

\[
E(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi_v(k, \omega) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(k, \omega) \, d\omega.
\]

This approach extends standard resolvent analysis (Trefethen et al., 1993; Jovanovic, 2004; McKeon and Sharma, 2010) to stochastically-forced flows and it allows the spatial covariance matrix \( W \) of the white-in-time stochastic forcing \( d \) to be embedded into the analysis. A recent reference (Towne et al., 2018) also establishes relation between spectral decomposition of \( S_v(k, \omega) \) and dynamic mode decomposition (Schmid, 2010).

The PSD of the boundary layer flow with \( Re_0 = 232 \) subject to near-wall stochastic excitation is shown in figure 4.8. While locally parallel analysis reveals isolated frequencies at which the PSD peaks, much broader frequency range is important in global analysis. In particular, locally parallel analysis for a flow with (i) \((k_x, k_z) = (7 \times 10^{-3}, 0.32)\) identifies nearly-steady streaks as dominant flow structures (the PSD peaks at \( \omega = 0.0063 \)); and (ii) \((k_x, k_z) = (0.19, 0.01)\) identifies two peaks at \( \omega = 0.08 \) and \( \omega = 0.19 \) which correspond to the TS waves and flow structures in the outer-layer, respectively. On the other hand, the peaks
Figure 4.8: Power spectral density $\Pi_v(k, \omega)$ as a function of temporal frequency $\omega$ for a boundary layer flow with $Re_0 = 232$ subject to near-wall white stochastic excitation. (a) Locally parallel analysis with $(k_x, k_z) = (7 \times 10^{-3}, 0.32)$ (blue) and $(k_x, k_z) = (0.19, 0.01)$ (red) corresponding to streaks and TS waves, respectively. (b) Global flow analysis with $k_z = 0.32$ (blue) and $k_z = 0.01$ (red). Black dots correspond to the temporal frequencies of the Fourier modes plotted in figure 4.9.

Figure 4.9: Fourier modes corresponding to the principal response directions of $T_{vd}(k, \omega)$ for a spatially evolving Blasius boundary layer flow with $Re_0 = 232$ and $k_z = 0.32$ subject to near-wall stochastic excitation. (a) $\omega = 10^{-5}$, (b) $\omega = 0.01$, and (c) $\omega = 0.02$.

are much less pronounced in the analysis of the spatially-evolving base flow. This suggests that the focus on isolated frequencies in global analysis may not capture the full complexity of the underlying flow structures. In fact, the shapes of spatial profiles associated with principal singular vectors of the frequency response $T_{vd}(k, \omega)$ change for different values of $\omega$. As shown in figure 4.9, even though the principal singular values of $T_{vd}(k, \omega)$ for $\omega = 10^{-5}$, 0.01, and 0.02 are comparable (5464, 4732, and 3565, respectively), the corresponding response directions change from streamwise streaks (for steady perturbations) to oblique modes (at larger frequencies). This trend is reminiscent of the various flow structures resulting from
the eigenvalue decomposition of the steady-state covariance matrix (cf. Section 3.2) and has been also recently observed in spatio-temporal analysis of hypersonic boundary layer flows (Dwivedi et al., 2019).
Chapter 5

Concluding remarks

In the present study, we have utilized the linearized NS equations to study energy amplification in the Blasius boundary layer flow subject to white-in-time stochastic forcing entering at various wall-normal locations. The evolution of flow fluctuations is captured by two models that arise from locally parallel and global perspectives, and the amplification of persistent stochastic disturbances is studied using the algebraic Lyapunov equation. Both parallel and global flow analyses predict the largest amplification of streamwise elongated streaks with similar spanwise wavelength. Moreover, TS wave-like flow structures arise from persistent near-wall stochastic excitation at long spanwise wavelengths. We have shown that as the region of excitation moves away from the wall, energy amplification reduces, suggesting that the near-wall region is more sensitive to external disturbances. We have also examined the spatial structure of characteristic eddies that result from stochastic excitation of the boundary layer flow. Our computational experiments demonstrate good agreement between the results obtained from parallel and global flow models and identify the importance of sub-optimal flow structures in global analysis. This agreement highlights the efficacy of using parallel flow assumptions in the receptivity analysis of boundary layer flows, especially when it is desired to evaluate the energetic contribution of individual streamwise scales.

In contrast to resolvent-mode analysis, which quantifies the energy amplification from monochromatic forcing, our stochastic approach incorporates a broad-band forcing model with known spatial correlations that captures the aggregate effect of all time scales. Our
Lyapunov-based framework generalizes the concept of receptivity to the amplification of velocity fluctuations from any external source of persistent excitation with known statistical properties. We note that the ability of the method to capture relevant flow physics relies on the spectral properties of the stochastic forcing that can be used to model the effect of, e.g., free-stream turbulence. In addition to white-in-time stochastic forcing with trivial (identity) spatial covariance operator, we have also investigated energy amplification arising from the streamwise-decaying forcing that corresponds to the spectrum of HIT. Our computations demonstrate the close correspondence between these two case studies. The spatio-temporal spectrum of stochastic excitation sources can be further determined in order to provide statistical consistency with the results of numerical simulations or experimental measurements of the boundary layer flow (Zare et al., 2017a,b). Implementation of such ideas to leverage statistical data and improve physics-based analysis is a topic for future research.
Part II

Modeling mode interactions in boundary layer flow
Chapter 6

Prabolized Floquet Equations

We are interested in the control-oriented modeling of spatially evolving flows. Due to their high complexity and large number of degrees of freedom, nonlinear dynamic models based on the Navier-Stokes (NS) equations are not suitable for analysis, optimization, and control. On the other hand, experimentally and numerically generated data sets are becoming increasingly available for a wide range of flow configurations. This has enabled data-driven techniques for the reduced-order modeling of fluid flow systems. Despite being computationally tractable, such models often lack robustness. Specifically, control actuation and sensing may significantly alter the identified modes, which introduces nontrivial challenges for model-based control design (Noack et al., 2011). In contrast, models that are based on the linearized NS equations are less prone to such uncertainty and are, at the same time, well-suited for analysis and synthesis using tools of modern robust control (Kim and Bewley, 2007).

At sufficiently large amplitudes, primary disturbances that are instigated via receptivity processes involving external or internal perturbations (Kachanov, 1994) lead to the parametric excitation of secondary instability mechanisms. Such mechanisms in turn trigger a strong energy transfer from the mean flow into secondary modes (Herbert, 1988). The physics of such transition mechanisms have been previously studied using Floquet analysis (Herbert, 1988; Andersson et al., 2001; Brandt, 2003), and the Parabolized Stability Equations (PSE) (Joslin et al., 1993; Herbert, 1994, 1997).
The PSE was introduced to account for non-parallel and nonlinear effects, which was not possible using eigenvalue problems arising from the Orr-Sommerfeld equations. In particular, the PSE was developed as a means to refine predictions of parallel flow analysis in slowly varying flows (Bertolotti et al., 1992; Herbert, 1997), e.g., the laminar boundary layer flow. In general, the linear PSE and their stochastically forced variant provide reasonable predictions for the evolution of primary modes such as Tollmien-Schlichting (TS) waves in boundary layer flow (Herbert, 1994; Ran et al., 2017a). However, secondary growth mechanisms that lead to the laminar-turbulent transition of the boundary layer flow originate from mode interactions (Herbert, 1994).

In the transitional boundary layer, primary instability mechanisms cause perturbations to grow to finite amplitudes and saturate at steady or quasi-steady states. Floquet stability analysis identifies secondary instability modes as the eigenmodes of the linearized NS equations around a modified base flow profile that contains the spatially periodic primary velocity fluctuations. In the corresponding eigenvalue problem, the operators inherit a lifted representation from the spatial periodicity of the base flow (Fardad et al., 2008) and, as a result, capture primary-secondary mode interactions. Such lifted representations also appear in the modeling of periodic flow control strategies in wall-bounded shear flows (Jovanovic, 2008; Moarref and Jovanovic, 2010, 2012).

In this chapter, we propose a framework that utilizes Floquet theory (Bittanti and Colaneri, 2009) to capture the dominant nonlinearity and adopt the assumptions of the linear PSE to account for the spatial evolution of the base flow. The resulting equations can be advanced downstream via a marching procedure. This framework thus inherits the ability to account for mode interactions from Floquet theory while maintaining the low complexity of the linear PSE.
6.1 Parabolized Stability Equations

We first present the equations that govern the dynamics of flow fluctuations in incompressible flows of Newtonian fluids and then provide details on our proposed model for the downstream marching of spatially growing fluctuations in the boundary layer flow.

In a flat-plate boundary layer, with geometry shown in figure 6.1, the dynamics of infinitesimal fluctuations around a two-dimensional base flow $\mathbf{\bar{u}} = [U(x,y) \ V(x,y) \ 0]^T$ are governed by the linearized NS equations

$$
\begin{align*}
\mathbf{v}_t &= - (\nabla \cdot \mathbf{\bar{u}}) \mathbf{v} - (\nabla \cdot \mathbf{v}) \mathbf{\bar{u}} - \nabla p + \frac{1}{Re_0} \Delta \mathbf{v} \\
0 &= \nabla \cdot \mathbf{v},
\end{align*}
$$

where $\mathbf{v} = [u \ v \ w]^T$ is the vector of velocity fluctuations, $p$ denotes pressure fluctuations, $u$, $v$, and $w$ are the streamwise $(x)$, wall-normal $(y)$, and spanwise $(z)$ components of the fluctuating velocity field, and $Re_0$ is the Reynolds number at the inflow location $x_0$. The Reynolds number is defined as $Re = U_{\infty} \delta/\nu$, where $\delta = \sqrt{\nu x/U_{\infty}}$ is the Blasius length scale at the streamwise location $x$, $U_{\infty}$ is the free-stream velocity, and $\nu$ is the kinematic viscosity. Spatial coordinates are non-dimensionalized by the Blasius length scale $\delta_0$ at the inflow location $x_0$, velocities by $U_{\infty}$, time by $\delta_0/U_{\infty}$, and pressure by $\rho U_{\infty}^2$, where $\rho$ is the fluid density.

It is customary to use the parallel-flow approximation to study the local stability of boundary layer flows to small amplitude disturbances (Schmid and Henningson, 2001). This
approximation, in conjunction with Floquet theory, has also been used to investigate sec-
ondary instabilities that inflict transition (Herbert, 1988; Schmid and Henningson, 2001).
However, the parallel-flow approximation excludes the effect of the evolution of the base
flow on the amplification of disturbances. This issue can be addressed via global stability
analysis which accounts for the spatially varying nature of the base flow by discretizing all
inhomogeneous directions. Nevertheless, global analysis of spatially-evolving flows may be
prohibitively expensive for analysis, optimization, and control purposes.

The PSE provide a computationally attractive framework for the spatial evolution of
perturbations in non-parallel and weakly nonlinear scenarios (Bertolotti et al., 1992; Herbert,
1994, 1997). They are obtained by removing terms of $O(1/Re^2)$ and higher from the NS
equations and are significantly more efficient than conventional flow simulations based on
the governing equations. In weakly non-parallel flows, e.g., in the pre-transitional boundary
layer, flow fluctuations can be separated into slowly and rapidly varying components via the
following decomposition for the fluctuation field $\mathbf{q} = [u \, v \, w \, p]^T$ in Eq. (6.1). For a specific
spanwise wavenumber and temporal frequency pair $(\beta, \omega)$, we consider

$$
\begin{align*}
\mathbf{q}(x,y,z,t) &= \hat{\mathbf{q}}(x,y) \chi(x,z,t) + \text{complex conjugate}, \\
\chi(x,z,t) &= \exp \left( i (\alpha(x) x + \beta z - \omega t) \right),
\end{align*}
$$

(6.2)

where $\hat{\mathbf{q}}(x,y)$ and $\chi(x,z,t)$ are the shape and phase functions, and $\alpha(x)$ is the streamwise
varying generalization of the wavenumber (Bertolotti et al., 1992). This decomposition
separates slowly ($\hat{\mathbf{q}}(x,y)$) and rapidly ($\chi(x,z,t)$) varying scales in the streamwise direction.
The ansatz in Eq. (6.2) provides a representation of oscillatory instability waves such as TS
waves.

The ambiguity arising from the streamwise variations of both $\hat{\mathbf{q}}$ and $\alpha$ is resolved by
imposing the condition

$$
\int_{\Omega_y} \hat{\mathbf{q}}^* \hat{\mathbf{q}}_z \, dy = 0,
$$
where \( \hat{q}^* \) denotes the complex conjugate transpose of the vector \( \hat{q} \). In practice, this condition is enforced through the iterative adjustment of the streamwise wavenumber (Herbert, 1994, Section 3.2.5). Following the slow-fast decomposition highlighted in Eq. (6.2), the linearized NS equations are parabolized under the assumption that the streamwise variation of \( \hat{q} \) and \( \alpha \) are sufficiently small to neglect \( \hat{q}_{xx}, \alpha_{xx}, \alpha_x \hat{q}_x, \alpha_x/Re_0 \), and their higher order derivates with respect to \( x \), resulting in the removal of the dominant ellipticity in the NS equations. The linear PSE take the form

\[
L \hat{q} + M \hat{q}_x = 0,
\]

where expressions for the operator-valued matrices \( L \) and \( M \) can be found in (Herbert, 1994).

We next propose a two-step modeling procedure to study the dominant mode interactions in weakly-nonlinear mechanisms that arise in spatially evolving flows.

### 6.2 Constructing Parabolized Floquet equations

In the transitional boundary layer flow, primary instabilities can cause disturbances to grow to finite amplitudes and get saturated by nonlinearity. Secondary stability analysis examines the asymptotic growth of the resulting modulated state and is based on the linearized NS equations around the modified base flow

\[
\tilde{u} = u_0 + u_{pr}.
\]

Here, \( u_0 \) denotes the original base flow and \( u_{pr} \) represents the primary disturbance field. Since \( \tilde{u} \) is typically spatially or temporally periodic, Floquet analysis is invoked to identify the spatial structure of exponentially growing fluctuations around \( \tilde{u} \). However, such analysis relies on a parallel flow assumption and it does not explicitly account for the spatially growing nature of the base flow. To account for the interactions of fluctuations with spatially growing modified base flow \( \tilde{u} \) in a computationally efficient manner, we introduce a framework which draws on Floquet theory to enhance the linear PSE. Our approach allows us to capture the
dominant mode interactions in the fluctuating velocity field while accounting for non-parallel effects in the base flow.

Starting from a spatially or temporally periodic initial condition the linear PSE can be marched downstream to obtain the primary disturbance field. For example, such an initial condition can be obtained using stability analysis of the two-dimensional Orr-Sommerfeld equation or transient growth analysis of streamwise constant linearized equations (under the locally-parallel base flow assumption). When the periodic solutions to the linear PSE computation are superposed to the Blasius boundary layer profile, the modulated base flow (6.4) takes the following form

\[ \bar{u}(x, y, z, t) = \sum_{m = -\infty}^{\infty} u_m(x, y) \phi_m(x, z, t). \]  

(6.5)

Here, \( u_0(x, y) = [U_B(x, y) \ V_B(x, y) \ 0]^T \) represents the Blasius boundary layer profile, \( \phi_0 = 1 \), \( u_m \) and \( \phi_m \) for \( m \neq 0 \) are the shape and phase functions corresponding to various harmonics that constitute flow structures of the primary disturbance field (such as TS waves or streaks), and \( u^*_m = u_{-m} \). Note that each harmonic \( u_m \) of the modified base flow \( \bar{u} \) inherits a similar slow-fast structure from PSE (cf. Eq. (6.2)) in which the phase function \( \phi_m \) is spanwise or streamwise/temporally periodic. For example, when TS waves are superposed to the Blasius boundary layer profile the phase functions \( \phi_m \) are streamwise and temporally periodic; see Section 7.1 for details. The evolution of fluctuations around the modulated base flow (6.5) can be studied using the following expansion

\[ q(x, y, z, t) = \sum_{n = -\infty}^{\infty} \hat{q}_n(x, y) \chi_n(x, z, t), \]  

(6.6)

which, similar to PSE, involves a decomposition of disturbances into slowly (\( \hat{q}_n \)) and rapidly (\( \chi_n \)) varying components. Note that we follow classical Floquet decomposition (Herbert, 1984, 1988) in assuming that the phase functions \( \chi_n \) represent various harmonics of the same fundamental frequency/wavenumber as \( \phi_m \) in Eq. (6.5). As a result of this assumption
the evolution of each harmonic mode in $q$ can contribute to the evolution of its neighboring harmonics via the periodicity of the modulated base flow (6.5). For spanwise periodic modulations to the base flow, a concrete example of the form of the fluctuation field (6.6) is discussed in Remark 3.

**Remark 3** When spanwise-periodic streaks with a fundamental wavenumber $\beta$ are superposed to the Blasius boundary layer profile, the modified base flow takes the form

$$\bar{u}(x, y, z) = \sum_{m=-\infty}^{\infty} u_m(x, y) e^{im\beta z},$$

and the spatial evolution of fluctuations that account for fundamental harmonics (in $z$) around this modulated base flow profile can be studied using the Fourier expansion

$$q(x, y, z) = \sum_{n=-\infty}^{\infty} \hat{q}_n(x, y) e^{i(\alpha_n(x)x + n\beta z)}.$$

Here, $\alpha_n(x)$ is the purely imaginary streamwise wavenumber of various harmonics, which can evolve in the streamwise direction similarly to linear PSE. Note that if $\alpha_n(x)$ is identical for all harmonics, we recover the nondispersive wavepacket assumed in Floquet stability analysis; see Section 7.2 for additional details.

Under the assumptions of linear PSE, the dynamics of fluctuations represented by (6.6) can be studied using the Parabolized Floquet Equations (PFE)

$$L_F \hat{q} + M_F \hat{q}_z = 0. \quad (6.7)$$

The state in Eq. (6.7),

$$\hat{q} = [ \cdots \hat{q}_{n-1}^T \hat{q}_n^T \hat{q}_{n+1}^T \cdots ]^T,$$
contains all harmonics of $q$ in the periodic direction, i.e.,

$$\hat{q}_n = [ u_n^T \ v_n^T \ w_n^T \ p_n^T ]^T,$$

and the operators $L_F$ and $M_F$ inherit the following bi-infinite structure from the periodicity of the phase functions $\phi_m$ in the modified base flow (6.5),

$$L_F := \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & L_{n-1,0} & L_{n-1,1} & L_{n-1,2} & \cdots \\ \cdots & L_{n-1,0} & L_{n,1} & L_{n,2} & \cdots \\ \cdots & L_{n+1,-2} & L_{n+1,-1} & L_{n+1,0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$ \hspace{1cm} (6.8)

Note that the operator $L_{i,j}$ captures the influence of the $j$th harmonic $\hat{q}_j$ on the dynamics of the $i$th harmonic $\hat{q}_i$. In practice, the generally bi-infinite structures of the state and operators in Eq. (6.7) are truncated to account for the spatial evolution of a finite number of essential modes.

In what follows, we employ the PFE to examine two canonical problems:

- the H-type transition scenario (Section 7.1); and
- the formation of streamwise elongated streaks in laminar boundary layer flow (Section 7.2).

In the wall-normal direction, homogenous Dirichlet boundary conditions are imposed,

$$u_n(0) = 0, \quad v_n(0) = 0, \quad w_n(0) = 0$$

$$u_n(L_y) = 0, \quad v_n(L_y) = 0, \quad w_n(L_y) = 0$$

where $L_y$ denotes the height of the computational domain. We discretize differential operators $L_F$ and $M_F$ using a pseudospectral scheme with $N_y$ Chebyshev collocation points in the
wall-normal direction (Weideman and Reddy, 2000) and employ an implicit Euler method to march the PFE (6.7) in the streamwise direction with constant step-size $\Delta x$.

### 6.2.1 Two-step modeling procedure

To model the effect of mode interactions in weakly nonlinear regimes we consider the following two-step procedure:

1. The linear PSE are used to march the primary harmonic and obtain the corresponding velocity profile $u_{pr}$ at each streamwise location.

2. The PFE are used to march all harmonics $\hat{q}$ and obtain the spatial evolution of velocity fluctuations around the modified base flow $\bar{u} = u_0 + u_{pr}$.

The PFE are thus used to study the effect of dominant harmonic interactions on the growth of disturbances in the streamwise direction. The block diagram in figure 6.2(a) illustrates our modeling procedure.

When the secondary disturbances contain the same temporal frequency and spatial wavenumbers as the primary disturbances, the dominant harmonics resulting from the PFE (e.g., $\hat{q}_0$, $\hat{q}_{\pm 1}$, and $\hat{q}_{\pm 2}$ in the formation of streaks) can be subsequently used to update the modulation to the base flow, iterate the PFE computation, and thereby provide an equilibrium configuration. The block diagram in figure 6.2(b) illustrates how our framework can be employed to correct primary disturbances that subsequently modulate the base flow. This should be compared and contrasted to the conventional Floquet analysis in which the frequencies and wavenumbers of the identified secondary instabilities are different from those in the primary disturbances. Our computational experiments in Section 7.2 demonstrate that the flow state of the feedback interconnection in figure 6.2(b) converges after a certain number of iterations. While the equilibrium configuration in figure 6.2(b) implies that our framework is inherently nonlinear, each step in our iterative procedure is linear. We note that a similar iterative method was successfully utilized for model-based design of spanwise wall oscillations in turbulent channel flows (Moarref and Jovanovic, 2012).
Figure 6.2: (a) The PFE are triggered with a primary disturbance $u_{pr}$ that results from linear PSE and modulates the base flow. The diagonal lines represent the base flows that enter as coefficients into the linear PSE and PFE, respectively. (b) When the secondary disturbances are of the same frequencies and spatial wavenumbers as the primary disturbances, the dominant harmonics resulting from each PFE iteration (e.g., $\hat{q}_0$, $\hat{q}_{\pm 1}$, and $\hat{q}_{\pm 2}$ in the formation of streaks) can be used to update the modulation to the base flow, iterate the PFE, and compute an equilibrium configuration.
Chapter 7

Applications of PFE in transitional boundary layer flows

In this chapter, we employ the Parabolized Floquet Equations (PFE) formulated in Chapter 6 to study the typical scenarios in the transitional boundary layers, i.e., H-type transition and streamwise elongated streaks, and compare the results with experiments and DNS to validate our PFE framework. We will see how the flow structures in these cases absorb energy from mean flow through the bridge of other harmonics and rapidly accumulate amplitudes, causing the laminar-turbulent transition.

7.1 H-type transition

We next apply our approach to model an H-type transition scenario in a zero-pressure-gradient boundary layer (Herbert, 1988). This route to transition begins with the exponential growth of two-dimensional TS waves which, upon reaching a critical amplitude, become unstable to secondary disturbances. The modulation of the Blasius profile by the TS waves induces the amplification of otherwise stable oblique modes which have half the frequency of the TS waves. While the linear PSE can be used to characterize the spatial evolution of modes arising from primary instabilities, secondary instabilities that trigger the growth of subharmonic modes call for an expansion in the harmonics of the modulated base flow. Such a growth mechanism cannot be identified via the normal-mode ansatz employed in the linear PSE but it can be captured by marching the PFE, a model resulting from a combination of linear PSE with Floquet decomposition.
7.1.1 Setup

At the initial position $x_0$ and for a real-valued temporal frequency $\omega$, the fundamental mode in the H-type scenario is identified as the eigenvector corresponding to the most unstable complex eigenvalue $\alpha$ from the discrete spectrum of the standard two-dimensional Orr-Sommerfeld equation; see (Schmid and Henningson, 2001, Section 7.1.2) for additional details. The linear PSE can be used to march this fundamental mode, which is in the form of a two-dimensional TS wave, and obtain a reasonable prediction of its spatial growth (Bertolotti et al., 1992). We use the resulting solution to a primary linear PSE computation to augment the Blasius boundary layer profile $u_0$ in the base flow for the PFE as

$$
\begin{align*}
U(x, y) &= U_B(x, y) + U_T(x, y) e^{i \alpha_r(x - ct)} + U^*_T(x, y) e^{-i \alpha_r(x - ct)} \\
V(x, y) &= V_B(x, y) + V_T(x, y) e^{i \alpha_r(x - ct)} + V^*_T(x, y) e^{-i \alpha_r(x - ct)} \\
W(x, y) &= 0.
\end{align*}
$$

(7.1)

Equivalently, the base flow $\bar{u}$ can be written in the following compact form

$$
\bar{u}(x, y, t) = \sum_{m=-1}^{1} u_m(x, y) e^{i m \alpha_r(x - ct)},
$$

where $u_0$ is the Blasius profile. In Eq. (7.1), $\alpha_r$ is the real part of the prescribed streamwise wavenumber of the fundamental mode $\alpha(x)$ resulting from linear PSE, $c = \omega / \alpha_r$ denotes the phase speed of the fundamental and subharmonic modes in the fixed (laboratory) frame, and $[U_T(x, y) \ V_T(x, y) \ 0]^T$ denotes the TS wave whose local amplitude and shape are obtained from the linear PSE computation. Note that the exponential growth resulting from the imaginary part of the wavenumber, $\alpha_i$, is absorbed into the amplitudes of $U_T(x, y)$ and $V_T(x, y)$.

To study the evolution of subharmonic modes that are triggered via secondary instability mechanisms, we follow the Floquet decomposition which was originally conducted in
the moving frame (Schmid and Henningson, 2001, Section 8.2) using the following Fourier expansion, but in the fixed (laboratory) frame

\[ q(x, y, z, t) = e^{i\beta z} \sum_{n = -\infty}^{\infty} \hat{q}_n(x, y) e^{i(n + 0.5)\alpha_r(x - ct) + i\gamma_n x - i\sigma_n t} \]  

(7.2)

where \( \hat{q}_n^* = \hat{q}_{-n-1} \), and \( \gamma_n \) and \( \sigma_n \) are the spatial and temporal detuning factors corresponding to the \( n \)th subharmonic. The imaginary and real parts of \( \gamma_n \) (\( \sigma_n \)) denote the spatial (temporal) growth rate and the detuning in the wavenumber (frequency), respectively. In practice, the detuning factors of the wavenumber and frequency are negligible, i.e., \( \gamma_n \) and \( \sigma_n \) can be assumed to be purely imaginary. A further assumption of nondispersive wavepackets in accordance with classical Floquet analysis (Herbert, 1988) brings the ansatz for the fluctuation field to the following form

\[ q(x, y, z, t) = e^{i(\beta z + c\gamma t)} \sum_{n = -\infty}^{\infty} \hat{q}_n(x, y) e^{i[(n + 0.5)\alpha_r + \gamma](x - ct)} \]

\[ = e^{i(\beta z + \gamma x)} \sum_{n = -\infty}^{\infty} \hat{q}_n(x, y) e^{i(n + 0.5)\alpha_r(x - ct)}, \]

(7.3)

where Gaster’s transformation (Gaster, 1962) has been used to replace \( \sigma \) with \( -c\gamma \) and all subharmonic modes are assumed to share a uniform detuning parameter \( \gamma \). In making these approximations, we have followed (Herbert, 1988) in assuming that all Fourier components have the same phase speed, which is consistent with experimental studies (Kachanov and Levchenko, 1984). While \( \alpha_r \) in the PFE computation is prescribed by the solution of linear PSE for the primary disturbance field, we update the spatial growth rate \( -\gamma \) via a similar scheme to the one used for the streamwise wavenumber update in PSE (Herbert, 1994, Section 3.2.5).

The PFE account for the interaction between different subharmonics by leveraging the slow-fast decomposition inherited from the solution of the linear PSE. By substituting the ansatz (7.3) and the modulated base flow (7.1) into the linearized NS equations, we arrive
at the PFE which take the form of Eq. (6.7). For this case study, the operators $L_F$ and $M_F$ in the PFE (6.7) are provided in Appendix C.1.

The procedure for obtaining the results presented in the next subsection can be summarized as follows:

1. Solve the spatial eigenvalue problem corresponding to the Orr-Sommerfeld equations to obtain the initial complex wavenumber $\alpha(x_0)$ and the shape of initial TS wave $u_{pr}(x_0)$.

2. Use linear PSE to march the TS wave downstream and obtain $\alpha(x)$ and $u_{pr}(x)$.

3. Augment the Blasius boundary layer profile with the TS wave $u_{pr}(x)$ to obtain the modulated base flow $\bar{u}$.

4. Obtain the initial uniform growth rate $\gamma$ and shape function $\hat{q}$ as the most unstable eigenvalue and eigenvectors from standard Floquet analysis (Herbert, 1988) at $x_0$.

5. Use PFE to compute the spatial evolution of all subharmonic modes around the modulated base flow.

### 7.1.2 Growth of subharmonic secondary instabilities

We next examine the interaction of TS waves with subharmonic secondary instabilities in the classical H-type transition scenario. This problem was initially studied in (Herbert, 1984) and has been further explored using both experiments (Kachanov and Levchenko, 1984) and numerical simulations (Joslin et al., 1993; Sayadi et al., 2013).

Following Refs. (Kachanov and Levchenko, 1984; Herbert, 1988), the primary disturbance field is generated by marching a TS wave with a root-mean-square (rms) amplitude of $4.8 \times 10^{-3}$ and frequency $\omega = 0.496$ from $Re_0 = 424$ to $Re = 700$ using linear PSE. We subsequently initialize the PFE with the most unstable eigen-mode from the classical Floquet analysis (Herbert, 1988) to study the growth of subharmonic secondary instabilities triggered by the TS wave. The initial rms amplitude of the subharmonic mode is $1.46 \times 10^{-5}$ and its frequency and spanwise wavenumber are $\omega = 0.248$ and $\beta = 0.132$, respectively. The
initial spatial growth rate $-\gamma$ is obtained by applying Gaster’s transformation to the temporal growth rate. We consider a truncation of the bi-infinite state $q$ with $2N$ modes, i.e., $n = -N \cdots N - 1$, and a computational domain with $L_x \times L_y = 1100 \times 40$. Our computations demonstrate that $N_y = 80$, $\Delta x = 15$, and $N = 2$ (i.e., 4 subharmonic modes), provide sufficient accuracy in capturing the physics of H-type transition; see Appendix C.3 for a discussion on wall-normal grid-convergence and the dependence of results on the number of harmonics.

Figure 7.1(a) shows the rms amplitude of individual modes resulting from experiments (Kachanov and Levchenko, 1984), DNS (Sayadi et al., 2013), nonlinear PSE (Joslin et al., 1993), along with the present PFE computations. Here, the modes are denoted by $(l,k)$, where $l$ stands for the temporal frequency of the harmonic and subharmonic modes as a multiple of the subharmonic mode frequency $\omega = 0.248$ and $k$ represents the spanwise wavenumber as a multiple of the fundamental wavenumber $\beta = 0.132$. The amplitude of the $(1,1)$ mode from the PFE is in excellent agreement with other results. While the amplitude of the $(3,1)$ mode is somewhat under-predicted, the general trend in the growth of this mode is captured by the PFE. We note that in the absence of interactions between harmonics, the linear PSE results in inaccurate predictions for the amplitude of subharmonic modes; see thin solid lines in figure 7.1(a). Figures 7.1(b-d) show the amplitude of the streamwise velocity profile of the modes considered in this study normalized by the results of nonlinear PSE. For all three modes, the profiles resulting from PFE are in good qualitative agreement with the result of nonlinear PSE.

### 7.2 Streamwise elongated laminar streaks

Bypass transition often originates from non-modal growth mechanisms that can lead to streamwise elongated streaks; see for example (Jacobs and Durbin, 2001). The streaks can attain substantial amplitudes (15-20% of the free-stream velocity) and make the flow susceptible to the amplification of high frequency secondary instabilities (Asai et al., 2007; Hack and Zaki, 2014). Secondary instability analysis of saturated streaks has been previously
used to analyze the breakdown stage in the transition process (Andersson et al., 2001; Brandt, 2003). However, nonlinear effects that influence the formation of streaks become prominent in earlier stages of transition and before the breakdown of streaks. In this section, we utilize the PFE to capture the interactions between various modes in the amplification of the streaks. We focus on the interaction between different spanwise harmonics and study
their contribution to the mean flow distortion (MFD), which in turn affects the energy balance among various harmonics that form streaks. We show that the linear PSE fail to predict such a phenomenon and demonstrate how the PFE provide the means to capture the correct trend in the MFD as well as the resulting velocity distribution.

7.2.1 Setup

We trigger the formation of streaks by imposing an initial condition computed via the PSE-based optimization approach introduced in (Hack and Moin, 2017). This optimal initial condition yields the highest amplification of perturbation kinetic energy and it is obtained from the singular value decomposition of a pseudo-propagator which advances arbitrary superpositions of the most unstable eigenfunctions in the non-parallel base flow. The initial perturbation field describes a set of counter-rotating streamwise vortices which give rise to the streaks by means of the lift-up mechanism. We compute the spatial evolution of the perturbation field via the linear PSE and use this solution to augment the Blasius boundary layer base flow profile \( u_0 \) for the subsequent PFE computations as

\[
\begin{align*}
U(x, y) &= U_B(x, y) + U_{S,1}(x, y) e^{i\beta z} + U^*_{S,1}(x, y) e^{-i\beta z} \\
V(x, y) &= V_B(x, y) \\
W(x, y) &= 0,
\end{align*}
\]

which can be written in the following compact form

\[
\bar{u}(x, y, z) = \sum_{m=-1}^{1} u_m(x, y) e^{im\beta z}.
\]

In the linear PSE computations, the real part of the complex wavenumber \( \alpha \) is set to zero in accordance with the nature of streamwise elongated streaks and its imaginary part is initialized with a small number (e.g., \( 10^{-10} \)). Moreover, the exponential growth resulting from the imaginary part of \( \alpha \) is absorbed into the amplitude of \( U_{S,1}(x, y) \) in Eq. (7.4).
The velocity field of streamwise elongated streaks is dominated by the growth of the streamwise component while wall-normal and spanwise components experience viscous decay. As a consequence, we disregard the normal and spanwise components of the solution to linear PSE, and only use the streamwise component \( U_{S,1}(x, y) \) in Eq. (7.4), which is also in agreement with the structure and amplitude of the initial condition. We represent the state in the PFE using the Fourier expansion

\[
\mathbf{q}(x, y, z) = e^{i\alpha x} \sum_{n=-\infty}^{\infty} \hat{q}_n(x, y) e^{in\beta z},
\]  

(7.6)

where \( \hat{q}_0 \) is the MFD, higher-order harmonics in the spanwise direction represent various streaks of wavelength \( 2\pi/(n\beta) \), and \( \alpha = i\alpha_i \) is the uniform streamwise wavenumber over all harmonics. Similar to the procedure in Section 7.1.1, we derive the PFE in the form of Eq. (6.7) by substituting the ansatz (7.6) and modulated base flow (7.4) into the linearized NS equations and rearranging the governing equations for each harmonic. For this case study, the operators \( \mathbf{L}_F \) and \( \mathbf{M}_F \) in PFE (6.7) are provided in Appendix C.2.

The procedure for obtaining the results presented in the next subsection can be summarized as follows:

1. Compute the initial fluctuation field for maximum streamwise growth using the methodology presented in (Hack and Moin, 2017).

2. March the initial fluctuation field downstream for the linear evolution of the optimal streak using linear PSE.

3. Augment the Blasius boundary layer profile with the solution to linear PSE to obtain the modulated base flow \( \bar{u} \) according to Eq. (7.5).

4. Use the same initial condition as the linear PSE in step 2 and its complex conjugate to initialize \( \hat{q}_{\pm 1} \) in Eq. (7.6) and initialize other harmonics with zero. Also, set the initial growth rate \( \alpha(x_0) \) to a small imaginary number.
5. Use the PFE to compute the spatial evolution of all harmonic modes around the modulated base flow.

7.2.2 Nonlinear evolution of optimal streaks

Although the initial condition imposed at the first downstream location only contains a single spanwise wavenumber, the appreciable amplitudes of the developing streaks lead to modal interactions that introduce additional harmonics and modulate the mean flow. To investigate these harmonic interactions, we consider truncations of the bi-infinite state \( \hat{q} \) in the PFE (6.7) to \( 2N + 1 \) harmonics in \( z \), i.e., \( n = -N, \ldots, N \). We set \( N = 3 \) and consider a computational domain with \( L_x \times L_y = 2000 \times 60 \), \( N_y = 80 \) collocation points in the wall-normal direction, and a step-size of \( \Delta x = 15 \); see Appendix C.3 for a discussion on wall-normal grid-convergence and the dependence of results on the number of harmonics.

The temporal frequency, streamwise and fundamental spanwise wavenumbers are set to \( \omega = 0 \), \( \alpha = -10^{-10} i \), and \( \beta = 0.4065 \), respectively. Note that the small imaginary-valued wavenumber \( \alpha \) corresponds to infinitely long structures in the streamwise direction that saturate after a particular streamwise location. Moreover, \( \alpha \neq 0 \) maintains a well-conditioned downstream progression for the PFE computations. We initialize the PFE computation at \( Re_0 = 467 \) with zero initial conditions for all \( \hat{q}_n \) with \( n \neq \pm 1 \). The fundamental harmonic \( \hat{q}_{\pm 1} \) is initialized with the same initial condition as the primary linear PSE computations and with an rms amplitude of \( 6.4 \times 10^{-4} \). Since this case study considers the evolution of disturbances with a slowly varying streamwise wavenumber \( \alpha \), we set \( \alpha_x = 0 \) for both the primary linear PSE and the subsequent PFE computations. To verify the predictions of our framework, we also conduct direct numerical simulations of the nonlinear NS equations (with the same initial conditions) using a second-order finite volume code with \( 2049 \times 257 \times 257 \) grid points in the streamwise, wall-normal, and spanwise dimensions, respectively.

As illustrated in figure 7.2, all harmonics undergo an initial algebraic growth followed by saturation. The solution to the linear PSE accurately predicts the evolution of the fundamental spanwise harmonic; cf. Eq. (7.6). The PFE accurately predict the growth of
Figure 7.2: The rms amplitudes of the streamwise velocity components for various harmonics with $\omega = 0$ and $\beta = 0.4065$ resulting from DNS ($\triangle$), PFE ($-$), and linear PSE ($--$). The MFD, first, second, and third harmonics are shown in black, blue, red, and green, respectively.

The dominant harmonics, and especially the MFD. While a discrepancy is observed for the third harmonic, its contribution to the overall structure of the streaks is negligible. The reasonable prediction of growth trends and generation of the MFD component is a direct consequence of accounting for interactions between different harmonics within our framework because, apart from $\hat{q}_{\pm 1}$, all other harmonics were initialized with zero.

Figure 7.3 shows the cross-plane spatial structure of the streaks comprised of all harmonics in the spanwise direction at $x = 2400$. Comparison of figures 7.3(a) and 7.3(b) indicates a significant discrepancy between the shape of the structures in the cross-plane if the interaction between modes is not taken into account. Since the first (fundamental) harmonic has much larger amplitude than the second and third harmonics, the velocity distribution resulting from the linear PSE is dominated by the structure of the first harmonic. Furthermore, in the absence of interactions between harmonics, the linear PSE would not be able to generate the MFD and would thus result in inaccurate predictions for the amplitude of higher-order harmonics; see dashed lines in figure 7.2. In figure 7.2, we have used a scaled version of the initial condition for the first harmonic to initialize the linear PSE computations for higher-order harmonics. Figure 7.3(c) demonstrates excellent agreement.
Figure 7.3: Cross-plane contours of the streamwise velocity of the streaks comprised of all harmonics in the spanwise direction at $x = 2400$ resulting from DNS (a), linear PSE (b), and PFE with (c) and without (d) the MFD component.

Since the evolution of streaks is highly influenced by nonlinear interactions, it is worth examining if the results would change with an increase in the streak amplitude. To test the robustness of our framework we consider a different initial condition which has twice the amplitude as the previous case. We use the same computational configuration as before and
initialize all harmonics apart from $\hat{q}_{\pm 1}$ with zero. As shown in figure 7.4(b), the linear PSE provide a poor prediction for the amplitude of the fundamental harmonic. We use the MFD $\hat{q}_0$, fundamental harmonics $\hat{q}_{\pm 1}$, and second harmonics $\hat{q}_{\pm 2}$ from each run of PFE to update the base flow modulation and rerun the PFE (cf. figure 6.2(b)). The base flow in subsequent iterations is thus given by

$$
\begin{align*}
U(x, y) &= U_B(x, y) + U_{S,0}(x, y) + U_{S,1}(x, y) e^{i\beta z} + U_{S,1}^*(x, y) e^{-i\beta z} + \\
&\quad U_{S,2}(x, y) e^{2i\beta z} + U_{S,2}^*(x, y) e^{-2i\beta z}, \\
V(x, y) &= V_B(x, y) + V_{S,0}(x, y), \\
W(x, y) &= 0,
\end{align*}
$$

(7.7)

and takes the compact form

$$
\bar{u}(x, y, z) = \sum_{m=-2}^{2} u_m(x, y) e^{im\beta z}.
$$

(7.8)

Here, $U_{S,0}$, $U_{S,1}$, and $U_{S,2}$ represent the MFD, first- and second-order harmonics corresponding to the solution of the previous PFE run, respectively, and $u_0$ contains both the Blasius profile and the MFD. We note that higher-order harmonics are omitted from the base flow modulation in Eq. (7.8) as they do not significantly influence the profiles that result from the iterative PFE procedure. Furthermore, similar to Eq. (7.4), special care is taken in modulating the base flow, i.e., higher-order harmonics ($|m| \geq 1$) are excluded from the wall-normal and spanwise components of the base flow modulation in agreement with the structure and amplitude of the initial condition.

Figure 7.4(a) demonstrates how iterating the PFE can improve our prediction of the predominantly nonlinear streak evolution. The rms curves for various harmonics converge after 7 iterations of the PFE feedback loop illustrated in figure 6.2(b). Note that in the previous case of moderate-amplitude streaks, subsequent iterations were not necessary and accurate results were obtained after one run of the PFE over the streamwise domain. The
Figure 7.4: (a) The rms amplitudes of the streamwise velocity components for various harmonics with $\omega = 0$ and $\beta = 0.4065$ after the first (⋯), third (−−), and seventh (−) run of the PFE. The MFD, first, and second harmonics are shown in black, blue, and red, respectively. (b) The rms amplitudes resulting from DNS (Δ), nonlinear PSE (○), and PFE (−). The evolution of the fundamental harmonic due to linear PSE is shown by the thin solid line.

High number of iterations required for convergence is indicative of the significant role nonlinear terms play in the more challenging case of high-amplitude streaks. In figure 7.4(b), we compare the result from the final iteration (solid lines in figure 7.4(a)) with the result of DNS and nonlinear PSE. We see that the PFE capture the initial algebraic growth, inhibition of growth, and general trend in the saturation of amplitudes. Nonlinear interactions generate an appreciable MFD that alters the mean flow profile and hampers the growth of the principal harmonic in comparison to the single mode computation of linear PSE (cf. thin blue line in figure 7.4(a)). Previous studies have reported the stabilizing effect of nonlinearity on the evolution of unsteady streaks (Ricco et al., 2011) and the boundary layer response to perturbations (Leib et al., 1999b; Zuccher et al., 2006). In figure 7.4(b), while a discrepancy is observed in the prediction of the MFD, dominant trends in the shape and amplitude of velocity profiles are consistently captured in the streamwise domain, and the final amplitudes are in close agreement with the result of nonlinear PSE. However, both nonlinear PSE and PFE seem to under-predict the growth of the fundamental harmonic and MFD. Figures 7.6 and 7.7 show the streamwise velocity component of the MFD and first harmonic at $x = 1700$ and $x = 2400$, which correspond to the largest error in matching the
MFD and the end of the longitudinal domain. Finally, as the cross-plane contour plots of figure 7.5 demonstrate, the PFE provide good predictions for the spatial structure of the streaks that are comprised of various harmonics.

Figure 7.5: Cross-plane contours of the streamwise velocity of the higher amplitude streaks at $x = 2400$, which is comprised of all harmonics in the spanwise direction; (a) DNS and (b) PFE.

In the present case study, nonlinear interactions play a crucial role in the growth of high amplitude streaks. The PFE capture the nonlinear interactions by allowing spanwise modulations to the base state and extending the state variable $\hat{q}$ over multiple harmonics in the spanwise direction. Subsequent iterations of the PFE feedback loop (figure 6.2(b)) refine our predictions of nonlinear interactions on a sweep-by-sweep basis, i.e., by treating the base flow as a streamwise varying parameter in each individual PFE run, and only updating it for the next run. This is in contrast to nonlinear PSE in which nonlinear interactions are captured by explicitly converging over the corresponding nonlinear terms at each step of the streamwise progression. Regardless of how nonlinear interactions are captured, our results demonstrate the difficulty in accurately capturing the correct growth of these optimal streaks (cf. figure 7.4(b)). While the approximation used by the PFE framework may be seen as a limitation, the encouraging performance of the PFE warrants future study into improving the predictive capability of models that capture harmonic interactions through iterative refinement of the base state and not by explicitly computation of nonlinear terms.
Figure 7.6: (a) The streamwise velocity component of the MFD; and (b) the magnitude of the streamwise component of the first harmonic at $x = 1700$ resulting from DNS ($\triangle$) and PFE ($-$).

Figure 7.7: (a) The streamwise velocity component of the MFD; and (b) the magnitude of the streamwise component of the first harmonic at $x = 2400$ resulting from DNS ($\triangle$) and PFE ($-$).

### 7.3 Comparison with nonlinear PSE

In contrast to the nonlinear PSE, which treat the interaction between various modes as a forcing, the PFE introduced in Eq. (6.7) account for a subset of dominant interactions between the primary and secondary modes while maintaining the linear progression of the governing equations. The implementation and evaluation of the PFE is thus less complex than that of the nonlinear PSE as the explicit evaluation of the nonlinear terms and the commonly
used transformations between physical and Fourier space are avoided. More specifically, at each downstream location, the PSE can be viewed as a predictor-corrector algorithm in iteratively converging over nonlinear terms. The PFE eliminate these iterations which can become difficult to converge as the amplitudes of the harmonics grow. There have been previous efforts to suppress the feedback from secondary to primary modes and to maintain the march of nonlinear PSE through the transitional region; see for example (Herbert, 1994, Section 3.4.3). The framework proposed in this chapter allows for the formal investigation of such effects by limiting the interactions within the PFE framework to a subset of dominant harmonics of the base flow. While in practice we observe that the PFE alleviate such issues that arise from nonlinear interactions, a rigorous proof of convergence for the PFE iterations is a topic of future research.

The computational cost of the PSE is dominated by the inner iterations that are required to evaluate nonlinear terms at each step of the marching procedure. In contrast, the PFE are advanced by inverting a sparse high-dimensional matrix at each step. Indeed, a worst-case complexity analysis would suggest that the PFE need more operations per iteration. However, even without exploiting the sparse structure of the matrices, our PFE computations have shown to require approximately the same amount of time to converge as our nonlinear PSE computations. Further improving the computational efficiency of our method in a way that would lead to a fair comparison to the PSE is nonetheless out of the scope of the current work.
Chapter 8

Concluding remarks

We have combined ideas from Floquet decomposition and the linear PSE to develop the Parabolized Floquet equations, which can be used to march primary and secondary instability modes while accounting for dominant mode interactions. Our modeling framework involves two steps: (i) the linear PSE are used to march the primary disturbances in the streamwise direction; (ii) the PFE are used to march velocity fluctuations around the modulated base flow profile while capturing weakly nonlinear effects and the interaction of modes. The developed framework can account for secondary instabilities as fluctuations around a modulated base flow that includes primary modes generated in step (i). The PFE involve a linear march of various harmonics and can be used as a tool to decipher the role of individual harmonics in the spatial evolution of the fluctuation field. Furthermore, subsequent iterations of the PFE, in which the base flow modulation results from the previous PFE computation, can provide a corrective sequence that improves the prediction quality. To demonstrate the utility of the proposed modeling framework, we have examined the secondary instability analysis of the H-type transition scenario and the evolution of streamwise streaks. Our computational experiments demonstrate good agreement with DNS and nonlinear PSE.

In the PFE calculations, the proposed method’s overall performance relies on the reasonable prediction of the evolution of primary disturbances using linear PSE. In cases where the linear PSE give a poor prediction, an additional source of white or colored stochastic excitation can be used to replicate the effect of nonlinearities and improve the outcome of linear PSE; see (Ran et al., 2017a, Section IV). For this purpose, the spatio-temporal
Figure 8.1: Block diagram illustrating the inclusion of control into the PFE loop. The controller uses the fluctuation field resulting from previous PFE iterations to dictate the control signal $f$ which perturbs the flow dynamics.

A spectrum of stochastic excitation sources can be identified using the recently developed theoretical framework outlined in (Zare et al., 2017a,b). This methodology can also be used to improve the accuracy of results when nonlinear interactions are critical in the evolution of multimodal dynamics. The implementation of such ideas to further improve the proposed method’s predictive capability is a topic for future research.

In the PFE framework, nonlinear interactions are captured via the interplay between the state and periodic base flow. The equilibrium configuration in figure 6.2(b) provides the means to better approximate nonlinear interactions through the iterative refinement of the base state. While this configuration implies that the PFE framework is inherently nonlinear, each run of the PFE is linear and is thus well-suited for feedback control design using the tools from linear systems theory. More specifically, the modes that are marched using PFE modulate the base state as a streamwise varying parameter in subsequent PFE runs and thus should not be assumed as variables that violate the premise of linearity. Based on the fluctuation field generated at each sweep of PFE, an optimal control strategy can be synthesized to perturb the dynamics of subsequent PFE runs; see schematic in figure 8.1.

While both the control strategy and dynamics are simultaneously updated, convergence of the fluctuation field $\hat{q}$ would ensure that the final control design is optimal. Analyzing the performance of this design strategy and providing theoretical justification for convergence calls for an additional in-depth examination.
Part III

Model-based analysis of turbulent flows over riblets
Chapter 9

Modeling the effect of corrugated surfaces in channel flows

Carefully designed surface corrugation can decrease skin-friction drag by more than 10% Bechert et al. (1997); Gad-el Hak (2000) and has been successfully employed in engineering applications Coustols and Savill (1989); Joslin (1998). Previous numerical and experimental studies have examined the effect of various design parameters, e.g., the shape (triangular, T-shaped, etc.) and size of riblets, on skin-friction drag in turbulent flows Walsh (1982); Walsh and Lindemann (1984); Bechert et al. (1997, 2000); García-Mayoral and Jiménez (2011). While these studies offer valuable insights, their reliance on costly experiments and simulations has hindered the model-based design of riblet-mounted surfaces. This motivates the development of low-complexity models that capture the essential physics of turbulent flows over riblets and are well-suited for analysis, optimization, and control design.

The linearized Navier-Stokes (NS) equations capture structural and statistical features of transitional Farrell and Ioannou (1993b); Bamieh and Dahleh (2001); Jovanovic and Bamieh (2005); Ran et al. (2019b) and turbulent Hwang and Cossu (2010b); Zare et al. (2017b) shear flows. An additive source of stochastic excitation is often used to model the effect of background disturbances and uncertainty in the linearized equations. This approach has enabled the model-based analysis of sensor-free strategies for suppressing turbulence via streamwise traveling surface blowing and suction Moarref and Jovanovic (2010); Lieu et al. (2010) or transverse wall oscillations Jovanovic (2008); Moarref and Jovanovic (2012). In this work, we extend the framework of Moarref and Jovanovic (2012) to account for the effect of spanwise-periodic surface modification using tools from control theory. We use
turbulence modeling in conjunction with stochastically-forced linearized NS equations to compute modifications to the turbulent mean velocity and skin-friction drag in a channel flow with corrugated walls. This is an unconventional sensor-free boundary control problem; the goal is to quantify the influence of the fluctuation velocity field induced by spatially-periodic boundary conditions on the mean velocity profile and the resulting skin-friction drag.

Receptivity of channel flow over riblets was recently studied using the $\mathcal{H}_2$ norm of the linearized dynamics Kasliwal et al. (2012), as well as frequency response analysis Chavarin and Luhar (2019). In Kasliwal et al. (2012), a change of coordinates was used to translate spatially-periodic boundary conditions into spatially-periodic differential operators and, in Chavarin and Luhar (2019), a volume penalization technique Khadra et al. (2000) was used to capture the effect of riblets as a feedback term in the dynamics. While we adopt the latter approach, in contrast to prior studies, we account for dynamical interactions among fluctuation harmonics in the spatially-periodic model and utilize the linearized NS equations to provide closure in the mean flow equations for flows over riblets.

In this chapter, we construct the model-based framework to quantify the effect of streamwise-aligned spanwise-periodic riblets on kinetic energy and skin-friction drag in turbulent channel flow. The effect of riblets is modeled as a volume penalization in the Navier-Stokes equations, and we use the statistical response of the eddy-viscosity-enhanced linearized equations to quantify the effect of background turbulence on the mean velocity and skin-friction drag.

### 9.1 Problem formulation

The pressure-driven channel flow of incompressible Newtonian fluid, with geometry shown in figure 9.1(a), is governed by the Navier-Stokes and continuity equations

\[
\partial_t \mathbf{u} = - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla P + \frac{1}{Re_b} \Delta \mathbf{u},
\]

\[
0 = \nabla \cdot \mathbf{u},
\]

(9.1)
where \( \mathbf{u} \) is the velocity vector, \( P \) is the pressure, \( \nabla \) is the gradient operator, \( \Delta = \nabla \cdot \nabla \) is the Laplacian, \((x, y, z)\) are the streamwise, wall-normal, and spanwise directions, and \( t \) is time. The friction Reynolds number \( Re_\tau = u_\tau \delta / \nu \) is defined in terms of the channel’s half-height \( \delta \) and the friction velocity \( u_\tau = \sqrt{\tau_w / \rho} \), where \( \tau_w \) is the wall-shear stress (averaged over horizontal directions and time), \( \rho \) is the fluid density, and \( \nu \) is the kinematic viscosity.

In (9.1) and throughout this part, spatial coordinates are nondimensionalized by \( \delta \), velocity by \( u_\tau \), time by \( \delta / u_\tau \), and pressure by \( \rho u_\tau^2 \). We also assume that the bulk flux, which is obtained by integrating the streamwise velocity over spatial dimensions and time, remains constant via adjustment of the uniform streamwise pressure gradient \( \partial_x P \).

When the lower channel wall is corrugated with a spanwise-periodic surface \( r(z) \) that is aligned with the flow, as shown in Fig. 9.1(b), boundary conditions on \( \mathbf{u} \) are given by the no-slip and no penetration conditions,

\[
\mathbf{u}(x, y = 1, z, t) = \mathbf{u}(x, y = -1 + r(z), z, t) = 0. 
\]  

Solving the NS equations (9.1) subject to these boundary conditions requires a stretched mesh that conforms to the geometry dictated by a shape function \( r(z) \). This approach is computationally inefficient because it requires a large number of discretization points to resolve the grid in the vicinity of the wall. This motivates the development of low-complexity models for analysis, optimization, and design. The key challenge is to capture the effect of riblets on the turbulent flow so that skin-friction drag is accurately predicted.
As skin-friction drag depends on the gradient of the turbulent mean velocity at the wall, a natural first step is to determine an approximation to the mean velocity in the presence of riblets. To this end, we adopt the Reynolds decomposition to split the velocity and pressure fields into their time-averaged mean and fluctuating parts as

\[
\begin{align*}
\mathbf{u} &= \bar{\mathbf{u}} + \mathbf{v}, & \langle \mathbf{u} \rangle &= \bar{\mathbf{u}}, & \langle \mathbf{v} \rangle &= 0, \\
P &= \bar{P} + p, & \langle P \rangle &= \bar{P}, & \langle p \rangle &= 0.
\end{align*}
\] (9.3)

Here, \( \bar{\mathbf{u}} = [U \; V \; W]^T \) is the vector of mean velocity components, \( \mathbf{v} = [u \; v \; w]^T \) is the vector of velocity fluctuations, \( p \) is the fluctuating pressure field around the mean \( \bar{P} \), and \( \langle \cdot \rangle \) denotes the expected value,

\[
\langle \mathbf{u}(x,y,z,t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{u}(x,y,z,t+\tau) \, d\tau.
\] (9.4)

Substituting the decomposition (9.3) into the NS equations (9.1) and taking the expectation yields the Reynolds-averaged NS equations

\[
\begin{align*}
\partial_t \bar{\mathbf{u}} &= - (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} - \nabla \bar{P} + \frac{1}{\text{Re}_\tau} \Delta \bar{\mathbf{u}} - \nabla \cdot \langle \mathbf{vv}^T \rangle, \\
0 &= \nabla \cdot \bar{\mathbf{u}}.
\end{align*}
\] (9.5)

The Reynolds stress tensor \( \langle \mathbf{vv}^T \rangle \) quantifies the transport of momentum arising from turbulent fluctuations (McComb, 1991), and its value significantly affects the solution of equations (9.5). The difficulty in obtaining the fluctuation correlations stems from closure problem. We overcome this challenge by utilizing the turbulent viscosity hypothesis (McComb, 1991), which considers the turbulent momentum to be transported in the direction of the mean rate of strain

\[
\langle \mathbf{vv}^T \rangle - \frac{1}{3} \text{trace}(\langle \mathbf{vv}^T \rangle) \mathbf{I} = - \frac{\nu_T}{\text{Re}_\tau} \left( \nabla \bar{\mathbf{u}} + (\nabla \bar{\mathbf{u}})^T \right),
\] (9.6)
where $\nu_T(y)$ is the turbulent eddy viscosity normalized by molecular viscosity $\nu$, and $I$ is the identity operator. As we discuss in what follows, our choice of turbulence model is motivated by Moarref and Jovanovic (2012) that demonstrated its utility in capturing the effect of transverse wall oscillations on turbulent drag and kinetic energy in channel flow.

### 9.1.1 Modeling surface corrugation

To account for the effect of riblets, we use the volume penalization technique proposed by Khadra et al. (2000). In this method, the solid obstruction of the flow is modeled as a spatially-varying permeability function $K$ that enters the governing equations through an additive body forcing term. This modulation along with the turbulent viscosity hypothesis (9.6) brings the mean flow equations (9.5) into the following form,

\[
\partial_t \bar{u} = - (\bar{u} \cdot \nabla) \bar{u} - \nabla \bar{P} - K^{-1} \bar{u} + \frac{1}{Re_T} \nabla \cdot \left( (1 + \nu_T) (\nabla \bar{u} + (\nabla \bar{u})^T) \right),
\]

\[
0 = \nabla \cdot \bar{u}.
\] (9.7)

The permeability function $K$ takes on two values: within the fluid, $K \rightarrow \infty$ yields back the original mean flow equations (9.5); and within the riblets, $K \rightarrow 0$ forces the velocity field to zero. Following Chavarin and Luhar (2019), we account for streamwise-constant, spanwise-periodic corrugation by considering the harmonic resistance

\[
K^{-1}(y, z) = \sum_{m \in \mathbb{Z}} a_m(y) \exp(\im \omega_z z).
\] (9.8)

Here, $\omega_z$ is the fundamental spatial frequency of the riblets and $a_m(y)$ are the Fourier series coefficients of $K^{-1}(y, z)$. The dependence of the resistance function $K^{-1}(y, z)$ on $y$ and $z$ follows from the geometry of riblets. Figure 9.2 shows a resistance function and corresponding dominant coefficients for triangular riblets.

In practice, we construct a resistance function $K^{-1}(y, z)$, e.g., the one shown in figure 9.2(a), and then compute $a_m(y)$ using the Fourier transform in $z$. Ideally, at any spanwise location $z$, the resistance should emulate a wall-normal step function at the interface.
Figure 9.2: (a) Resistance function $K^{-1}(y,z)$ given by equation (9.9) with $h = 0.0804$, $R = 1.5 \times 10^5$, $s_f = 141$, $\omega_z = 30$, and $r_p = 0.7434$. (b) The first five Fourier coefficients $a_m(y)$ with successively decreasing amplitudes corresponding to riblets shown in (a).

of the solid riblet surface and the fluid; see figure 9.3. However, in favor of wall-normal differentiability, we use the hyperbolic approximation

$$K^{-1}(y,z) = \frac{R}{2} \left(1 - \tanh \left(s_f (y + 1 - r(z))\right)\right)$$ (9.9)

where $-1 + r(z)$ indicates the location of the lower corrugated wall (cf. Eq. (9.2)), $s_f$ is a smoothness factor that modifies the slope of the hyperbolic curve, and $R$ is a resistance rate that controls the accuracy of the solution in the solid region. While larger values of $s_f$ yield a better approximation of the step function, they require the use of a larger number of harmonics to maintain the smoothness of the resistance field. Herein, we choose $s_f$ to be inversely proportional to the height of the riblets $h$, i.e., $s_f = 3.6\pi/h$. On the other hand, while large values of the resistance rate $R$ induce a smaller velocity field within the riblets, they may trigger spurious negative solutions. In view of this fundamental trade-off, we relax the non-negativity constraint on $\bar{u}$ and choose $R$ to guarantee that the solution to (9.7) is larger than $-1 \times 10^{-6}$. In particular, for turbulent channel flow with $Re = 186$ over the triangular lower-wall riblets with frequency $\omega_z = 30$ and height to spacing ratio $h/s = 0.38$, our computational experiments show that $R = 1.5 \times 10^5$, $s_f = 141$, and 25 spanwise harmonics ($m = -12, \ldots, 12$) yield small negative mean velocity while preserving
the smoothness of the resistance field. For triangular riblets with \( \omega_z = 30 \) and height \( h = 0.0804 \), figure 9.2(a) shows the resistance field \( K^{-1} \) resulting from Eq. (9.9) with,

\[
r(z) = -h r_p + \frac{h \omega_z}{\pi} \left| z - \frac{2\pi}{\omega_z} \left( 1 + \left\lfloor \frac{z \omega_z}{2\pi} - \frac{1}{2} \right\rfloor \right) \right|.
\]

(9.10)

Here, \( \cdot \) is the absolute value, \( \lfloor \cdot \rfloor \) is the floor function, and \( r_p \) denotes the proportion of the riblet height in the extended channel, i.e., below \( y = -1 \). In this study, we tune \( r_p \), and thereby adjust the wall-normal position of riblets, so that the mean velocity profile resulting from (9.7) has the same bulk as the channel flow with smooth walls. While the choice of \( r(z) \) in equation (9.10) corresponds to triangular riblets, which are used as a case study throughout this part, the shape function \( r(z) \) can be selected to account for an arbitrary spanwise-periodic surface corrugation.

For a given smoothness factor \( s_f \), we start from an initial choice of \( r_p \) and resistance rate \( R \) and iterate steps (i)-(iii) below to identify \( r_p \) and the largest \( R \) that ensure that the mean velocity is greater than \(-1 \times 10^{-6}\) and that it satisfies the constant bulk flux condition.

1. Determine the shape function \( r(z) \) to capture the desired geometry of riblets.

2. Use the shape function \( r(z) \) to construct the resistance function \( K^{-1}(y, z) \) using the hyperbolic approximation (9.9) and derive the Fourier series coefficients \( a_m(y) \).

3. Solve (9.7) for \( \bar{u} \) and check to see if it has the same bulk as the turbulent channel flow with smooth walls.

9.1.2 The turbulent mean velocity

We approach the problem of quantifying the influence of riblets on skin-friction drag by developing robust models that approximate the turbulent viscosity \( \nu_T \) in equations (9.7). Several studies have offered expressions for \( \nu_T \) that yield the turbulent mean velocity in the flow over smooth walls (Malkus, 1956; Cess, 1958; Reynolds and Tiederman, 1967).
The wall-normal dependence of the resistance function $K^{-1}(y, z)$ at the tip ($z = \pi/30$) of the triangular riblet given in figure 9.2(a). The dashed curve results from equation (9.9) and it represents a smooth hyperbolic approximation to the step function (solid line). Here, $h = 0.0804$, $R = 1.5 \times 10^5$, $s_f = 141$, $\omega_z = 30$, $r_p = 0.7434$, and the function $r(z)$ represents triangular riblets.

The following turbulent viscosity model for channel flow was developed by Reynolds and Tiederman (1967) as an extension of the model introduced by Cess (1958) for pipe flow:

$$\nu_T(y) = \frac{1}{2} \left( 1 + \left( \frac{c_2}{3} Re_T (1 - y^2)(1 + 2y^2) \left( 1 - e^{-|y| Re_T/c_1} \right) \right)^2 \right) \frac{1}{2} - 1. \quad (9.11)$$

In this expression, parameters $c_1$ and $c_2$ are selected to minimize the least squares deviation between the mean streamwise velocity obtained in experiments and simulations and the steady-state solution to Eq. (9.7) without riblets using the averaged wall-shear stress $\tau_w = 1$ and $\nu_T$ given by Eq. (9.11). For turbulent channel flow with $Re_T = 186$, the optimal parameters $c_1 = 46.2$ and $c_2 = 0.61$ provide the best fit to the mean velocity in a turbulent channel flow resulting from DNS (Del Álamo and Jiménez, 2003; Del Álamo et al., 2004). For the turbulent channel flow with $Re_T = 547$ discussed in § 10.1.1 and § 10.2.3 these parameters are $c_1 = 29.4$ and $c_2 = 0.45$. Even though the turbulent viscosity model given by Eq. (9.11) does not hold in the presence of riblets, we use $\nu_T$ as a starting point for determining the mean flow in the presence of riblets. Furthermore, in the vicinity of the solid wall the flow
is dominated by viscosity and, for small-size riblets, the flow in the grooved region can be assumed to be laminar. Thus, we consider small-size riblets and set $\nu_T = 0$ for $y \leq -1$.

As shown in § 9.1.1, a harmonic resistance function $K^{-1}(y, z)$ is used to model a spatially periodic surface corrugation; cf. (9.8). The corresponding base flow, i.e., the solution to the steady-state mean flow equations (9.7), can be also decomposed into the Fourier series

$$\tilde{u}(y, z) = \sum_{m \in \mathbb{Z}} \tilde{u}_m(y) \exp(i m \omega_z z). \quad (9.12)$$

The steady-state solution to the nonlinear mean flow equations (9.7) is obtained via Newton’s method and it only contains a streamwise velocity component, $\tilde{u} = [\bar{U}(y, z) \ 0 \ 0]^T$. Since the spanwise and wall-normal base flow components are zero, the nonlinear terms in mean flow equation (9.7) vanish and the equation for $\bar{U}(y, z)$ is linear,

$$(1 + \nu_T) \Delta \bar{U} + \nu_T' \bar{U}' - K^{-1} \bar{U} = Re_\tau \bar{P}_x. \quad (9.13)$$

Here $\bar{U}'$ denotes the wall-normal derivative of $\bar{U}$ and $\bar{P}_x$ is the mean pressure gradient. Inclusion of the harmonics of $K^{-1}$ yields the equation for the $m$th harmonic $\bar{U}_m$,

$$\left[(1 + \nu_T) \left(\partial_y^2 - m^2 \omega_z^2\right) + \nu_T' \partial_y - a_0\right] \bar{U}_m + \sum_{n \in \mathbb{Z}\{0\}} a_n \bar{U}_{m-n} = \begin{cases} Re_\tau \bar{P}_x, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

which amounts to the following bi-infinite matrix form:

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & L_{m-1,0} & L_{m-1,+1} & L_{m-1,+2} & \cdots \\ \cdots & L_{m,-1} & L_{m,0} & L_{m,+1} & \cdots \\ \cdots & L_{m+1,-2} & L_{m+1,-1} & L_{m+1,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \bar{U}_{-1} \\ \bar{U}_0 \\ \bar{U}_1 \\ \cdots \end{bmatrix} = \begin{bmatrix} \cdots \\ Re_\tau \bar{P}_x \end{bmatrix}. \quad (9.14)$$
A pseudospectral scheme with Chebyshev polynomials (Weideman and Reddy, 2000) is used to discretize the differential operators in the wall-normal direction. To avoid numerical oscillations in the solution to equations (9.14), we divide the wall-normal extent of the computational domain into two parts using block operators (Aurentz and Trefethen, 2017) and use \( N_i \) collocation points for \( y \in [-1, 1] \) and \( N_o \) collocation points for \( y \in [-1 - r_p h, -1] \).

We impose no-slip boundary conditions (9.2) on the upper wall. Ideally, the adopted volume penalization method should automatically enforce immersed boundary conditions on the non-smooth lower wall without the need for additional boundary conditions. However, in practice, since the resistance rate \( R \) in (9.9) is a finite number, the immersed boundary conditions cannot be exactly enforced. To ensure that the operators in (9.14) are well-defined, we employ additional no-slip conditions at the lower boundary (\( y = -1 - r_p h \)). The boundary conditions at the intersection of the aforementioned wall-normal regimes (\( y = -1 \)) enforce smoothness on physical quantities, i.e.,

\[
\begin{align*}
\bar{U}(y = -1^+, z) &= \bar{U}(y = -1^-, z) \\
\frac{\partial \bar{U}}{\partial y}(y = -1^+, z) &= \frac{\partial \bar{U}}{\partial y}(y = -1^-, z).
\end{align*}
\]

Figure 9.4 shows the solution \( \bar{U} \) to equation (9.7) for a turbulent channel flow with \( \text{Re}_x = 186 \) subject to a streamwise pressure gradient \( \bar{P}_x = -1 \) over triangular riblets. Here, \( N_i = 179, \) \( N_o = 20, \) 25 harmonics have been used to approximate the solution, i.e., \( m = -12, \ldots, 12, \) and the triangular riblets are characterized by \( K^{-1}(y, z) \) with \( h = 0.0804, \omega_z = 30, R = 1.5 \times 10^5, \) \( s_f = 141, \) and \( r_p = 0.7434. \) Figure 9.4 demonstrates that the solution respects the shape of riblets and is approximately zero in the solid region.

### 9.1.3 Prediction of drag reduction

In what follows, subscripts \( s \) and \( c \) are used to signify channel flow with smooth walls and the correction that arises from surface corrugation. We use a variation in the driving pressure
Figure 9.4: The streamwise mean velocity $\bar{U}(y, z)$ for turbulent channel flow with $Re_{\tau} = 186$ over the triangular riblets given in figure 9.2(a).

gradient to enforce a constant bulk flux requirement. This introduces a correction to the 0th harmonic of the mean velocity as

$$\bar{U}_c(y) = \bar{U}_0(y) - \left(1 - \bar{P}_{x,c}\right)\bar{U}_s(y). \quad (9.15)$$

Here, $\bar{U}_s(y)$ denotes the mean velocity profile in channel flow with smooth walls and the additional bulk introduced by the 0th harmonic of the solution to equation (9.13) is used to compute the correction to pressure gradient $\bar{P}_{x,c}$, i.e.,

$$\bar{P}_{x,c} = 1 - \frac{1}{U_B} \int_{-h_p}^{1} \bar{U}_0(y) \, dy. \quad (9.16)$$

The form of $\bar{P}_{x,c}$ ensures that the mean velocity correction $\bar{U}_c(y)$ in equation (9.15) has zero bulk. The corrections to the pressure gradient and mean velocity are used to compute the variation in skin-friction drag.

The rate of drag reduction caused by riblets is given by

$$\Delta D := (D - D_s) / D_s,$$

where $D_s$ denotes the slope of the mean velocity at the lower wall in a flow without riblets. In a flow with riblets, the skin-friction drag at the lower wall can be computed using the
pressure gradient, which maintains a constant bulk, and the well-defined slope of the mean velocity at the upper wall,
\[ D = \tilde{P}_x - \frac{\omega_z}{2\pi} \int_0^{2\pi/\omega_z} \frac{\partial \bar{U}}{\partial y} (y = 1, z) \, dz. \]

Since \( \tilde{P}_x = 2D_s \), \( \Delta D \) is determined by the difference between the pressure gradient adjustment and the drag reduction at the upper wall, i.e.,
\[ \Delta D = \frac{1}{D_s} \left[ \tilde{P}_{x,c} - \left( \frac{\omega_z}{2\pi} \int_0^{2\pi/\omega_z} \frac{\partial \tilde{U}}{\partial y} (y = 1, z) \, dz - D_s \right) \right]. \quad (9.17) \]

Even though the mean velocity profile shown in figure 9.4 respects the shape of riblets and goes to zero within the solid region, the resulting drag reduction does not follow trends reported in literature. As demonstrated in figure 9.5, the mean velocity profile resulting from the use of \( \nu_{Ts} \) implies a reduction in drag regardless of the size of riblets. Furthermore, no optimal spacing that maximizes drag reduction is identified. To improve predictions of the mean velocity and the resulting skin-friction drag, in § 9.2 we extend the framework proposed in Moarref and Jovanovic (2012) to account for the effect of velocity fluctuations in a flow over riblets on the turbulent viscosity \( \nu_T \).

### 9.2 Stochastically-forced dynamics of velocity fluctuations

In this section, we compute a correction to the turbulent viscosity and, subsequently, the mean velocity of a turbulent channel flow over riblets using second-order statistics of velocity fluctuations. To this end, we examine the dynamics of fluctuations around the mean velocity profile computed in § 9.1.3. As illustrated in figure 9.6, our model-based framework for studying the effect of riblets involves the following steps:

1. [§ 9.1.2] The turbulent mean velocity \( \bar{u} \) is obtained from equations (9.7), where closure is achieved using the turbulent viscosity \( \nu_{Ts} \) for the channel flow with smooth walls.
Figure 9.5: Prediction of drag reduction in a turbulent channel flow with $Re_T = 186$ resulting from the steady-state solution of equations (9.7) with $\nu_{T_s}(y)$ given by (9.11). Triangular riblets, shown in figure 10.1, with different peak-to-peak spacing but a constant height to spacing ratio $h/s = 0.38$ are considered and the spacing is reported in inner viscous units, i.e., $s^+ = Re_s s$.

2. [§ 9.2.4] The stochastically forced linearized NS equations around the mean flow $\bar{u}$ resulting from step (i) are used to compute the second-order statistics of the fluctuating velocity field and provide a correction to $\nu_{T_s}$.

3. [§ 9.1.2 and § 9.1.3] The modification to turbulent viscosity is used to correct the mean velocity and compute skin-friction drag.

In § 10.1.1, we show that the correction to the mean velocity significantly improves our prediction of the optimal size of triangular riblets for drag reduction. The separation of steps (i) and (iii), in which the mean velocity is updated, from step (ii), in which the statistics of velocity fluctuations are computed, is justified by the slower time evolution of the mean velocity compared to fluctuations (Moarref and Jovanovic, 2012). While the turbulent viscosity and the mean velocity can be updated in an iterative manner, a theoretical justification for the convergence of such an iterative procedure requires additional examination and is outside of the scope of the current study. Even though our discussion focuses on spanwise-periodic triangular riblets, the methodology and theoretical framework that we develop can be used to study turbulent flows over a much broader class of periodic surface corrugations.
9.2.1 Model equation for $\nu_T$

As described in § 9.1.3, $\nu_{Ts}$ does not provide the proper eddy viscosity model for the channel flow with riblets. Establishing a relation between $\nu_T$ and the second-order statistics of velocity fluctuations represents the main challenge for identifying the appropriate eddy viscosity model. With appropriate choices of velocity and length scales, turbulent viscosity can be expressed as (Pope, 2000)

$$\nu_T(y) = c R e_T^2 \frac{k^2(y)}{\epsilon(y)}$$  \hspace{1cm} (9.18)

where $c = 0.09$, $k$ is the turbulent kinetic energy, and $\epsilon$ is the rate of dissipation. The $k$-$\epsilon$ model (Jones and Launder, 1972; Launder and Sharma, 1974) provides two differential transport equations for $k$ and $\epsilon$, but is computationally demanding and does not offer insight into analysis, design, and optimization. On the other hand, wall-normal profiles for $k$ and
\( \epsilon \) can be obtained by averaging the second-order statistics of velocity fluctuations over the streamwise coordinate and one period of the spanwise surface corrugation:

\[
k(y) = \frac{1}{2} \left( \langle uu \rangle + \langle vv \rangle + \langle ww \rangle \right)
\]

\[
\epsilon(y) = 2 \left( \langle u_x u_x \rangle + \langle v_y v_y \rangle + \langle w_z w_z \rangle + \langle u_y v_x \rangle + \langle u_z w_x \rangle + \langle v_z w_y \rangle \right)
+ \langle u_y u_y \rangle + \langle w_y w_y \rangle + \langle v_x v_x \rangle + \langle w_x w_x \rangle + \langle u_z u_z \rangle + \langle v_z v_z \rangle.
\]

(9.19)

Here, overline denotes averaging in \( x \) and one period in \( z \). We next demonstrate how second-order statistics, e.g., \( uu \), can be computed using the stochastically forced linearized NS equations.

### 9.2.2 Stochastically forced linearized Navier-Stokes equations

The dynamics of infinitesimal velocity \( \mathbf{v} = [u \ v \ w]^T \) and pressure \( p \) fluctuations around \( \mathbf{u} = [\bar{U}(y,z) \ 0 \ 0]^T \) and \( \bar{P} \) are governed by the linearized NS and continuity equations:

\[
\begin{aligned}
\partial_t \mathbf{v} &= - (\nabla \cdot \mathbf{u}) \mathbf{v} - (\nabla \cdot \mathbf{v}) \mathbf{u} - \nabla p - K^{-1} \mathbf{v} \\
&\quad + \frac{1}{Re_x} \nabla \cdot \left((1 + \nu_T)(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)\right) + \mathbf{f}, \\
0 &= \nabla \cdot \mathbf{v},
\end{aligned}
\]

(9.20)

where \( \mathbf{f} \) is a zero-mean white-in-time additive stochastic forcing. The normal modes in \( x \) are given by \( e^{i k_x x} \), where \( k_x \) is the streamwise wavenumber, and the normal modes in \( z \) are given by the Bloch waves (Odeh and Keller, 1964; Bensoussan et al., 1978), which are determined by the product of \( e^{i \theta z} \) with \( \theta \in [0, \omega_z/2) \) and a \( 2\pi/\omega_z \) periodic function in \( z \). For example, the forcing field in (9.20) can be represented as

\[
\begin{align*}
\mathbf{f}(x,y,z,t) &= e^{i k_x x} e^{i \theta z} \hat{f}(k_x, y, z, t) \\
\hat{f}(k_x, y, z, t) &= \hat{f}(k_x, y, z + 2\pi/\omega_z, t)
\end{align*}
\]

(9.21)
where real parts are used to represent physical quantities. The Fourier series expansion of
the $2\pi/\omega_z$-periodic function $\hat{f}(k_x, y, z, t)$ can be used to obtain,

$$f(x, y, z, t) = \sum_{n \in \mathbb{Z}} \hat{f}_n(k_x, y, \theta, t) e^{i(k_x x + \theta_n z)}, \quad \theta_n = \theta + n\omega_z,$$

$$k_x \in \mathbb{R}, \theta \in [0, \omega_z/2) \tag{9.22}$$

where $\{\hat{f}_n(k_x, y, \theta, t)\}_{n \in \mathbb{Z}}$ are the Fourier coefficients of the function $\hat{f}(k_x, y, z, t)$ in (9.21).

Substituting (9.22) into the linearized equations (9.20) and eliminating pressure through
a standard conversion (Schmid and Henningson, 2001) yields the evolution form

$$\partial_t \varphi_\theta(k_x, y, t) = [A_\theta(k_x) \varphi_\theta(k_x, \cdot, t)](y) + d_\theta(k_x, y, t),$$

$$v_\theta(k_x, y, t) = [C_\theta(k_x) \varphi_\theta(k_x, \cdot, t)](y), \tag{9.23}$$

with the state $\varphi_\theta$ consisting of the wall-normal velocity $v$ and vorticity $\eta = \partial_z u - \partial_x w$. The state-space representation (9.23) is parameterized by the streamwise wavenumber $k_x$ and the spanwise wavenumber offset $\theta$: for each $k_x$ and $\theta$, $\varphi_\theta$, $v_\theta$, and $d_\theta := B_\theta f_\theta$ are bi-infinite column vectors, e.g., $\varphi_\theta(k_x, y, t) = \text{col}\{\hat{\varphi}_n(k_x, y, \theta, t)\}_{n \in \mathbb{Z}}$, and $A_\theta(k_x)$, $B_\theta(k_x)$, and $C_\theta(k_x)$ are bi-infinite matrices whose elements are operators in $y$, e.g.,

$$A_\theta := \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & A_{n-1,0} & A_{n-1,1} & A_{n-1,2} & \cdots \\
\cdots & A_{n,0} & A_{n,1} & \cdots \\
\cdots & A_{n+1,-2} & A_{n+1,-1} & A_{n+1,0} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{9.24}$$

where the off-diagonal term $A_{n,m}$ denotes the influence of the $(n + m)$th harmonic $\varphi_{n+m}$ on
the dynamics of the $n$th harmonic $\varphi_n$. Apart from accounting for an extended wall-normal region, the block operators on the main diagonal of $A_\theta$ are identical to the operators for the channel flow without riblets; see Appendix D.1 for details. At the upper wall of the channel,
homogenous Dirichlet boundary conditions are imposed on $\eta$, and homogeneous Dirichlet and Neumann boundary conditions are imposed on $v$. Similar to the mean flow equations (9.7), the boundary conditions at the corrugated surface are automatically satisfied via volume penalization. Finally, smoothness of all physical quantities at the intersection of the inner and outer wall-normal regimes ($y = -1$) is imposed by enforcing the following conditions:

$$
\begin{align*}
v(y = -1^+, z) &= v(y = -1^-, z), & \frac{\partial v}{\partial y}(y = -1^+, z) &= \frac{\partial v}{\partial y}(y = -1^-, z), \\
\frac{\partial^2 v}{\partial y^2}(y = -1^+, z) &= \frac{\partial^2 v}{\partial y^2}(y = -1^-, z), & \frac{\partial^3 v}{\partial y^3}(y = -1^+, z) &= \frac{\partial^3 v}{\partial y^3}(y = -1^-, z), \\
\eta(y = -1^+, z) &= \eta(y = -1^-, z), & \frac{\partial \eta}{\partial y}(y = -1^+, z) &= \frac{\partial \eta}{\partial y}(y = -1^-, z).
\end{align*}
$$

A pseudospectral scheme used for discretizing the mean flow equations (9.7) is utilized to discretize the wall-normal operators in (9.23). In addition, a change of variables is employed to obtain a state-space representation in which the kinetic energy is determined by the Euclidean norm of the state vector in a finite-dimensional approximation of the evolution model (Zare et al., 2017b, Appendix A),

$$
\begin{align*}
\dot{\psi}_\theta(k_x, t) &= A_\theta(k_x) \psi_\theta(k_x, t) + d_\theta(k_x, t), \\
v_\theta(k_x, t) &= C_\theta(k_x) \psi_\theta(k_x, t).
\end{align*}
\tag{9.25}
$$

For $N_i$ and $N_o$ collocation points in the inner and outer wall-normal regimes, respectively, and a Fourier series expansion (9.8) with $M$ harmonics, $\psi_\theta(k_x, t)$ and $v_\theta(k_x, t)$ are vectors with $2 \times M \times (N_i + N_o)$ and $3 \times M \times (N_i + N_o)$ complex-valued entries, respectively. The state-space matrices $A_\theta(k_x)$ and $C_\theta(k_x)$ are discretized versions of the operators in (9.23) that incorporate the aforementioned change of coordinates.
9.2.3 Second-order statistics of velocity fluctuations and forcing

Let the linearized dynamics (9.25) be driven by zero-mean stochastic forcing \(d_\theta(k_x, t)\) that is white in time, with covariance matrix \(M_\theta(k_x) = M_\theta^*(k_x) \succeq 0\), i.e.,

\[
\langle d_\theta(k_x, t_1) d_\theta^*(k_x, t_2) \rangle = M_\theta(k_x) \delta(t_1 - t_2),
\]

(9.26)

where \(\delta\) is the Dirac delta function. Following the bi-infinite structure of \(d_\theta(k_x, t)\), \(M_\theta(k_x)\) takes the bi-infinite form,

\[
M_\theta(k_x) := \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \ddots \\
\vdots & M(k_x, \theta_{n-1}) & 0 & 0 & \ddots \\
\vdots & 0 & M(k_x, \theta_n) & 0 & \ddots \\
\vdots & 0 & 0 & M(k_x, \theta_{n+1}) & \ddots \\
\ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

(9.27)

with the block operator \(M(k_x, \theta_n)\) representing the covariance of the \(n\)th harmonic of the forcing \(d_\theta(k_x, t)\). The off-diagonal blocks of \(M_\theta(k_x)\) are zero because the stochastic forcing is uncorrelated over various spanwise harmonics.

The steady-state covariance of the state in equations (9.25) can be determined from the solution \(X_\theta(k_x)\) to the Lyapunov equation (Fardad et al., 2008; Moarref and Jovanovic, 2010)

\[
A_\theta(k_x) X_\theta(k_x) + X_\theta(k_x) A_\theta^*(k_x) = -M_\theta(k_x),
\]

(9.28)

where the \((i, j)\)th block of \(X_\theta(k_x)\) determines the correlation matrix associated with the \(i\)th and \(j\)th harmonics of the state \(\psi_\theta\).
As mentioned in § 9.2.2, the block operators on the main diagonal of $A_\theta$ contain the dynamical generators of the turbulent channel flow with smooth walls at various wavenumber pairs $(k_x, \theta_n)$. Based on this, $A_\theta$ can be decomposed as

$$A_\theta(k_x) = A_{\theta,s}(k_x) + A_{\theta,c}(k_x).$$

(9.29)

where $A_{\theta,s} = \text{diag}\{\ldots, A_s(k_x, \theta_{n-1}), A_s(k_x, \theta_n), A_s(k_x, \theta_{n+1}), \ldots\}$ accounts for the dynamical generator of the turbulent channel flow with smooth walls and $A_{\theta,c}$ captures the contribution of the spatially periodic surface corrugation. The block-diagonal structure of $M_\theta(k_x)$ implies that the solution $X_\theta(k_x)$ to (9.28) can also be decomposed as

$$X_\theta(k_x) = X_{\theta,s}(k_x) + X_{\theta,c}(k_x).$$

(9.30)

Here, $X_{\theta,s} = \text{diag}\{\ldots, X_s(k_x, \theta_{n-1}), X_s(k_x, \theta_n), X_s(k_x, \theta_{n+1}), \ldots\}$ is a block-diagonal covariance operator whose entries are determined by the steady-state covariance matrix of turbulent channel flow over smooth walls parameterized by $(k_x, \theta_n)$, and $X_{\theta,c}$ denotes the modification resulting from the presence of riblets. This follows from substitution of (9.29) and (9.30) into the Lyapunov equation (9.28),

$$(A_{\theta,s}(k_x) + A_{\theta,c}(k_x))(X_{\theta,s}(k_x) + X_{\theta,c}(k_x))$$

$$+ (X_{\theta,s}(k_x) + X_{\theta,c}(k_x))(A_{\theta,s}(k_x) + A_{\theta,c}(k_x))^* = -M_\theta(k_x),$$

from which the Lyapunov equation corresponding to the turbulent channel flow with smooth walls can be extracted as

$$A_{\theta,s}(k_x) X_{\theta,s}(k_x) + X_{\theta,s}(k_x) A_{\theta,s}(k_x) = -M_\theta(k_x).$$

Following Moarref and Jovanovic (2012), we select the block-diagonal covariance matrix $M_\theta(k_x)$ to guarantee equivalence between the two-dimensional energy spectrum of the turbulent channel flow with smooth walls and the flow governed by the stochastically-forced
NS equations linearized around $\bar{\mathbf{u}} = [\bar{U}_s(y) \ 0 \ 0]^T$. This is achieved by scaling the block covariances in (9.27) as

$$M(k_x, \theta_n) = \frac{\bar{E}_s(k_x, \theta_n)}{E_{s,0}(k_x, \theta_n)} M_s(k_x, \theta_n).$$

Here, $\bar{E}_s(k_x, \theta_n) = \int_{-1}^{1} E_s(y, k_x, \theta_n) \, dy$ is the two-dimensional energy spectrum of turbulent channel flow with smooth walls, which is obtained from the DNS-based energy spectrum $E_s(y, k_x, \theta_n)$ (Del Álamo and Jiménez, 2003; Del Álamo et al., 2004), and $\bar{E}_{s,0}(k_x, \theta_n)$ is the energy spectrum resulting from the linearized NS equations (9.25) subject to white-in-time stochastic forcing with the covariance matrix

$$M_s(k_x, \theta_n) = \begin{bmatrix} \sqrt{E_s(y, k_x, \theta_n)} I & 0 \\ 0 & \sqrt{E_s(y, k_x, \theta_n)} I \end{bmatrix} \begin{bmatrix} \sqrt{E_s(y, k_x, \theta_n)} I & 0 \\ 0 & \sqrt{E_s(y, k_x, \theta_n)} I \end{bmatrix}^*.$$

Finally, the energy spectrum of velocity fluctuations is determined from the solution to the Lyapunov equation (9.28) as

$$\bar{E}(k_x, \theta) = \sum_{n \in \mathbb{Z}} \text{trace} (X_d(k_x, \theta_n)). \quad (9.31)$$

where $X_d(k_x, \theta_n)$ represent the block covariance matrices on the main diagonal of $X_{\theta}(k_x)$ confined to the wall-normal range of $y \in [-1, 1]$. The correction to the energy spectrum that arises from the presence of riblets is determined by

$$\bar{E}_c(k_x, \theta) = \bar{E}(k_x, \theta) - \bar{E}_s(k_x, \theta) \quad (9.32)$$

where

$$\bar{E}_s(k_x, \theta) = \sum_{n \in \mathbb{Z}} \bar{E}_s(k_x, \theta_n) \quad (9.33)$$
represents the reference energy spectrum from DNS of channel flow in the absence of riblets.

9.2.4 Correction to turbulent viscosity

The turbulent viscosity $\nu_T(y)$ is determined by the second-order statistics of velocity fluctuations, i.e., the kinetic energy $k(y)$ and its rate of dissipation $\epsilon(y)$; see equation (9.18). The statistics can be computed using the covariance matrix $X_d(k_x, \theta_n)$ and $k(y)$, $\epsilon(y)$ can be decomposed as

$$k(y) = k_s(y) + k_c(y), \quad \epsilon(y) = \epsilon_s(y) + \epsilon_c(y),$$  \hspace{1cm} (9.34)

where, again, the subscript $s$ signifies channel flow with smooth walls, and the subscript $c$ quantifies the influence of fluctuations in the flow over riblets. The DNS results for turbulent channel flow yield $k_s$. On the other hand, $\epsilon_s$ is computed using $\epsilon_s(y) = \Re c^2 k_s^2(y)/\nu_{Ts}(y)$ and the corrections $k_c$ and $\epsilon_c$ can be determined from the second-order statistics in $X_{\theta,c}(k_x)$; see Appendix D.2 for details. Substitution of $k(y)$ and $\epsilon(y)$ from (9.34) into equation (9.18) and application of the Neumann series expansion yields

$$\nu_T(y) = \nu_{Ts}(y) + \nu_{Tc}(y),$$  \hspace{1cm} (9.35)

where the correction $\nu_{Tc}(y)$ to turbulent viscosity $\nu_{Ts}(y)$ is given by

$$\nu_{Tc}(y) = \nu_{Ts}(y) \left( \frac{2k_c(y)}{k_s(y)} - \frac{\epsilon_c(y)}{\epsilon_s(y)} \right).$$  \hspace{1cm} (9.36)

Since we primarily focus on small size riblets, this expression is obtained by neglecting higher-order terms that involve multiplication of $k_c(y)$ and $\epsilon_c(y)$.

The influence of fluctuations on the turbulent mean velocity and, consequently, skin-friction drag can be evaluated by substituting $\nu_T(y)$ from (9.35) and solving equations (9.7);
see § 9.1.2 for details regarding the correction to the mean flow profile and § 9.1.3 for the subsequent computation of the drag.
Chapter 10

Results and analysis of triangular riblets

In this chapter, we study the triangular riblets using the low-complexity model constructed in chapter 9 and compare drag-reducing trends predicted by our simulation-free approach with that from experiments and DNS. We also investigate the effect of height and spacing on drag reduction as well as discover the correlation between energy suppression and drag-reduction for appropriately sized riblets. In the end, we analyze the effect of riblets on drag reduction mechanisms and typical turbulent flow structures, including very large scale motions.

10.1 Turbulent drag reduction and energy suppression

In this section, we use the framework developed in § 9.2 to examine the effect of triangular riblets shown in figure 10.1 on the mean velocity, skin-friction drag, and kinetic energy in turbulent channel flow with $Re_\tau = 186$. We assume that the influence of small-size riblets on the channel height and shear velocity is negligible, thereby implying that the Reynolds number remains unchanged over various case studies. By letting the ratio between the height and spacing of riblets be fixed, the riblets of different sizes are obtained by modifying the frequency $\omega_z$; see Table 10.1 for a list of cases considered in our study. In the absence of riblets, DNS results (Del Álamo and Jiménez, 2003; Del Álamo et al., 2004) provide second-order statistics which are used to determine the covariance of stochastic forcing $d_\theta(k_x, t)$ in equation (9.26) and to compute the kinetic energy $k_s(y)$; see § 9.2.3.

We use a total of 199 Chebyshev collocation points to discretize the operators in the wall-normal direction ($N_i = 179, N_o = 20$). Furthermore, we parameterize the linearized
Figure 10.1: Triangular riblets with height \( h \), spacing \( s = 2\pi/\omega_z \), and tip angle \( \alpha \).

<table>
<thead>
<tr>
<th>( Re_\tau )</th>
<th>( h/s )</th>
<th>( \alpha )</th>
<th>( \omega_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.38</td>
<td>105°</td>
<td>30, 35, 40, 45, 50, 60, 80, 100, 160</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>90°</td>
<td>30, 35, 40, 45, 50, 60, 80, 100, 160</td>
<td></td>
</tr>
<tr>
<td>186</td>
<td>0.65</td>
<td>35, 40, 45, 50, 60, 80, 100, 160</td>
<td></td>
</tr>
<tr>
<td>0.87</td>
<td>60°</td>
<td>50, 60, 70, 80, 100, 120, 160</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>45°</td>
<td>60, 80, 100, 120, 160, 210</td>
<td></td>
</tr>
<tr>
<td>547</td>
<td>0.5</td>
<td>90, 115, 145, 175, 210, 250, 300, 360</td>
<td></td>
</tr>
</tbody>
</table>

Table 10.1: Triangular riblets with different height to spacing ratios \((h/s)\), tip angles \( \alpha \), and spanwise frequencies \( \omega_z \) that we examine in our study.

equations (9.25) using 48 logarithmically spaced streamwise wavenumbers with \( 0.03 < k_z < 40 \) and utilize 25 harmonics of \( \omega_z (n = -12, \ldots, 12) \) with 50 equally spaced offset points \( \theta \in [0, \omega_z/2] \) to parameterize \( \theta_n = \theta + n\omega_z \). Finally, to capture the triangular shape of riblets via (9.8), we use 25 harmonics in \( z \) \((m = -12, \ldots, 12)\).

10.1.1 Drag reduction

We first examine the effect of riblet size on turbulent drag. In our parametric study, we follow García-Mayoral and Jiménez (2011) and refer to the regime of vanishing riblet spacing, in which the drag reduction is proportional to the size of riblets, as the viscous regime. For a turbulent channel flow with \( Re_\tau = 186 \) subject to triangular riblets on the lower wall, figure 10.2 shows the influence of the height and peak-to-peak spacing of riblets on \( \Delta D \) in equation (9.17). In this figure, the height and spacing are reported in inner viscous units, i.e. \( h^+ = Re_\tau h \) and \( s^+ = Re_\tau s \), and various curves represent different tip angles \( \alpha \) as a measure of riblet geometry. As shown in figure 10.1, a particular tip angle \( \alpha \) corresponds to a specific height to spacing ratio.
Figure 10.2: Turbulent drag reduction for triangular riblets with different tip angles $\alpha$ as a function of (a) spacing $s^+$; and (b) height $h^+$ in a channel flow with $Re_\tau = 186$ and $\alpha = 105^\circ$ (▽); $90^\circ$ (△); $75^\circ$ (○); $60^\circ$ (★); and $45^\circ$ (×), as well as $Re_\tau = 547$ and $\alpha = 90^\circ$ (□).

Clearly, small-size riblets can indeed reduce skin-friction drag. Figure 10.2 shows the percentage of drag reduction obtained in turbulent channel flows with $Re_\tau = 186$ and $Re_\tau = 547$ over the different triangular riblets listed in Table 10.1. Herein, we focus on the channel flow with $Re_\tau = 186$ to analyze the dependence of drag reduction on the spacing and height of riblets. Figure 10.2(a) demonstrates that, for $s^+ < 20$, the drag reduction first increases as $h^+$ increases, saturates, and then decreases. This trend, however, slows down for smaller values of $s^+$. For $\alpha = 90^\circ$ and $\alpha = 60^\circ$, the drag reduction trends and optimal $s^+$ values resulting from our method reliably capture the trends reported in experimental studies (Bechert et al., 1997). On the other hand, as shown in figure 10.2(b), for a fixed height, as the spacing $s^+$ of riblets decreases, drag reduction increases, saturates, and then decreases. Furthermore, figures 10.2(a) and 10.2(b) show that as the riblet tip angle $\alpha$ decreases, maximum drag reduction is achieved for less separated and taller riblets, respectively. Finally, as $\alpha$ decreases, the maximum value of drag reduction first increases and then decreases, which is also in agreement with experimental observations (Bechert et al., 1997). The trends predicted by our framework indicate an optimal height to spacing ratio of $h/s \approx 0.65$ ($\alpha = 75^\circ$) for triangular riblets, which over-predicts the previously reported optimal tip angle $\alpha = 54/\circ$ (Dean and Bhushan, 2010).
As demonstrated in figure 10.2, the optimal height and spacing can be quite different for riblets of different shape (i.e., different values of $\alpha$), thereby indicating that the height and spacing may not be suitable metrics for characterizing the breakdown of the linear viscous regime. Instead, the groove cross-section area $l_g^+ := \sqrt{A^+}$ (for triangular riblets, $A^+ = h^+ s^+/2$) provides the best collapse of the critical breakdown dimension across different riblet shapes (García-Mayoral and Jiménez, 2011; García-Mayoral and Jiménez, 2011). Furthermore, to remove the effect of riblets’ shape on their slope in the viscous regime $m_l := \lim_{l_g^+ \to 0} \Delta D/l_g^+$, we normalize the drag reduction curves by $m_l$ (García-Mayoral and Jiménez, 2011).

For turbulent channel flow over triangular riblets, figure 10.3 shows the $m_l$-normalized drag reduction as a function of $l_g^+$. The normalization factor $m_l$ is computed by averaging the slope obtained from the first two points on each curve, which are both in the viscous regime. The shaded region represents the envelope of normalized drag reduction values resulting from prior experimental and numerical studies (García-Mayoral and Jiménez, 2011). For riblets with $\alpha = 105^\circ, 90^\circ, 75^\circ, 60^\circ,$ and $45^\circ$, figure 10.3 shows the collapse of drag reduction curves with the largest drag reduction occurring within a tight range of cross-section areas: $l_g^+ = 9.7, 11.1, 11.3, 11.3,$ and $10.2$, for $\alpha = 105^\circ, 90^\circ, 75^\circ, 60^\circ,$ and $45^\circ$, respectively. This prediction agrees well with the values reported by García-Mayoral and Jiménez (2011), $l_g^+ \in [9.7, 11.7]$. Moreover, the drag reduction curves resulting from our framework are located within the shaded region and they reliably predict the overall trend.

For turbulent channel flow over triangular riblets with $Re_\tau = 547$, $h/s = 0.5$, and $\alpha = 90^\circ$, our predictions of the normalized drag reduction remain within this shaded region and are very similar to the results obtained for $Re_\tau = 186$; see square symbols in figure 10.3.

Figure 10.4 compares the $m_l$-normalized drag reduction resulting from our framework and the experimental results of Bechert et al. (1997). Our method captures the overall trends and even the optimal size of riblets with tip angle $\alpha = 60^\circ$ ($2.9\%$ relative error). For riblets with tip angle $\alpha = 90^\circ$, the optimal size is over predicted by $14.3\%$. We note that while the maximum amount of drag reduction is reasonably captured in both cases ($13.1\%$ and
Figure 10.3: Drag reduction normalized with its slope in the viscous regime $m_l$. Different shapes of triangular riblets are represented by $\alpha = 105^\circ (\triangledown); 90^\circ (\triangle); 75^\circ (\bigcirc); 60^\circ (\odot); 45^\circ (\times)$ for $Re_\tau = 186$; and $\alpha = 90^\circ (\square)$ for $Re_\tau = 547$. The shaded area shows the envelope of experimental and numerical results (Bechert et al., 1997; García-Mayoral and Jiménez, 2011).

7.6\% relative errors for $\alpha = 60^\circ$ and $\alpha = 90^\circ$, respectively), the quality of our predictions deteriorates for larger riblets with $\alpha = 90^\circ$. As we discuss in § 10.1.2, an over predicted turbulence suppression in wall-normal regions away from the riblet-mounted lower surface can be a source of this mismatch.

The performance deterioration for large riblets observed in figures 10.3 and 10.4 is associated with the breakdown of the viscous regime within the grooves. This breakdown arises from the lodging of near-wall vortices (Lee and Lee, 2001; Suzuki and Kasagi, 1994), the generation of secondary flow vortices (Goldstein and Tuan, 1998), or the emergence of spanwise coherent rollers (García-Mayoral and Jiménez, 2011). When turbulence moves into the grooves, a turbulence model, which assumes that the wall-normal region with $y < -1$ is laminar (i.e., $\nu_T = 0$), loses its validity. To support an extended turbulent regime and go beyond the breakdown of the viscous regime, our model of surface corrugation assumes the tip of riblets to be located within the original channel region (i.e., $y > -1$); see § 9.1.1. The parameter $r_p$ in (9.10), which controls the level of protrusion into the turbulent regime, is determined to satisfy a constant bulk assumption. Because of this, our model remains valid even for riblets that are larger than the optimal ($l_g^+ \lesssim 20$).
10.1.2 Effect of riblets on turbulent viscosity and turbulent mean velocity

We next examine the effect of riblets on turbulent viscosity and mean velocity. Figure 10.5 shows the turbulent eddy viscosity $\nu_{Ts}$ and mean velocity $\bar{U}_s$ of channel flow over smooth walls with $Re_\tau = 186$ along with the corresponding corrections, $\nu_{Tc}$ and $\bar{U}_c$, introduced by riblets with $\alpha = 90^o$ on the lower wall. The wall-normal coordinate is given in inner (viscous) units, i.e., $y^+ = Re_\tau(1 + y)$. Among the cases listed in table 10.1, cases with maximum drag reduction ($\omega_z = 50$), minimum drag reduction ($\omega_z = 30$), and smallest riblets ($\omega_z = 160$) are chosen. For $\omega_z = 30$, figure 10.5(c) shows that turbulence is promoted at the beginning of the buffer layer, but is then suppressed in regions farther away from the wall. On the other hand, for $\omega_z = 50$, turbulence is always suppressed ($\nu_{Tc} \leq 0$) and the region of suppression shifts closer to the wall ($y^+ \gtrsim 3$). We observe similar trends for riblets with $\omega_z = 160$ to riblets with $\omega_z = 50$, but with smaller amplitude. Figure 10.5(d) shows that, in all cases, riblets reduce the mean velocity gradient in the immediate vicinity of the wall ($y^+ \lesssim 6$). These results demonstrate that for the same shape of riblets (i.e., same tip angle $\alpha$), riblets of sizes that are larger than the optimal yield smaller amounts of turbulence suppression and mean shear reduction.
To illustrate the influence of the shape of riblets on turbulent viscosity and mean velocity, figure 10.6 shows $\nu_{Tc}$ and $\bar{U}_c$ for turbulent channel flow over riblets with different tip angles $\alpha$ and with spanwise frequencies $\omega_z$ that correspond to the maximum drag reduction. The largest turbulence suppression and mean velocity reduction is achieved for $\alpha = 75^\circ$, which is in agreement with the drag reduction trends observed in figure 10.2. Between $\alpha = 45^\circ$ and $\alpha = 105^\circ$, the reduction in turbulent viscosity is more pronounced for the latter, which again reflects the drag reduction trends reported in figure 10.2.

Figure 10.5: (a) The turbulent viscosity, $\nu_{Ts}(y^+)$; and (b) the turbulent mean velocity $\bar{U}_s(y^+)$, in uncontrolled channel flow with $Re_\tau = 186$. The correction to (c) turbulent viscosity, $\nu_{Tc}(y^+)$; and (d) the mean velocity, $\bar{U}_c(y^+)$, in the presence of riblets with $\alpha = 90^\circ$. Red solid lines correspond to riblets with $\omega_z = 30$ (minimum drag reduction), blue dashed lines correspond to riblets with $\omega_z = 50$ (maximum drag reduction), and black dotted lines correspond to riblets with $\omega_z = 160$ (smallest riblets).

For riblets with $\alpha = 60^\circ$ and $\omega_z = 60$ ($s^+ \approx 20$), figure 10.7 shows the variation of the mean velocity in the spanwise plane. No variation is found above $y > -0.9$, which is
Figure 10.6: Correction to (a) turbulent viscosity $\nu_{Tc}(y^+)$; and (b) mean velocity $\bar{U}_c(y^+)$ in a turbulent channel flow with $Re_\tau = 186$ over triangular riblets on the lower wall. The spanwise frequency $\omega_z$ associated with different shapes is selected to maximize drag reduction: $\alpha = 105^{\circ}$, $\omega_z = 50$ (solid black); $\alpha = 75^{\circ}$; $\omega_z = 60$ (dotted red); $\alpha = 45^{\circ}$, $\omega_z = 100$ (dot-dashed blue).

in agreement with the result of numerical simulations (Choi et al., 1993). However, our predictions of the mean velocity profiles deviate from the result of DNS in the vicinity of the wall. This is because we have set the location of riblets to be slightly lower than the DNS computations in order to satisfy the constant bulk criteria. On the other hand, our results show a consistent promotion in the lower-half and suppression in the upper-half of the channel resulting in a slight lack of symmetry with respect to the centerline. This is mainly because of an over-predicted correction to turbulent eddy viscosity $\nu_{Tc}$ in the buffer layer and the inertial sublayer; cf. figure 10.5(c). Such over-predicted levels of turbulence suppression and drag reduction (cf. figures 10.2 and 10.4(b)) are caused by high amplitude stochastic forcing to the linearized equations (9.25), which is shaped to match the two-dimensional DNS energy spectrum (§ 9.2.3). Analyzing the efficacy of more sophisticated forcing schemes (Zare et al., 2016, 2017a,b) that may refine mean velocity predictions is a topic for future research.

10.1.3 Effect of riblets on turbulent kinetic energy

We next examine the effect of triangular riblets on the fluctuations’ kinetic energy. Figure 10.8 compares the reference energy spectrum of turbulent channel flow with smooth
Figure 10.7: Mean velocity profiles $\bar{U}(y,z)$ normalized with the laminar centreline velocity $U_l$ for $\alpha = 60^\circ$ and $\omega_z = 60$ ($s^+ \approx 20$): (a) One-dimensional view for different spanwise locations ($z \in [0, \pi/60]$) from our model (black curves) and the profile corresponding to $z \approx \pi/120$ resulting from DNS of Choi et al. (1993) (blue circles). The direction of the arrow points to velocity profiles corresponding to spanwise locations farther away from the tip of riblets. (b) Color-plot of the streamwise mean velocity in the cross-plane.

walls (equation (9.33)) to the changes in the energy spectrum caused by equally shaped riblets of different sizes ($\omega_z = 160, 50, \text{and } 30$) with $\alpha = 90^\circ$ (equation (9.32)). The energy spectra are premultiplied by the logarithmically scaled streamwise wavenumber $k_x$ so that the areas under the plots determine the total kinetic energy. Since the spanwise direction involves the parameterization $\theta_n = \theta + n\omega_z$, summation over $n$ is performed to integrate the energetic contribution of various harmonics in $\omega_z$ and identify the dependence of the energy spectrum on $\theta$; see § 9.2.3.

For channel flow over smooth walls with $Re_x = 186$, the color plots in the left column of figure 10.8 show that the most energetic modes take place at $(k_x, \theta) = (2.5, 3.5)$. As blue regions in figures 10.8(b), 10.8(d), and 10.8(f) illustrate, riblets reduce the energy content of flow structures with smaller streamwise wavenumbers. Moreover, yellow and red regions in figures 10.8(d) and 10.8(b) demonstrate that larger riblets increase the energy content of flow structures with larger streamwise wavenumbers. For these three cases, the largest energy amplification takes place around $(k_x, \theta) = (5.5, 2.4), (6.4, 4.6)$, and $(6.4, 0.6)$, respectively. On
Figure 10.8: The premultiplied reference energy spectrum $k_x \bar{E}_s(k_x, \theta)$ (equation (9.33)) from DNS of a turbulent channel flow with $Re_\tau = 186$ (Del Álamo and Jiménez, 2003) (left column), and the correction to the premultiplied spectrum $k_x \bar{E}_c(k_x, \theta)$ (equation (9.32)) due to the presence of various sizes of triangular riblets with $\alpha = 90^\circ$ (right column): (a, b) $\omega_z = 160$ ($l_g^+ = 3.6$); (c, d) $\omega_z = 50$ ($l_g^+ = 11.7$, optimal); and (e, f) $\omega_z = 30$ ($l_g^+ = 19.5$).

The other hand, the maximum energy reduction occurs around $(k_x, \theta) = (4.9, 0.9), (4.0, 0.5)$, and $(4.0, 0.5)$, respectively. Although the peak points are different, figures 10.8(b), 10.8(d), and 10.8(f) demonstrate similar amplification/suppression trends: riblets suppress/increase
Figure 10.9: Correction to the premultiplied energy spectrum $k_x\bar{E}_c(k_x, \theta)$ in a turbulent channel flow with $Re_\tau = 186$ triangular riblets of various sizes. The spanwise frequency $\omega_z$ associated with different shapes of riblets corresponds to maximum drag reduction: (a) $\alpha = 105^\circ$, $\omega_z = 50$; (b) $\alpha = 75^\circ$, $\omega_z = 60$; and (c) $\alpha = 45^\circ$, $\omega_z = 100$.

energy content of long/short streamwise length scales. These results provide evidence that the analysis of spatially-periodic systems, e.g., the one considered in this study, cannot be limited to a single horizontal wavenumber pair associated with the peak of the energy spectrum or the dominant near-wall cycle. We note that similar conclusions were reached in the analysis of turbulent channel flow subject to transverse wall oscillations (Moarref and Jovanovic, 2012). Finally, the dependence of correction $\bar{E}_c(k_x, \theta)$ on the shape of riblets is shown in figure 10.9. For all cases shown in this figure, similar modes are affected by the presence of riblets and the suppression of kinetic energy is more pronounced for riblets with $\alpha = 75^\circ$ and $\omega_z = 60$, which also yield the largest drag reduction (cf. figure 10.2). This suggests a synchrony between the dependence of drag reduction and energy suppression on the geometry of triangular riblets.

Figure 10.10(a) shows the percentage of kinetic energy variation $\Delta E := \hat{E}_c/\hat{E}_s$ for triangular riblets as a function of the riblet groove area $l_g^+$. Here, $\hat{E}_c$ and $\hat{E}_s$ denote the correction to kinetic energy due to the presence of riblets and the kinetic energy of velocity fluctuations in the absence of riblets, respectively. These two quantities can be computed by integrating the corresponding energy spectra, i.e., $\bar{E}_c(k_x, \theta)$ and $\bar{E}_s(k_x, \theta)$, over all horizontal wavenumbers $k_x$ and $\theta$. On the other hand, figure 10.10(b) shows the percentage of drag reduction for the same values of $l_g^+$. Our computations demonstrate similar trends in the dependence of
Figure 10.10: (a) Kinetic energy suppression; and (b) turbulent drag reduction in a channel flow with $Re_{\tau} = 186$ over triangular riblets of various sizes. Symbols denote different shapes of triangular riblets with $\alpha = 105^\circ (\triangledown); 90^\circ (\Delta); 75^\circ (\bigcirc); 60^\circ (\diamond); \text{and } 45^\circ (\times)$.

$\Delta E$ and $\Delta D$ on $l_g^+$ (cf. figures 10.10(a) and 10.10(b)), especially for the riblets in the viscous regime. Based on the various cases considered in figure 10.10, the linear regression model $\Delta D = 1.7152 \Delta E + 1.0907$ can be extracted with a coefficient of determination $R^2 = 0.9925$ for riblets with $l_g^+ \leq 14$. Strong correlation between changes in turbulent drag and kinetic energy suggests that energy can be used as a surrogate for predicting the effect of riblets on skin-friction drag; see figure 10.11(a). We note that a similar linear relation (but with a different slope) can be observed at $Re_{\tau} = 547$. These results are not reported here for brevity.

As shown in figure 10.5(c), riblets can suppress or enhance turbulence near the wall. Small riblets can disturb the near-wall cycle in the turbulent flow by generating and preserving laminar regions within the grooves. On the other hand, for larger riblets, streamwise rollers penetrate into the grooves which enhances turbulence close to the wall. As a result, nonlinear effects take over and the linear relation between drag/energy reduction and any metric of the riblet size (e.g., $l_g$) becomes compromised. As illustrated in figure 10.11(b), the quality of a linear regression model drops (i.e., $R^2$ becomes smaller) when data for larger-size riblets is taken into account.
10.2 Turbulent flow structures

In this section, we use the stochastically forced linearized model (9.20) to examine the effect of riblets of different sizes and shapes on typical turbulent flow structures and relevant drag reduction mechanisms. First, we study the distortion of the dominant near-wall cycle that arises from the presence of riblets on the lower wall. We also examine the K-H instability, which is related to the breakdown of the viscous regime, and the performance deterioration for large riblets. Finally, we consider a channel flow with higher-Reynolds number to investigate the wall-normal support of very large scale motions (VLSM) in the presence of riblets.

In this section, in addition to the wall-normal coordinate, wavelengths are also given in inner (viscous) units with $\lambda^+_x = 2\pi Re_{\tau}/k_x$ and $\lambda^+_z = 2\pi Re_{\tau}/\theta$.

Following the proper orthogonal decomposition of Bakewell and Lumley (1967); Moin and Moser (1989), we extract flow structures from our model using the eigenvalue decomposition of the velocity covariance matrix in statistical steady-state,

$$\Phi_\theta(k_x) = C_\theta(k_x) X_\theta(k_x) C^*_\theta(k_x)$$  \hspace{1cm} (10.1)
where \( X_\theta(k_x) \) represents the solution of Lyapunov equation (9.28). The eigenvectors associated with the principal pair of eigenvalues form flow structures that are located in the vicinity of the upper and lower channel walls. The first pair of eigenvalues are usually one order of magnitude larger than the second pair. This indicates that the flow structures that correspond to the principal eigenvectors are energetically dominant and representative of the essential dynamics. The velocity components of flow structures are constructed by integrating over all spanwise harmonics and by accounting for the symmetry in the streamwise direction as

\[
\begin{align*}
  u(x, y, z) &= \sum_{n \in \mathbb{Z}} \cos(\theta_n z) \Re \left( \tilde{u}(k_x, \theta_n) e^{ik_x x} \right), \\
  v(x, y, z) &= \sum_{n \in \mathbb{Z}} \cos(\theta_n z) \Re \left( \tilde{v}(k_x, \theta_n) e^{ik_x x} \right), \\
  w(x, y, z) &= -\sum_{n \in \mathbb{Z}} \sin(\theta_n z) \Im \left( \tilde{w}(k_x, \theta_n) e^{ik_x x} \right).
\end{align*}
\]

Here, \( \Re \) and \( \Im \) denote real and imaginary parts, and \( \tilde{u} \), \( \tilde{v} \), and \( \tilde{w} \) correspond to the streamwise, wall-normal, and spanwise components of an eigenvector of the matrix \( \Phi_\theta(k_x) \), given in equation (10.1).

In a turbulent channel flow with smooth walls, the dominant eigenmodes of the velocity covariance matrix appear in pairs and represent symmetric flow structures that reside in the vicinity of the upper and lower walls. Surface corrugation on the lower wall breaks this symmetry and can cause a suppression of near-wall structures in the lower half of the channel. In other words, the flow structures that dominate the flow close to the riblets can be less energetic than the flow structures close to the upper wall. As a result, physically relevant flow structures near the riblets, e.g., the dominant flow structures associated with the near-wall cycle over riblets of optimal size, are often associated with the second, less energetic, eigenmode of \( \Phi_\theta(k_x) \).
### 10.2.1 Near-wall cycle in turbulent channel flow with $Re_\tau = 186$

In the absence of riblets, the so-called near-wall cycle dominates the physics of the turbulent channel flow by generating streamwise streaks from the advection of the mean shear by streamwise vortices and the formation of streamwise vortices through streak instability and nonlinear interactions (Robinson, 1991; Hamilton et al., 1995; Jiménez and Pinelli, 1999). Riblets can break this near-wall cycle and push the streamwise vortices and streaks away from the wall so that a laminar region is retained within the grooves. This ultimately reduces skin-friction drag. The typical wavelength of the flow structures in the near-wall cycle are reported as $(\lambda_x^+, \lambda_z^+) \approx (1000, 100)$, which corresponds to $(k_x, \theta) \approx (1.1687, 11.687)$ in a turbulent channel flow with $Re_\tau = 186$.

For different sizes of riblets, figure 10.12 compares the flow structures that correspond to the near-wall cycle in a turbulent channel flow with $Re_\tau = 186$. Flow patterns resulting from the combination of streaks and vortices can be clearly observed. In particular, it is evident that the quasi-streamwise vortices and regions of high and low streamwise velocity are pushed above the riblet tips creating a region of limited turbulence in the riblet grooves, and effectively impeding the transfer of mean momentum toward the lower wall. The first two rows of figure 10.12 illustrate the flow structures over small- and optimal-sized riblets, respectively. The dominance of streamwise elongated structures that follow the length-scales of the near-wall cycle is evident in these two scenarios. However, as shown in figures 10.12(e) and 10.12(f), the flow over larger riblets is contaminated by multiple energetically relevant spanwise length-scales. This indicates energy distribution across multiple Fourier modes beyond the ones that are relevant in the near-wall cycle. This distribution of energy is caused by the interaction of near-wall turbulence with the spanwise-periodic surface, which leads to the generation of secondary flow structures that follow the spatial frequency of riblets close to their tips (Goldstein and Tuan, 1998); see figure 10.12(f).

The cross-plane views in figure 10.12 also show that, for larger riblets, secondary flow structures begin to penetrate into the grooves. This induces high-momentum flow into the viscous flow regime, which is reflected by an increase in the amplitude of velocity fluctuations.
Figure 10.12: Dominant flow structures in the vicinity of the lower wall of a turbulent channel flow with $Re_\tau = 186$ over triangular riblets with $\alpha = 90^\circ$ and (a)(b) $\omega_z = 160$; (c)(d) $\omega_z = 50$; and (e)(f) $\omega_z = 30$. These flow structures correspond to $(\lambda_x^+, \lambda_z^+) = (1000, 100)$, typical scales of the near-wall cycle, and are extracted from the dominant eigenmode pair of the covariance matrix $\Phi(k_x)$. Left column: $x-z$ slice of the streamwise velocity $u$ at $y^+ \approx 6$; Right column: $y-z$ slice of $u$ along with the vector field $(v, w)$ at $x^+ = 500$, which corresponds to the thick black lines in the left column. Color plots are used for the streamwise velocity fluctuation $u$ and vector fields identify streamwise vortices.

at $y^+ \approx 6$ and the breakdown stage of the viscous regime which precedes the deterioration of drag reduction. Moreover, the kinetic energy corresponding to the length-scale of the near-wall cycle varies from 0.0498, for a channel flow with smooth walls, to 0.0468, 0.0441, and 0.0609 (given by $\bar{E}(k_x, \theta)$ defined in (9.31)), for small, optimal, and large riblets, respectively. This shows that small and optimally sized riblets suppress the energy of near-wall cycle flow structures while larger ones can promote their energy, which is consistent with the trend observed in § 10.1.3.
10.2.2 Spanwise rollers resembling Kelvin-Helmholtz vortices

In addition to the influence of secondary flow structures around the tip of riblets, the breakdown of the viscous regime and decrease in drag reduction can also result from the amplification of spanwise rollers that are induced by a two-dimensional K-H instability (García-Mayoral and Jiménez, 2011). The amplification of long spanwise scales for large riblets is evident from the energy spectra shown in figure 10.8. Figure 10.13 shows the DNS-based premultiplied streamwise co-spectra of various Reynolds stresses for infinitely wide scales ($\theta = 0$) in a turbulent channel flow with $Re_\tau = 186$ (Del Álamo and Jiménez, 2003; Del Álamo et al., 2004), as well as the corresponding co-spectra resulting from our analysis of the flow over triangular riblets of different size. For larger riblets, the amplification of the co-spectra corresponding to streamwise and wall-normal intensities become stronger and occur closer to the wall. However, this trend is not observed for the co-spectrum corresponding to the spanwise turbulence intensity, which shows smaller amplification of channel-wide scales for riblets of larger size. Nevertheless, as the size of riblets increases, the co-spectrum corresponding to the wall-shear stress, $-k_x E_{uw}$, starts to show signs of suppression in the vicinity of the wall; the penetration of negative shear stress into the riblet grooves is evident from the co-spectrum associated with spatial frequency $\omega_z = 30$. The co-spectrum $-k_x E_{uw}$ shows the largest suppression of shear stress for streamwise wavelengths $\lambda_x^+ \approx 200$. Our results demonstrate that large riblets result in the suppression of shear stress within the grooves, which is consistent with our earlier findings that showed a degradation of drag reduction for such sizes (cf. figure 10.3). We note that our computations illustrate that the trends observed for the energy spectra do not vary for different shapes of riblets (i.e., different values of $\alpha$).

For largest riblets ($\omega_z = 30$), figure 10.13 illustrates that the streamwise ($k_x E_{uu}$) and wall-normal ($k_x E_{vv}$) contributions to the energy spectra are significantly larger than the spanwise ($k_x E_{ww}$) contribution. Furthermore, the wall-normal spectrum shows significant amplification of wall-separated flow structures. Figure 10.14 shows the premultiplied energy spectrum of wall-normal velocity, $k_x k_z E_{vv}$, at $y^+ = 5$ for different sizes of riblets. We note that energy amplification becomes larger as the size of riblets increases. For the riblets with
Figure 10.13: Premultiplied streamwise co-spectra at $\theta = 0$ for a turbulent channel flow with $Re_\tau = 186$ over triangular riblets of tip angle $\alpha = 90^\circ$. The four rows correspond to $k_x E_{uu}$, $k_x E_{vv}$, $k_x E_{ww}$, and $-k_x E_{uv}$; the four columns represent DNS data for the uncontrolled channel flow (Del Álamo and Jiménez, 2003; Del Álamo et al., 2004) and data resulting from our computations for the flow with riblets of spatial frequency $\omega_z = 160$, 50, and 30, respectively.

Figure 10.14: Premultiplied energy spectra of wall-normal velocity, $k_x k_z E_{uv}$, at $y^+ = 5$ in turbulent channel flow with $Re_\tau = 186$ over triangular riblets of tip angle $\alpha = 90^\circ$ and (a) $\omega_z = 160$; (b) $\omega_z = 50$; and (c) $\omega_z = 30$. 

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\( \omega_z = 30 \), figure 10.14(c) shows that the band of streamwise and spanwise scales corresponding to \( \lambda_x^+ \in [100, 300] \) and \( \lambda_z^+ > 300 \), is significantly more amplified than the other two cases, which is consistent with the DNS-result of García-Mayoral and Jiménez (2011).

![Figure 10.15](image)

**Figure 10.15:** Turbulent flow structures corresponding to spanwise elongated rollers with \( \lambda_x^+ \approx 200 \) that are extracted from the dominant eigenmode of the covariance matrix \( \Phi_\theta(k_x) \) for a turbulent channel flow with \( Re_x = 186 \) over triangular riblets of \( \alpha = 90^\circ \) and \( \omega_z = 30 \). (a) Vector field denotes the in-plane velocities \( (u, v) \); and (b) \( x - z \) slice of the wall-normal velocity \( v \) at \( y^+ = 5 \).

Figure 10.15 shows the flow structures that are extracted from our model for \( (k_x, \theta) = (5.76, 0) \) which correspond to spanwise-averaged infinitely long scales in \( z \) and the peak in the shear stress co-spectrum for riblets with \( \omega_z = 30 \). These flow structures indicate that the dominant eigenmode of the covariance matrix (10.1) resembles a spanwise-constant roller centered at \( y^+ = 17.7 \), which penetrates well into the grooves, thereby causing the breakdown of the viscous regime (figure 10.15(a)). The streamwise wavelength of these flow structures extracted from the premultiplied co-spectrum \( -k_x E_{uv} \) is \( \lambda_x^+ \approx 200 \). The wall-normal velocity
These flow structures are reminiscent of the spanwise rollers identified using DNS of turbulent channel flow over square riblets; see figure 14 in García-Mayoral and Jiménez (2011).

Figure 10.16: Wall-normal locations of the core of spanwise elongated rollers with $\lambda_x^+ \approx 200$ in a turbulent channel flow with $Re_\tau = 186$ over triangular riblets with $\alpha = 90^\circ$ and sizes specified by $l_g^+$. For these riblets, the spanwise rollers have similar dominant streamwise length scales ($\lambda_x^+ \approx 200$). For larger riblets, the core of the spanwise rollers moves down toward the riblets. Thus, as the size of riblets increases and the grooves become deeper, the dominant turbulent flow structures penetrate further down into the viscous region in the grooves.

10.2.3 Very large scale motions in turbulent channel flow with $Re_\tau = 547$

Apart from the dominant flow structures associated with the near-wall cycle that are centered at $y^+ \approx 10$, other flow structures within the logarithmic and wake region become significant in wall-bounded shear flows with high Reynolds numbers. An important class of streamwise elongated flow structures that reside in the logarithmic region, i.e., $3\sqrt{Re_\tau} < y^+ < 0.15 Re_\tau$, determine very large scale motions (VLSM) (Hutchins and Marusic, 2007; Monty et al., 2007;
Marusic et al., 2013). Relative to the near-wall cycle, VLSMs are centered farther away from the wall. Yet, they exhibit a wall-normal reach that can influence the energy transfer at the wall (Marusic et al., 2010). It is thus relevant to explore the effect of riblets on the energy and locality of these flow structures.

For a turbulent channel flow with $Re_\tau = 547$, VLSMs are characterized by wall-parallel wavelengths $(\lambda_x^+, \lambda_z^+) \approx (1400, 700)$ that can be extracted from the premultiplied one-dimensional energy spectrum generated from DNS data (Del Álamo and Jiménez, 2003; Del Álamo et al., 2004). For various sizes of riblets with $\alpha = 90^\circ$, figure 10.17 shows the spatial structure of the principal eigenvector of the steady-state covariance matrix $\Phi_\theta(k_x)$ (equation (10.1)) corresponding to such VLSMs. In this figure, riblets with $\omega_z = 175$ (figure 10.17(b)) provide the maximum drag reduction; cf. figure 10.3. Figure 10.17 demonstrates that small- and optimal-size riblets have little influence on the shape of the VLSMs. In contrast, large riblets distort the shape of such flow structures close to the wall. Moreover, the core of flow structures (i.e., the wall-normal location of the maximum streamwise velocity) shifts from $y^+ \approx 85$ in the flow over smooth walls to $y^+ \approx 90.2$, 95.7, and 86.0 for small, optimal, and large riblets, respectively. Thus, larger size riblets allow the VLSMs that are otherwise pushed away to settle into the riblet valleys; see figure 10.17(c). This is consistent with the downward shift of near-wall flow structures predicted in § 10.2.1, which results in an increase in skin-friction drag.

In a turbulent channel flow with smooth walls at $Re_\tau = 547$, the kinetic energy associated with $(\lambda_x^+, \lambda_z^+) = (1400, 700)$ is 0.0606. In the presence of small, optimal, and large riblets the kinetic energy for this wavelength pair shifts to 0.0602, 0.0599, and 0.0611 (given by $\bar{E}(k_x, \theta)$ defined in (9.31)), respectively. This is consistent with the experimental study of Lee and Lee (2001) where small changes to the turbulent kinetic energy and velocity fluctuations in the outer layer of turbulent channel flow over drag-reducing surfaces with semi-circular grooves were observed. The same reference also reports an increase in the turbulent kinetic energy that arises from a drag-increasing surface with larger grooves.
Figure 10.17: Spatial structure of the streamwise velocity (red and blue colors denoting regions of high and low velocity) and the \((v, w)\) vector field (quiver lines) corresponding to VLSMs with \((\lambda_x^+, \lambda_z^+) = (1400, 700)\) in a turbulent channel flow with \(Re_\tau = 547\) over triangular riblets with \(\alpha = 90^\circ\) and (a) \(\omega_z = 360\); (b) \(\omega_z = 175\); and (c) \(\omega_z = 90\).
Chapter 11

Concluding remarks

We have developed a model-based framework for evaluating the effect of surface corrugation on skin-friction drag and kinetic energy in turbulent channel flows. The influence of the corrugated surface is captured via a volume penalization technique that enters as a feedback term into the governing equations. Our simulation-free approach utilizes eddy-viscosity-enhanced NS equations and it consists of two steps: (i) we use the turbulent viscosity of the turbulent channel flow with smooth walls to capture the effect of the corrugated surface on the turbulent base velocity; and (ii) we use second-order statistics of stochastically forced equations linearized around this base velocity profile to assess the role of velocity fluctuations and correct the turbulent viscosity model. This correction perturbs the turbulent base velocity profile obtained in the first step and refines our prediction of skin-friction drag.

For a turbulent channel flow with streamwise-aligned spanwise-periodic triangular riblets on the lower wall, we demonstrate that the base flow computed in the first step of our approach does not capture drag-reducing trends reported in experiments and simulations. Incorporating the influence of fluctuations on the turbulent viscosity significantly improves our predictions. Our results demonstrate good agreement with experimental and numerical results both in capturing drag-reducing trends and in identifying optimal shapes and sizes of riblets for the largest drag reduction. We also investigate the dependence of the turbulent kinetic energy of fluctuations on the size of riblets and demonstrate similar trends to what we observe for drag reduction. Building on this similarity and data obtained through a parametric study for riblets of various shapes and sizes, we extract a linear regression model.
and show that energy can be used as a surrogate for predicting the effect of riblets on skin-friction drag in the viscous regime.

The steady-state covariance matrices that we compute also allow us to examine the impact of riblets on dominant turbulent flow structures. We show that small-size triangular riblets limit the wall-normal transfer of momentum associated with the near-wall cycle and the generation of secondary flow structures around the tips. Our model captures the penetration of secondary vortices into riblet grooves and predicts that drag reduction reduces for large-size riblets. We also investigate the amplification of spanwise rollers that resemble K-H vortices and show good agreement between our predictions of their streamwise length-scale and core location with previous numerical studies. Finally, for turbulent channel flow with $Re_{\tau} = 547$, we study the influence of riblet size on the wall-normal reach of VLSMs and demonstrate that drag-reducing riblets have little effect on the energy of these flow structures, but large riblets can increase their strength.

Our long-term objective is to develop a framework for the low complexity modeling of turbulent flows over corrugated surfaces that can bypass the need for costly numerical simulations and experiments and guide the optimal design of drag-reducing surfaces. The present work represents a step in this direction in that it provides a method for improving predictions to the turbulent viscosity and the mean velocity in channel flows over spanwise periodic surfaces. We anticipate that incorporation of data from numerical simulations and experiments can further improve the predictive capability of the framework based on stochastically forced linearized NS equations (Zare et al., 2017a,b, 2020). On the other hand, the current framework supports any size of the solid obstacles in the corrugated surfaces region, which results in the full coupling of all harmonics together. Thus, exploiting the small size of drag-reducing riblets within a framework that utilizes perturbation analysis is a valuable future extension since such a framework will significantly reduce the computational resource requirement due to the decoupling of the harmonics.
Chapter 12

Discussion and future directions

12.1 Complex fluids and complex flows

Other than the simple Newtonian fluids discussed in this study, complex fluids and complex flows are common in practical engineering applications. The input-output analysis, such as resolvent analysis and Lyapunov equations employed in this dissertation, were proven can be successfully applied to quantify the influence of deterministic and stochastic inputs in these complex flows.

On the one hand, non-Newtonian fluids are essential in industrial applications. For example, the low-inertia flow of viscoelastic fluids is often utilized in extrusion, coating, and other materials processing operations. The transition of these viscoelastic flows from a laminar to a disordered state can result in defective end-products, while the elastic turbulence after the transition can promote the mixing by enhancing transport in microfluidic flows. The input-output analysis is successfully utilized for discovering mechanisms for transition and elastic turbulence in viscoelastic fluids (Hoda et al., 2008, 2009; Jovanovic and Kumar, 2010, 2011; Lieu et al., 2013; Hariharan et al., 2018). In the future, it would be interesting to look at the nonlinear mode interactions using our PFE framework. Furthermore, it is also worth studying the drag-reducing property of turbulent flows by combining the polymer fluids and small riblets.

On the other hand, the high-speed compressible flows, especially the hyper- and ultrasonic flows, are crucial for investigating the safety and efficiency of high-speed vehicles in the aerospace industry. At hypersonic Mach numbers, the coupling of inertial and thermal
effects (e.g., vorticity interaction from entropy layer and viscous interaction induced by
temperature-thickening boundary layer), and chemical reactions, are significant, which makes the
flow extremely complicated (Anderson, 2006). Conventional receptivity based on a local
spatial analysis is not applicable to analyze the flows around hypersonic vehicles due to this
complexity. As a result, the transition process around these bodies of aerodynamic interest
with a complex shape or shock interactions with control surfaces is poorly understood. As a
contrast, success has been achieved by the input-output analyses around spatially-evolving
hypersonic flows for shock/boundary-layer interactions in (Hildebrand et al., 2018; Dwivedi
et al., 2020). Moreover, the amplification of exogenous disturbances is quantified in (Dwivedi
et al., 2019) using global input-output analysis, and they also explained the reattachment
streaks in a hypersonic compression ramp flow that observed in experiments. A possible
future direction is adopting the Lyapunov equation to investigate the asymptotic properties
of stable hypersonic flows at the impact of exogenous disturbances.

12.2 Flow control

Flow control strategies are designed to either suppress the adverse influence from turbulence,
e.g., delaying the laminar-turbulent transition to reduce skin-friction drag in turbulent flows
or improve turbulence, e.g., promote the transition to avoid a stall, and enhance turbulence
for mixing. They can be classified into two main categories, the sensor-less and feedback
control.

Sensor-less control strategies that are usually inspired by nature. For example, a typical
passive control strategy, the riblets for reducing turbulent drag that studied in this dissert-
tation, is originally arising from the observation of drag-reducing shark skin (Walsh, 1983).
Similarly, super-hydrophobic surface coatings’ design is also motivated by the water repel-
lent properties of lotus leaf surface (Rothstein, 2010). Since both strategies reduce drag
by their particular surface properties, our modeling framework that studies riblets in this
dissertation can also be applied to investigate the super-hydrophobic surface with a proper
representation of the porous surface.
Compared with the sensor-less control, the feedback control is more applicable for dealing with uncertainties that usually appear in engineering flows. However, in the piratical flow control scenario, the sensors and actuators are typically restricted to the surface. Thus, the measurements are noisy and limited close to the surface region. As a result, flow fields away from the surface require estimation (typically Kalman filter) using the limited near-surface information to form a control actuation. These estimation procedures have been investigated in wall-bounded turbulent flows (Chevalier et al., 2006; Illingworth et al., 2018). A possible direction in the future is to estimate velocity status using the measurement of shear and pressure at the wall and design actuation like blow and suction to suppress particular flow structures such as the Near-wall cycle so that the near-wall turbulence can be mitigated.
Bibliography


Appendices
Appendix A

The effect of sponge layers on global stability analysis of Blasius boundary layer flow

In this Appendix, we conduct a parametric study on the influence of sponge layer strength on temporal eigenvalue problems arising from the one-dimensional wave equation and the linearized Navier-Stokes equations. Sponge layers have shown to stabilize eigenmodes and introduce additional spatial growth to eigenfunctions. As the strength of sponge layers increases, temporal eigenvalues are displaced, and the spatial growth rates of their associated eigenfunctions are modified. In both wave and linearized Navier-Stokes equations, the linear relationship between temporal damping and spatial growth can be specified as an approximate dispersion relation. It can also be shown that an over strengthened sponge layer can reflect spatially propagating waves. This reflection can lead to a destabilization of the otherwise stable eigenspectrum with alteration of eigenfunction wavelengths. We provide an empirical guideline for determining the desirable sponge layer strength and demonstrate the efficacy of our method in the global stability analysis of the linearized Navier-Stokes equations.

A.1 Wave equation

In this section, we study the eigenvalue problem for the one dimensional linear wave equations. Since the essence of the complex dynamics of boundary layer flows can be captured by the spatial growth (decay) of propagating waves, we use this simple model to gain insight into the influence of sponge layers on the dynamics.

The two-way wave equation can be considered as a coupled system of two first-order equations

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{\partial v}{\partial x} - \sigma(x) u \\
\frac{\partial v}{\partial t} &= -\frac{\partial u}{\partial x} - \sigma(x) v,
\end{align*}
\]

(A.1.1)

where \(u\) and \(v\) are the wave functions defined in the domain \(x \in [-1, 1]\). Homogeneous Dirichlet boundary conditions are imposed for \(u\) at both ends of the domain. Without sponge layers, waves striking either end of the domain are perfectly reflected back into the domain. To model open boundaries, however, we include sponge layers represented by the function \(\sigma(x)\), which is non-zero only close to the boundaries. When \(\sigma(x) > 0\), waves are gradually driven back to zero.
To form the temporal eigenvalue problem, the wave functions are expressed in wave form as $u(x, t) = \hat{u}(x) \exp(-i\omega t)$ and $v(x, t) = \hat{v}(x) \exp(-i\omega t)$. This yields the eigenvalue problem

$$
-i\omega \hat{u} = -\frac{\partial \hat{v}}{\partial x} - \sigma(x) \hat{u}
$$

$$
-i\omega \hat{v} = -\frac{\partial \hat{u}}{\partial x} - \sigma(x) \hat{v}.
$$

When the sponge layer is not applied ($\sigma(x) = 0$), eigenvalues and corresponding eigenfunctions can be analytically derived as $\omega = n\pi/2$ ($n \in \mathbb{Z}$), $\hat{u} = \sin(\omega x)$ and $\hat{v} = -i \cos(\omega x)$, respectively.

Parabolic shaped sponge layers are placed at both ends of the computational domain. The thickness of each sponge layer is $L_s = 0.1225$ ($\sigma(x) = 0, x \in [-0.8775, 0.8775]$). The strength of the sponge layer is defined as $S_s := \sigma(\pm 1)$. Chebyshev polynomials are adopted to discretize the computational domain with 200 collocation points. We study 20000 cases of sponge layer strengths growing linearly from 1 to $10^6$.

Figure A.1(a) shows the trajectories of eigenvalues. Blue arrows show the approximate path of eigenvalues starting from $\omega_0 = 9\pi/2$ ($S_s = 0$, red square) and end at $\omega = 5.2923 - 0.1114i$ ($S_s = 10^6$, black dot); (b) Imaginary parts of eigenvalues from three different trajectories starting at $\omega_0 = \pi/2$ ($\circ$), $10\pi$ ($\times$), and $50\pi$ ($\triangle$) as a function of sponge layer strength.

**Figure A.1:** Displacement of eigenvalues as the sponge layer strength increases; (a) the displacement trajectories of eigenvalues start from $\omega_0 = 9\pi/2$ ($S_s = 0$, red square) and end at $\omega = 5.2923 - 0.1114i$ ($S_s = 10^6$, black dot); (b) Imaginary parts of eigenvalues from three different trajectories starting at $\omega_0 = \pi/2$ ($\circ$), $10\pi$ ($\times$), and $50\pi$ ($\triangle$) as a function of sponge layer strength.
Figure A.2: Real (dashed) and imaginary (solid) parts of the eigenfunction $\hat{v}$ corresponding to the trajectory originating at $\omega_0 = 9\pi/2$ with (a) no sponge layer ($S_s = 0$), and with (b) a sponge layer of strength $S_s = 10^6$.

The increase in $\Im(\omega)$ is caused by a reflection of waves from the sponge layer. The damping rate of waves passing though the sponge layer is proportional to $\exp(-2 \int_0^{L_s} \sigma(x) \, dx)$ (Mani, 2012). This implies that stronger sponge layers better absorb waves. However, it is also observed that significantly strong sponge layer can cause wave reflection (Mani, 2012). This is a similar phenomenon to the echo of sound (waves) when they hit a solid wall, or an infinitely strong sponge layer. As shown in figure A.2, in the extreme case ($S_s = 10^6$), the eigenfunction retains its sinusoidal shape, but with a different wavelength. Figure A.2(b) shows that infinitely strong sponge layers form new boundaries with homogeneous Dirichlet boundary conditions, ultimately shrinking the computational domain. We thus infer that strong sponge layers can shrink the domain and detune eigenfunctions.

Figure A.3 shows the amplitude of two eigenfunctions from the trajectory that starts at $\omega_0 = 10\pi$. Even though the wave equation provides no instability mechanism, figures A.3(a) and A.3(b) show that the eigenfunctions experience significant spatial growth. This growth is required for each crest to maintain constant amplitude as it propagates away from the center of the domain, even though the entire mode is damped in time. Figures A.3(c) and A.3(d) illustrate the exponential spatial growth in the amplitude of $\hat{v}$. In the exponential growth region, the amplitude of the wave perfectly collapses on lines denoting a $|\Im(\omega)|$ growth-rate for upstream and downstream traveling waves. To explain this, without loss of generality, we only consider the right propagating waves. The solution to the wave equation can be formed as $\exp(ikx) \times \exp(-i\omega t) + \text{c.c.}$, where $k$ is the complex wavenumber and $\omega$ is the eigenvalue. The phase speed of traveling waves is 1 in the non-sponged domain with no dispersion, i.e., $\omega/k = c = 1$, and as a result $k = \omega$. As a consequence the spatial growth rate $-\Im(k)$ is the negative of the temporal damping rate $\Im(\omega)$, i.e., $-\Im(k) = -\Im(\omega)$. Moreover, the spatial wavenumber $\Re(k)$ is equal to temporal frequency $\Re(\omega)$. As a result, the nontrivial shift in the real part of the eigenvalues $\Re(\omega)$ entails a shift in $\Re(k)$, which can be related to the reflecting property of the sponge layer. While figure A.3(c) shows a
Figure A.3: Eigenfunctions $\hat{v}$ from the trajectory starting at $\omega_0 = 10\pi$; left column is at $\omega = 31.4126 - 10.2352i$ with $S_s = 251$ and right column is at $\omega = 32.3432 - 10.5801i$ with $S_s = 871$. (a, b) Envelopes (thin black lines), real (red thick lines) and imaginary (thick blue dashed lines) parts of eigenfunctions. (c, d) Amplitudes of eigenfunctions (thick black lines) and the estimated amplitudes of downstream (thin red lines) and upstream (thin blue dashed lines) propagating eigenfunctions, which are respectively obtained as $|\hat{v}(0.5)| \exp(-\omega_i(x - 0.5))$ and $|\hat{v}(-0.5)| \exp(\omega_i(x + 0.5))$. Here, $\omega_i$ is the imaginary part of $\omega$.

The lack of streamwise homogeneity significantly complicates the dynamics of spatially evolving flows. In this section, we consider the global stability analysis of temporal eigenvalues in the Blasius boundary layer flow. The Reynolds number $Re = U_\infty \delta_0 / \nu$ based on the free stream velocity $U_\infty = 1$ and the boundary layer thickness at the inlet $\delta_0 = \delta(0)$ is 400. The Navier-Stokes (NS) equations are linearized around the Blasius velocity profile.
\[
[U(x, y) \ V(x, y)]^T, \text{ which yields the governing equations for fluctuations in velocity } [u \ v]^T \\
\text{and pressure } p. \text{ Assuming wavelike solutions, } u(x, y, t) = \hat{u}(x, y) \exp(-i\omega t), \text{ we arrive at the temporal eigenvalue problem}
\]

\[
-i\omega \hat{u} = \left( C - \frac{\partial U}{\partial x} \right) \hat{u} - \frac{\partial U}{\partial y} \hat{v} - \frac{\partial \hat{p}}{\partial x} - \sigma(x) \hat{u}
\]

\[
i\omega \hat{v} = \left( C - \frac{\partial V}{\partial y} \right) \hat{v} - \frac{\partial V}{\partial x} \hat{u} - \frac{\partial \hat{p}}{\partial y} - \sigma(x) \hat{v} \quad (A.2.1)
\]

\[
0 = \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y},
\]

where \( C = \Delta/Re - U \partial/\partial x - V \partial/\partial y \) and \( \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 \).

We consider a computational domain of \( L_x = 900\delta_0 \) and \( L_y = 35\delta_0 \). We employ a pseudospectral scheme with \( N_x = 200 \) and \( N_y = 50 \) Chebyshev collocation points in the streamwise \((x)\) and wall-normal \((y)\) directions, respectively (Weideman and Reddy, 2000). Homogeneous Dirichlet boundary conditions for velocity fluctuations are applied at the inflow \((x = 0)\) as well as the wall \((y = 0)\) and in the free-stream \((y = 35\delta_0)\). Linear extrapolation is employed as the boundary condition at the outflow \((x = 900\delta_0)\) (Alizard and Robinet, 2007),

\[
\hat{v}(N_x, y) = a_1 \hat{v}(N_x - 1, y) + a_2 \hat{v}(N_x - 2, y)
\]

\[
\hat{u}(N_x, y) = a_1 \hat{u}(N_x - 1, y) + a_2 \hat{u}(N_x - 2, y)
\]

\[
a_1 = \frac{x(N_x) - x(N_x - 2)}{x(N_x - 1) - x(N_x - 2)}, \quad a_2 = \frac{x(N_x - 1) - x(N_x)}{x(N_x - 1) - x(N_x - 2)}.
\]

We adopt the similar sponge layer shape as in section A.1; parabolas at both ends of \( x \) with a thickness of \( L_s = 55 \), i.e., \( \sigma(x) = 0 \) for \( x \in [55, 845] \) (cf. Eq. (A.1.1)), and solve 50 temporal eigenvalue problems with different sponge layer strengths growing exponentially from \( 10^{-2} \) to \( 10^{0.8} \). We employ the shift and invert technique for solving Eq. (A.2.1), which is a common strategy for solving large-scale generalized eigenvalue problems (Theofilis, 2003b; Ehrenstein and Gallaire, 2005; Alizard and Robinet, 2007; Theofilis, 2011).

Figure A.4(a) shows the temporal eigenspectra of Eq. (A.2.1) with different sponge layer strengths. The eigenfunctions of these modes correspond to Tollmien-Schlichting (TS) waves. Similar to section A.1, the displacement trajectories can be considered as starting at red squares and ending at black dots. Similar to figure A.1(a), a turn-around phenomenon can also be observed, which is an indicator of significant reflection from the sponge layer. As for the wave equation, an desirable sponge layer strength can be determined. We consider the desirable sponge strength as the strength that corresponds to the turn-around point beyond which reflection becomes significant. This value is \( S_s \approx 0.235 \) for the problem considered in this section.

In figure A.4(a), two modes (on the same trajectory), which are marked with black squares, are selected for further investigation: the upper eigenvalue \( \omega = 0.04058 - 0.004325i \) corresponds to \( S_s = 0 \) and the lower eigenvalue \( \omega = 0.03944 - 0.008787i \) corresponds to the desirably-sponged case with \( S_s = 0.235 \). Figures A.4(c) and A.4(d) show the real parts of the streamwise velocity component \( \hat{u} \) of the eigenfunctions for the non-sponged and desirably
Figure A.4: (a) Eigenspectra of TS modes are displaced as the sponge layer strength increases: The red crosses (×) denote the spectrum obtained without a sponge layer ($S_s = 0$); Red squares represent the starting points on the trajectories ($S_s = 10^{-2}$); black dots are the final points on the trajectories ($S_s = 10^{0.8}$); the two black boxes (□) represent the non-sponged and desirably sponged cases selected for studying the properties of eigenfunctions. (b) Amplitude of two wall-normal velocity component $\hat{v}$ from the non-sponged case ($S_s = 0$, thick red dashed line) with corresponding eigenvalue $\omega = 0.04058 - 0.004325i$ and desirably sponged case ($S_s = 0.235$, thick black line) with corresponding eigenvalue $\omega = 0.03944 - 0.008787i$. (c) The real part of the streamwise velocity component $\hat{u}$ from the eigenfunction of the non-sponged problem; the amplitude data in (b) is extracted from the black dashed line ($y = 3\delta_0$). (d) The real part of $\hat{u}$ from the eigenfunction of the desirably sponged case.

sponged boundary conditions. Both show significant spatially evolving TS waves. The spatial growth of the TS wave for the desirably sponged problem is more pronounced. This demonstrates the effect of the sponge layer and is in harmony with the observations made for the wave equation.

To determine the spatial growth of the selected modes, we consider the amplitude of the wall-normal velocity component $\hat{v}$ at the wall-normal location $y = 3\delta_0$ marked by black
Figure A.5: (a) Relationship between the imaginary part of eigenvalues $\omega$ ($\circ$) and the imaginary part of the wavenumber $\alpha$ for the downstream propagating wave when the sponge layer strength grows from $10^{-2}$ to $10^{0.8}$. The red square denotes the beginning of the trajectory and the black dot denotes the end. The non-sponged case is marked by the red cross ($\times$). The red line is the fitted linear relationship. (b) Evolution of amplitudes of non-sponged and desirably sponged modes: non-sponged mode at $t = 0$ (thick red dashed line); desirably sponged mode at $t = 0$ (thick black line); non-sponged mode at $t = 200$ (thin red dashed line); desirably sponged mode at $t = 200$ (thin black line).

dashed lines in figures A.4(c) and A.4(d). In figure A.4(b), the amplitude of the mode corresponding to the non-sponged problem is denoted by the thick red dashed line. Its growth in the streamwise direction is almost exponential. The oscillations in the vicinity of the inlet are a result of numerical error. The black solid line in figure A.4(b) corresponds to the desirably sponged mode with $S_s = 0.235$. For this case, a branch of significant upstream propagating waves is observed close to the inlet. Unlike the wave equation results, however, the spatial growth rates of downstream and upstream propagating waves are different. This is because the phase speeds of downstream and upstream propagating waves are different. On the other hand, the spatial growth rate of the downstream branch from the desirably sponged mode is significantly larger than that of the non-sponged mode. The weak upstream propagating waves experience very slow growth, which is negligible. Thus, only the downstream propagating branches are considered in the later analyses.

As we observed in the previous section, sponge layers can suppress the temporal growth of eigenfunctions while amplifying their spatial growth. This hints at a relationship between temporal and spatial growth rates. However, since the dispersion relation is more complicated here, the relationship is not as obvious as in the wave equation. To identify this relationship, the eigenfunctions are approximated with the wave packet form: the phase function is $\exp(i(\alpha x - \omega t))$, where $\Im(\omega)$ is the temporal damping rate and $-\Im(\alpha)$ is the spatial growth rate of eigenfunctions. This is a reasonable approximation because the temporal frequency $\Re(\omega)$ and spatial wavenumber $\Re(\alpha)$ of eigenfunctions from the same displacement trajectory are approximately equal, and the amplitude of eigenfunctions grow exponentially in the streamwise direction. Using this approximation, we can extract $\Im(\alpha)$
from the amplitude of each eigenfunction via linear fitting and as the slope of the red line in figure A.4(b).

Figure A.5(a) shows the relation between $\text{Im}(\omega)$ and $\text{Im}(\alpha)$ for all eigenvalues on a single trajectory. We obtain an almost perfect linear mapping between the temporal damping rate and the spatial growth rate as

$$\text{Im}(\omega) = 0.372 \text{Im}(\alpha) + 0.000475,$$

which holds even after the turn-around phenomenon. The relationship is likely to show the dispersion relation since the slope 0.372 is close to the phase velocity of TS waves which is typically $0.3 \sim 0.4$. To verify this, the two eigenfunctions considered in figure A.4(b) are adopted again. For visual convenience, we decrease the amplitude of the eigenfunction corresponding to the non-sponged problem by a factor of 10. The eigenfunctions are then evolved from $t = 0$ to $t = 200$. Figure A.5(b) shows the amplitude of the wall-normal component of the eigenfunctions at $t = 0$ and $t = 200$. As shown in this figure, the amplitude of eigenfunctions for both non-sponged and desirably sponged cases decreases. On the other hand, this decrease in time can be viewed as the propagation of a wave packet in space. Since the phase speed of a wave does not change in the domain with no sponge layers, we choose the point of crossing of red and black lines as two points of interest with the first located at $x = 579.7$ (+) and the second at $x = 654.2$ ($\times$). The wall-normal amplitude of the mode is $3.27 \times 10^{-5}$ at $x = 579.7$ and is $3.61 \times 10^{-5}$ at $x = 654.2$. Based on this, a phase speed can be computed for the two eigenfunctions as $c = 0.3725$. This value is equivalent to the slope identified for the in Eq. (A.2.2) from figure A.5(a).

The offset in the linear relation (A.2.2) denotes a factor of 1.104 increase in the amplitude of the wave. From the phase function $\exp(i(\alpha x - \omega t))$, the convective amplitude growth can be computed as $\exp(-\alpha_i \Delta x + \omega_i \Delta t)$, where $\alpha_i$ and $\omega_i$ denote the imaginary parts of $\alpha$ and $\omega$. From linear relation (A.2.2) and $\Delta x = c \Delta t$ we arrive at an amplitude growth of $\exp(0.001279 \Delta x)$. This results in a factor of 1.0997 amplitude growth after $\Delta x = 74.5$ (the distance between two points of interest on figure A.5(b). This is in agreement with the actual growth of 1.104. We can thus write the linear relationship between $\text{Im}(\omega)$ and $\text{Im}(\alpha)$ as:

$$\text{Im}(\omega) = c \text{Im}(\alpha) + c_0,$$

where $c$ is the phase speed and $c_0$ is the convective spatial growth rate. For the wave equation, the phase speed is 1 and the convective spatial growth rate is 0, which satisfy this relationship. In fact, if the spatial eigenproblem of the Orr-Sommerfeld equation at $x = 579.7$ ($Re = 626$) is solved using the eigenvalue $\omega$ from the global stability analysis in figure A.5(a), the approximate linear dispersion relation is obtained as

$$\text{Im}(\omega) = 0.3707 \text{Im}(\alpha) + 0.000458,$$

which is very close to the linear relationship we obtained above.
The flat-plate boundary layer flow, e.g., the Blasius boundary layer flow, is globally stable, i.e., all eigenvalues of the dynamic generator matrix are in the stable half-plane (Monokrousos et al., 2010). This allows us to study the long-term response of velocity fluctuations to continuous input disturbances using the steady-state Lyapunov equation corresponding to the linearized NS equations. However, the discretized operators in the linearized NS equations are ill-conditioned and unstable non-physical modes often appear, which limits the utility of the Lyapunov framework. Herein, we build on the insight gained in previous sections to design desirable sponge layers that can stabilize such non-physical modes.

The NS equations are linearized around the Blasius boundary layer profile $[U \ V \ 0]^T$. We use a similar computational region $(L_x, L_y)$ and free stream velocity $(U_\infty)$ as in section A.2. The Reynolds number is also defined in a similar manner. We bring the linearized equations into its evolution form (Jovanovic and Bamieh, 2005) with state $\psi = [v, \eta]^T$. Here, $v$ and $\eta$ denote wall-normal velocity and vorticity, respectively. We leverage spatial homogeneity in the spanwise direction and apply Fourier transform to the governing equations. The state can thus be expressed as $\hat{\psi} = \hat{\psi}(x,y,t) \exp(i\beta z)$, where $\beta$ is the spanwise wavenumber and $\hat{\psi} = [\hat{v}, \hat{\eta}]^T$. This brings the linearized NS equations into the following form:

$$
\begin{bmatrix}
\dot{\hat{v}} \\
\dot{\hat{\eta}}
\end{bmatrix} =
A
\begin{bmatrix}
\hat{v} \\
\hat{\eta}
\end{bmatrix} +
B
\begin{bmatrix}
d_v \\
d_\eta
\end{bmatrix}
$$

(A.3.1)

$$
\begin{bmatrix}
\dot{\hat{u}} \\
\dot{\hat{v}} \\
\dot{\hat{w}}
\end{bmatrix} =
C
\begin{bmatrix}
\hat{v} \\
\hat{\eta}
\end{bmatrix}
$$

(A.3.2)

The definition of operators $A$, $B$ and $C$ are:

$$
A_{11} = \Delta^{-1} \left[ \frac{1}{Re} \Delta^2 - U \Delta \frac{\partial}{\partial x} - V \Delta \frac{\partial}{\partial y} - \frac{\partial V}{\partial y} \Delta - 2 \frac{\partial U}{\partial x} \frac{\partial^2}{\partial y^2} - \frac{\partial^2 V}{\partial y^2} \frac{\partial}{\partial x} + \frac{\partial^2 U}{\partial y^2} \frac{\partial}{\partial x} 
- \frac{\partial^3 V}{\partial y^3} - 2 \left( \frac{\partial^2 U}{\partial x \partial y} \frac{\partial}{\partial x} + \frac{\partial U}{\partial x} \frac{\partial^2}{\partial x \partial y} \right) \left( \frac{\partial^2}{\partial x^2} - \beta^2 \right)^{-1} \frac{\partial^2}{\partial x \partial y} \right] - \sigma(x),
$$

$$
A_{12} = 2i\beta \Delta^{-1} \left[ \left( \frac{\partial^2 U}{\partial x \partial y} \frac{\partial}{\partial x} + \frac{\partial U}{\partial x} \frac{\partial^2}{\partial x \partial y} \right) \left( \frac{\partial^2}{\partial x^2} - \beta^2 \right)^{-1} \right],
$$

$$
A_{21} = -i\beta \frac{\partial U}{\partial y}, \quad A_{22} = \frac{1}{Re} \Delta - U \frac{\partial}{\partial x} - V \frac{\partial}{\partial y} - \frac{\partial U}{\partial x} - \sigma(x),
$$

$$
B_{11} = \Delta^{-1} \left( f \Delta + \frac{\partial f}{\partial y} \frac{\partial}{\partial y} \right), \quad B_{22} = f, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2,
$$
\[ C_{11} = \left( \frac{\partial^2}{\partial x^2} - \beta^2 \right)^{-1} \frac{\partial^2}{\partial x \partial y} \quad , \quad C_{12} = -i\beta \left( \frac{\partial^2}{\partial x^2} - \beta^2 \right)^{-1} , \]
\[ C_{31} = i\beta \left( \frac{\partial^2}{\partial x^2} - \beta^2 \right)^{-1} \frac{\partial}{\partial y} \quad , \quad C_{32} = \left( \frac{\partial^2}{\partial x^2} - \beta^2 \right)^{-1} \frac{\partial}{\partial x} . \]

Here, \([\dot{\vec{v}}, \dot{\vec{\eta}}]^T\) denotes the time derivative of the state and \( \mathbf{d} = [d_v, d_\eta]^T \) represent the input forcing corresponding to the dynamics of \( v \) and \( \eta \). We use a similar pseudospectral scheme as in section A.2 to discretize the operators in the streamwise and wall-normal directions using \( N_x = 101 \) and \( N_y = 50 \) collocation points in \( x \) and \( y \), respectively. Configuration of boundary conditions and sponge layers follow section A.2.

When the dynamic generator \( A \) is stable, the steady-state covariance of the state \( \mathbf{\psi} \)
\[
X = \lim_{t \to \infty} \langle \mathbf{\hat{\psi}}(t) \mathbf{\hat{\psi}}^*(t) \rangle ,
\]
can be obtained as the solution to the algebraic Lyapunov equation
\[
AX + XA^* = -B \Omega B^* \quad (A.3.3)
\]
where \( \langle \cdot \rangle \) is the expectation operator, \( \ast \) denotes the adjoints of operators \( A \) and \( B \), \( \Omega \) is the covariance of zero mean white-in-time stochastic forcing
\[
\langle \mathbf{d}(t_1) \mathbf{d}^*(t_2) \rangle = \Omega \delta(t_1 - t_2) .
\]

For simplicity, we consider \( \Omega = I \). The operator \( B \) specifies the wall-normal region in which the forcing enters. In this section, we restrict the forcing to a region in the vicinity of the wall (from \( y = 0 \) to \( y = 5\delta_0 \)) (Ran et al., 2017a); see the appendix for details. The covariance of velocity fluctuations can be obtained from \( X \) as \( \mathbf{\Phi} = CXC^* \). The steady-state flow structures in the boundary layer flow can be extracted from eigenfunctions of the matrix \( \mathbf{\Phi} \). The corresponding eigenvalues represent the energy of flow structures (Jovanovic and Bamieh, 2005; Bagheri et al., 2009).

The temporal eigenspectrum for the linearized NS equations with \( \beta = 0.01 \) is shown in figure A.6(a). Due to the presence of unstable (non-physical) modes in the upper left region of the spectrum, it is essential to design sponge layers that can stabilize the spectrum and at the same time have minimal effect on the physics of relevant modes. Figure A.6(b) shows the displacement trajectory of eigenvalues that correspond to TS modes. The location corresponding to the desirable sponge layer strength is marked by the black squares (sponge layer strength \( S_s = 0.2 \)), and provides the largest temporal decay. Application of the desirable sponge layer stabilizes the eigenspectrum and sufficiently suppresses the non-physical modes; see figure A.6(a). In contrast, the eigenvalues corresponding to the TS modes are less decayed.

Figure A.7 shows the real part of the streamwise velocity component of the principle and second most energetic modes of \( \mathbf{\Phi} \). The covariance matrix \( \mathbf{\Phi} \) results from solving the Lyapunov equation in the presence of sponge layers with strength \( S_s = 0.2 \). The physical structure of these modes is similar to TS waves extracted from the corresponding dynamic
Figure A.6: (a) The temporal eigenspectrum of the linearized NS equations with no sponge layer (×), and with the desirably strengthened sponge layer $S_s = 0.2$ (○). (b) The displacement of the eigenvalues corresponding to the TS modes as the sponge layer strength increases; the spectrums without the sponge layer (×), red squares show the beginning of trajectories ($S_s = 10^{-2}$), black dots are the final points on the trajectories ($S_s = 10^{0.8}$). The black boxes (□) represent the desirably sponged case ($S_s = 0.2$).

generator $A$. Due to this visual similarity, one may conclude that they are resonant forms of multiple TS waves excited by white stochastic forcing. However, this remains to be proven.

We next investigate the influence of different sponge strengths ($S_s = 0.2, 4$) on the steady-state response. Figure A.8(a) shows the normalized envelope of the wall-normal component of the second eigenfunction of $\Phi$ at $y = 3$ as a function of $x$. This figure shows that the strength of the sponge layer has minimal influence on the envelope of the second mode. Our observations show that unlike the eigenfunctions studied in section A.2, no additional spatial growth is introduced by the sponge layers. In addition, as shown in figure A.8(b), the first 50 eigenvalues of the covariance matrix $\Phi$ do not significantly change due to different sponge layer strengths. Figure A.8(a) also shows the effect of a strong sponge layer in shrinking the computational domain in the streamwise direction, as well as inducing wave reflections into the solution. As shown in figure A.8(b), when the sponge layer strength increases, the energy of the principle mode increases whereas the energy of the second mode decreases. As a result, the total energy ($\sum \lambda_i$) is kept approximately intact. This shows the ability of the sponge layer in affecting the receptivity of individual modes to exogenous disturbances.
Figure A.7: Flow structures (real part of streamwise velocity fluctuation) extracted from the eigenvalue decomposition of the covariance matrix $\Phi$; (a) the principle (most energetic) mode, (b) the second mode.

Figure A.8: (a) The envelope of $\hat{v}$ at $y = 3\delta_0$ extracted from the second most significant mode of $\Phi$ with a sponge layer strength of $S_s = 0.2$ (dashed line) and $S_s = 4$ (solid line). (b) The first 50 eigenvalues of $\Phi$ with a sponge layer strength of $S_s = 0.2$ (○) and $S_s = 4$ (×).
Appendix B

Stochastic forced boundary layer flows

B.1 Operator valued matrices in Eqs. (2.2)

Equation (2.2) is of the following form:

\[
\begin{bmatrix}
v_t \\
\eta_t \\
u \\
v \\
w \\
\eta
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
v \\
\eta
\end{bmatrix}
+ \begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & 0 & B_{23}
\end{bmatrix}
\begin{bmatrix}
d_u \\
d_v \\
d_w
\end{bmatrix}
\]

with operators defined as

\[
A_{11} = \frac{1}{Re} \Delta^2 - U \Delta \partial_x - V \Delta \partial_y - \partial_y V \Delta - 2 \partial_x U \partial_{xx} - \partial_{yy} V \partial_y + \partial_{yy} U \partial_x
- \partial_{yyy} V - 2 (\partial_{xy} U \partial_x + \partial_x U \partial_{xy}) (\partial_{xx} - k_z^2)^{-1} \partial_{xy}
- \sigma(x),
\]

\[
A_{12} = 2i k_z \Delta^{-1} \left[ (\partial_{xy} U \partial_x + \partial_x U \partial_{xy}) (\partial_{xx} - k_z^2)^{-1} \right],
\]

\[
A_{21} = -ik_z \partial_y U,
\]

\[
A_{22} = \frac{1}{Re} \Delta - U \partial_x - V \partial_y - \partial_x U - \sigma(x),
\]

\[
B_{11} = -\Delta^{-1} (f \partial_{xy} + \partial_y f \partial_x),
\]

\[
B_{12} = \Delta^{-1} (f \partial_{xx} - k_z^2 f),
\]

\[
B_{13} = -ik_z \Delta^{-1} (\partial_y f + f \partial_y),
\]

\[
B_{21} = ik_z f,
\]

\[
B_{23} = -f \partial_x,
\]

\[
C_{11} = - (\partial_{xx} - k_z^2)^{-1} \partial_{xy},
\]

\[
C_{12} = ik_z (\partial_{xx} - k_z^2)^{-1},
\]

\[
C_{31} = -ik_z (\partial_{xx} - k_z^2)^{-1} \partial_y,
\]

\[
C_{32} = - (\partial_{xx} - k_z^2)^{-1} \partial_x.
\]

Here, \(\sigma(x)\) determines the strength of sponge layers as a function of \(x\); see (Ran et al., 2017b) for additional details. For parallel flows, Fourier transform in \(x\) can be used to further
parameterize the operators over streamwise wavenumbers; see (Jovanovic and Bamieh, 2005) for the expressions of $A$, $B$, and $C$ in such instances.

**B.2 Change of variables**

The kinetic energy of velocity fluctuations in the linearized NS equations (2.2) is defined using the energy norm

$$E = \langle \varphi, \varphi \rangle_e = \frac{1}{2} \int_\Omega \varphi^* Q \varphi \, dy =: \langle \varphi, Q \varphi \rangle$$

where $\Omega$ is the computational domain, $\langle \cdot, \cdot \rangle$ is the $L_2$ inner product and $Q$ is the operator which determines kinetic energy of the state $\varphi = [v \ \eta]^T$ on the appropriate state-space (Reddy and Henningson, 1993; Jovanovic and Bamieh, 2005). With proper discretization of the inhomogeneous directions, the kinetic energy is given by $E = \varphi^* Q \varphi$. Here, $Q$ is the discrete representation of operator $Q$ and is a positive definite matrix. The coordinate transformation $\psi = Q^{1/2} \varphi$ can thus be employed to obtain the kinetic energy via the standard Euclidean norm: $E = \psi^* \psi$ in the new coordinate space. Equation (2.3) results from the application of this change of variables on the discretized state-space matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$

$$A = Q^{1/2} \bar{A} Q^{-1/2}, \quad B = Q^{1/2} \bar{B} I_W^{-1/2}, \quad C = I_W^{1/2} \bar{C} Q^{-1/2},$$

and the discretized input $\bar{d}$ and velocity $\bar{v}$ vectors

$$d = I_W^{1/2} \bar{d}, \quad v = I_W^{1/2} \bar{v}.$$  

Here, $I_W$ is a diagonal matrix of integration weights on the set of Chebyshev collocation points.

The operator $Q$ in the global model is of the form:

$$Q = \begin{bmatrix} \partial_{xy}^\dagger \Theta^\dagger \Theta \partial_{xy} + I + k_z^2 \partial_y^\dagger \Theta^\dagger \Theta \partial_y & 0 \\ 0 & k_z^2 \partial_y^\dagger \Theta^\dagger + \partial_x^\dagger \Theta^\dagger \Theta \partial_x \end{bmatrix}$$

where $\Theta = (\partial_x^2 - k_z^2)^{-1}$, $I$ is the identity operator and $\dagger$ represents the adjoint of an operator. The representation of $Q$ for parallel flows can be found in (Jovanovic and Bamieh, 2005, Appendix A).

**B.3 Global analysis using the descriptor form**

The descriptor form of the linearized NS equations around the Blasius boundary layer profile is given by

$$F \dot{\psi}(t) = A \psi(t) + B \bar{d}(t),$$

$$v(t) = C \psi(t),$$

(B.3.1)
where \( \psi = [u \ v \ w \ p]^T \) and

\[
F = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
A = \begin{bmatrix}
K + \partial_y V & -\partial_y U & 0 & -\partial_x \\
0 & K - \partial_y V & 0 & -\partial_y \\
0 & 0 & K & -ik_z \\
\partial_x & \partial_y & ik_z & 0
\end{bmatrix},
B = \begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I \\
0 & 0 & 0
\end{bmatrix},
\]

and \( C = B^T \), where \( I \) is the identity operator and

\[
K = \frac{1}{Re} \left( \partial_x^2 + \partial_y^2 - k_z^2 \right) - U\partial_x - V\partial_y - \sigma(x).
\]

Here, \( \sigma(x) \) determines the strength of sponge layers as a function of \( x \). The width and strength of the sponge layers are selected to guarantee the stability of the generalized dynamics (B.3.1) in their discretized form, while having minimal influence on velocity fluctuation field. The energy of velocity fluctuations in Eqs. (B.3.1) can be determined by

\[
E = \text{trace} \left( C (G_c + G_{nc}) C^\dagger \right),
\]

which is analogous to expression (2.7) for the evolution model with \( \psi = [v \ \eta]^T \). Here, \( \dagger \) represents the adjoint of an operator and \( G_c \) and \( G_{nc} \) are the causal and non-causal reachability Gramians that satisfy the following generalized Lyapunov equations:

\[
FRG_c\dagger A + AG_c F = -P_l BB^\dagger P_l^\dagger,
FG_{nc} F + AG_{nc} A = -Q_l BB^\dagger Q_l^\dagger,
\]

where \( P_l \) and \( Q_l \) are the projection operators that project the state-space into causal and non-causal subspaces; see (Moarref, 2012, Appendix.E) for additional details.

After proper spatial discretization of the state-space, the procedure for solving the generalized Lyapunov equations (B.3.2) consists of the following steps: (i) compute the generalized Schur form of the discretized pair \((A,F)\); (ii) computing the solution to a system of generalized Sylvester equations; and (iii) solving the generalized Lyapunov equations (B.3.2) for Gramian matrices \( G_c \) and \( G_{nc} \). The Schur decomposition and the solution to the Sylvester equations are required to split the state into slow (causal) and fast (non-causal) parts and to form projection matrices \( P_l \) and \( Q_l \). For a spatial discretization that involves \( n = 4N_xN_y \) states, the overall computational complexity of this procedure is \( O(n^3) \), which is significantly higher than the computational complexity of solving the Lyapunov equation (2.6) with \( n = 2N_xN_y \). Moreover, since the state-space of the descriptor form has twice the number of states as the evolution model (2.3), computations based on this representation require more memory. We refer the interested reader to (Moarref, 2012, Appendix.E) for additional details on computing energy amplification using the descriptor form.

In order to demonstrate the close agreement between the outcome of receptivity analysis based on the evolution model of Section 2.1 and the descriptor form (B.3.1), we focus on the energy amplification of flow structures with \( k_z = 0.32 \). Similar to Section 2.1, we
discretize system (B.3.1) by applying Fourier transform in \(z\) and using a Chebyshev collocation scheme in the wall-normal and streamwise directions. In the wall-normal direction, we enforce homogenous Dirichlet boundary conditions on all velocity components. In the streamwise direction, we use homogeneous Dirichlet boundary conditions at the inflow and spatial extrapolation at the outflow for all velocity components. Moreover, sponge layers are applied at the inflow and outflow to mitigate the influence of boundary conditions on the fluctuation dynamics. As shown in Fig. B.1, the dominant flow structures that result from near-wall excitation closely resemble the results presented in Fig. 3.9.

Figure B.1: (a) The streamwise component of the principle eigenmode of output covariance matrix \(\Phi\) resulting from near-wall stochastic excitation (case 1 in Table 2.1) of the linearized model (B.3.1) with \(k_z = 0.32\) and \(Re_0 = 232\). (b) Streamwise velocity at \(z = 0\). (c) Slice of streamwise velocity (color plots) and vorticity (contour lines) at \(x = 750\), which corresponds to the cross-plane slice indicated by the black dashed lines in (b).

B.4 Matching the HIT spectrum with stochastically forced linearized NS equations

We briefly describe how the spectrum of HIT can be matched using stochastically forced linearized NS equations; see (Moarref, 2012, Appendix C) for additional details. The dynamics of velocity fluctuations \(v\) around a uniform base flow \(\bar{u} = [1 \ 0 \ 0]^T\) subject to the solenoidal forcing \(d_s = [d_u \ d_v \ d_w]\) \((\nabla \cdot d_s = 0)\) are governed by the linearized NS equations

\[
v_t(k, t) = A(k)v(k, t) + d_s(k, t),
\]

where \(k = [k_x \ k_y \ k_z]^T\) is the spatial wavenumber vector and

\[
A(k) = -\left(ik_x + \frac{k^2}{Re}\right)I_{3 \times 3},
\]

is the linearized operator. Here, \(k^2 = k_x^2 + k_y^2 + k_z^2\) and \(I_{3 \times 3}\) is the identity operator. The steady-state covariance of velocity fluctuations \(\Phi(k) = \lim_{t \to \infty} \langle v(k, t)v^*(k, t)\rangle\) satisfies the following Lyapunov equation

\[
A(k)\Phi(k) + \Phi(k)A^*(k) = -M(k),
\]  

\[(B.4.1)\]
where \( M(k) \) denotes the covariance of white-in-time stochastic forcing. The steady-state covariance matrix \( \Phi \) corresponding to HIT is given by (Batchelor, 1953)

\[
\Phi(k) = \frac{E(k)}{4\pi k^2} \left( I_{3 \times 3} + \frac{k k^T}{k^2} \right).
\]

where \( E(k) \) is the energy spectrum of the HIT based on the von Kármán spectrum (Durbin and Reif, 2011),

\[
E(k) = L C_{vk} \frac{(k L)^4}{(1 + k^2 L^2)^{17/6}}.
\]

Here, \( C_{vk} = \frac{\Gamma(17/6)}{\Gamma(5/2)\Gamma(1/3)} \approx 0.48 \) is a normalization constant in which \( \Gamma(\cdot) \) is the gamma function and the integral length-scale \( L = 1.5 \) corresponds to numerical simulations of HIT (Wang et al., 1996). The input forcing covariance can be derived by substituting \( \Phi(k) \) into Eq. (B.4.1), which yields

\[
M(k) = \frac{E(k)}{2\pi Re} \left( I_{3 \times 3} + \frac{k k^T}{k^2} \right).
\]

After finite dimensional approximation of all operators, the covariance of forcing \( d_s \), parameterized by \( k_z \), is obtained via inverse Fourier transform in \( x \) and \( y \). The resulting covariance matrix \( M(k_z) \) includes two-point correlations of the white stochastic forcing in the streamwise and wall-normal directions and it replaces \( W \) in Eq. (2.6).
Appendix C

Parabolized Floquet Equations

C.1 Operators for H-type transition

For the case study considered in Sec. 7.1, the operators $L_{n,m}$ and $M_{n,m}$ in $L_F$ and $M_F$ from Eq. (6.7) are of the form:

\[ L_{n,0} = \begin{bmatrix}
\Gamma_n - \partial_y V_B & \partial_y U_B & 0 & i((n + 0.5)\alpha_r + \gamma) \\
0 & \Gamma_n + \partial_y V_B & 0 & \partial_y \\
0 & 0 & \Gamma_n & i\beta \\
i((n + 0.5)\alpha_r + \gamma) & \partial_y & i\beta & 0
\end{bmatrix}, \]

\[ L_{n,-1} = \begin{bmatrix}
\Gamma_{n-} - \partial_y V_T & \partial_y U_T & 0 & 0 \\
i\alpha_r V_T & \Gamma_{n-} + \partial_y V_T & 0 & 0 \\
0 & 0 & \Gamma_{n-} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad L_{n,1} = \begin{bmatrix}
\Gamma_{n+} - \partial_y V_T^* & \partial_y U_T^* & 0 & 0 \\
-i\alpha_r V_T^* & \Gamma_{n+} + \partial_y V_T^* & 0 & 0 \\
0 & 0 & \Gamma_{n+} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \]

and

\[ M_{n,0} = \begin{bmatrix}
\Omega_n & 0 & 0 & I \\
0 & \Omega_n & 0 & 0 \\
0 & 0 & \Omega_n & 0 \\
I & 0 & 0 & 0
\end{bmatrix}, \quad M_{n,-1} = \begin{bmatrix}
U_T & 0 & 0 & 0 \\
0 & U_T & 0 & 0 \\
0 & 0 & U_T & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad M_{n,1} = \begin{bmatrix}
U_T^* & 0 & 0 & 0 \\
0 & U_T^* & 0 & 0 \\
0 & 0 & U_T^* & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}. \]
where
\[
\Gamma_n = -\frac{1}{Re} \left( \partial_{yy} - ((n + 0.5)\alpha_r + \gamma)^2 + \beta^2 \right) + i((n + 0.5)\alpha_r + \gamma)U_B - i(n + 0.5)\alpha_r c + V_B \partial_y,
\]
\[
\Gamma_{n-} = i((n + 1.5)\alpha_r + \gamma)U_T + V_T \partial_y,
\]
\[
\Gamma_{n+} = i((n - 0.5)\alpha_r + \gamma)U_T^* + V_T^* \partial_y,
\]
\[
\Omega_n = U_B - \frac{2i}{Re} ((n + 0.5)\alpha_r + \gamma).
\]

### C.2 Operators for streamwise streaks

For the case study considered in Sec. 7.2, the operators \( L_{n,m} \) and \( M_{n,m} \) in \( L_F \) and \( M_F \) from Eq. (6.7) are of the form:

\[
pL_{n,0} = \begin{bmatrix}
\Gamma_n - \partial_y V_B & \partial_y U_B & 0 & i \alpha \\
0 & \Gamma_n + \partial_y V_B & 0 & \partial_y \\
0 & 0 & \Gamma_n & in \beta \\
i \alpha & \partial_y & in \beta & 0
\end{bmatrix},
\]

\[L_{n,-1} = \begin{bmatrix}
i \alpha U_{S,1} & \partial_y U_{S,1} & i \beta U_{S,1} & 0 \\
0 & i \alpha U_{S,1} & 0 & 0 \\
0 & 0 & i \alpha U_{S,1} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad L_{n,+1} = \begin{bmatrix}
i \alpha U_{S,1}^* & \partial_y U_{S,1}^* & -i \beta U_{S,1}^* & 0 \\
0 & i \alpha U_{S,1}^* & 0 & 0 \\
0 & 0 & i \alpha U_{S,1}^* & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad (C.2.1a)
\]

\[
M_{n,0} = \begin{bmatrix}
U_B - \frac{2i \alpha}{Re} & 0 & 0 & 0 \\
0 & U_B - \frac{2i \alpha}{Re} & 0 & 0 \\
0 & 0 & U_B - \frac{2i \alpha}{Re} & 0 \\
i & 0 & 0 & 0
\end{bmatrix},
\]

\[M_{n,-1} = \begin{bmatrix}
U_{S,1} & 0 & 0 & 0 \\
0 & U_{S,1} & 0 & 0 \\
0 & 0 & U_{S,1} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad M_{n,+1} = \begin{bmatrix}
U_{S,1}^* & 0 & 0 & 0 \\
0 & U_{S,1}^* & 0 & 0 \\
0 & 0 & U_{S,1}^* & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad (C.2.1b)
\]
where
\[
\Gamma_n = -\frac{1}{Re} \left( \partial_{yy} \alpha^2 + (n \beta)^2 \right) + \left( -i \omega + i \alpha U_B + V_B \partial_y \right).
\]

Note that consistent with nonlinear PSE, the boundary conditions on the MFD require special treatment; see (Bertolotti et al., 1992). Following (Haj-Hariri, 1994; Li and Malik, 1996; Day et al., 2001) which showed that the streamwise pressure gradient is the main contributor to the residual ellipticity in the PSE, we remove the pressure gradient from the streamwise velocity momentum \((M_{n,0}(1,4) = 0)\) to ensure a well-posed streamwise march.

In subsequent PFE runs, the second harmonic and MFD also augment the base flow (cf. Eq. (7.7)) and \(U_B\) and \(V_B\) in Eqs. (C.2.1a) and (C.2.1b) denote the combination of the Blasius profile and the MFD. Interactions with the second harmonic generated from previous PFE runs are facilitated by the off-diagonal operators:

\[
L_{n,-2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \alpha U_{S,2} & 0 & 0 \\
0 & 0 & \alpha U_{S,2} & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad L_{n,+2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \alpha U_{S,2} & 0 & 0 \\
0 & 0 & \alpha U_{S,2} & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
M_{n,-2} = \begin{bmatrix}
U_{S,2} & 0 & 0 & 0 \\
0 & U_{S,2} & 0 & 0 \\
0 & 0 & U_{S,2} & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad M_{n,+2} = \begin{bmatrix}
U_{S,2} & 0 & 0 & 0 \\
0 & U_{S,2} & 0 & 0 \\
0 & 0 & U_{S,2} & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

### C.3 Grid-convergence and dependence on the number of harmonics

We examine the influence of the wall-normal grid-resolution \((N_y)\) and the number of harmonics \((N)\) considered in the PFE progression on the convergence of results obtained in Secs. 7.1 and 7.2. To quantify convergence, the kinetic energy of the most important mode, i.e., the \((1,1)\)-subharmonic mode in H-type transition and the MFD of streaks are computed at various streamwise locations and stored in the vector \(E\). The total energy in Tables C.1 and C.2 denotes the aggregate kinetic energy in the streamwise dimension and is computed using the Euclidean norm of the vector \(E\), i.e., \(\|E\|_2\). To quantify grid-independence, we use the relative error \(\|E - E_r\|_2/\|E_r\|_2\), where \(E_r\) is the kinetic energy obtained by refining resolution (in \(N_y\) or in the number of (sub)harmonics).
### Table C.1: Convergence of results in the study of H-type transition

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N_y$</th>
<th>$E$</th>
<th>$|E - E_r|_2 / |E_r|_2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>40</td>
<td>0.15818</td>
<td>15.6</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>0.18266</td>
<td>0.54</td>
</tr>
<tr>
<td>2</td>
<td>160</td>
<td>0.18366</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>0.18266</td>
<td>0.31</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
<td>0.18209</td>
<td>0.0008</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>0.18209</td>
<td>...</td>
</tr>
</tbody>
</table>

### Table C.2: Convergence of results in the study of laminar streaks

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N_y$</th>
<th>$E$</th>
<th>$|E - E_r|_2 / |E_r|_2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>40</td>
<td>$1.5375 \times 10^{-3}$</td>
<td>0.6</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
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Appendix D

Analysis of turbulent channel flow with riblets

D.1 Operators $A_\theta$, $B_\theta$, and $C_\theta$ in equations (9.23)

The dynamical generator $A_\theta$ in equations (9.23) has a bi-infinite structure shown in equation (9.24), in which elements $A_{n,m}$ contain four operators,

$$ A_{n,m} = \begin{bmatrix} A_{n,m,1,1} & A_{n,m,1,2} \\ A_{n,m,2,1} & A_{n,m,2,2} \end{bmatrix}. $$

For the operators on the main diagonal, $A_{n,0}$, we have

$$ A_{n,0,1,1} = \Delta_n^{-1} [(1 + \nu_T)\Delta^2_n + \nu''_T(\partial_y^2 + k_n^2) + 2\nu'_T\Delta_n] / Re + \Gamma_{n,0,1,1} $$

$$ A_{n,0,1,2} = \Gamma_{n,0,1,2} $$

$$ A_{n,0,2,1} = \Gamma_{n,0,2,1} $$

$$ A_{n,0,2,2} = [(1 + \nu_T)\Delta_n + \nu'_T] / Re + \Gamma_{n,0,2,2} $$

and for the off-diagonal ones, $A_{n,m}$ with $m \neq 0$, we have

$$ A_{n,m,1,1} = \Gamma_{n,m,1,1}, \quad A_{n,m,1,2} = \Gamma_{n,m,1,2}, $$

$$ A_{n,m,2,1} = \Gamma_{n,m,2,1}, \quad A_{n,m,2,2} = \Gamma_{n,m,2,2} $$

where

$$ \Gamma_{n,m,1,1} = \Delta_n^{-1} \left[ 2im k_x \omega_z \theta_{n+m} \frac{\theta_{n+m}}{k_{n+m}^2} \left( \bar{U}_{-m}\partial_y + \bar{U}_{-m}\partial_{yy} \right) + ik_x \left( \bar{U'}_{-m} - \bar{U}_{-m}\Delta_{n+m} \right) 

+ ik_x (m^2 \omega_z)^2 \bar{U}_{-m} - 2m k_x \omega_z \theta_{n+m} \bar{U}_{-m} + m \omega_z (m \omega_z - 2\theta_{n+m})a_{-m} 

- a_{-m}\Delta_{n+m} - a'_{-m}\partial_y + m \omega_z \frac{\theta_{n+m}}{k_{n+m}^2} \left( a'_{-m}\partial_y + a_{-m}\partial_{yy} \right) \right] $$
Similarly, the output operator \( N \) is given by 

\[
\Gamma_{n,m,1} = \Delta_n^{-1} \left[ 2 \frac{im k^2}{k_{n+m}} \omega_z \left( \bar{U}'_m + \bar{U}_m \partial_y \right) + \frac{m k_x \omega_z}{k^2_{n+m}} \left( A'_m + a_m \partial_y \right) \right]
\]

\[
\Gamma_{n,m,2} = im \omega_z \left( \bar{U}'_m - \bar{U}_m \partial_y \right) - i \theta_{n+m} \bar{U}'_m + \left[ i(m \omega_z)^2 \theta_{n+m} \bar{U}'_m - \frac{m k_x \omega_z}{k^2_{n+m}} a_m \right] \partial_y
\]

\[
\Gamma_{n,m,2} = -ik_x \bar{U}_m - a_m + \frac{i k_x (m \omega_z)^2}{k^2_{n+m}} \bar{U}_m + m \omega_z \theta_{n+m} a_m
\]

Here, \( \theta_{n+m} = (n + m) \omega_z + \theta, k^2_{n+m} = k^2_x + \theta^2_{n+m}, \) and \( \Delta_n = \partial_{yy} - k^2_{n+m}. \)

The input operator \( B_\theta \) takes the form

\[
B_n = \begin{bmatrix} B_v \\ B_y \end{bmatrix} = \begin{bmatrix} -ik_x \Delta_n^{-1} \partial_y & -ik_x \Delta_n^{-1} \partial_y \\ i \theta_n I & 0 \\ -ik_x I \end{bmatrix}.
\]

Similarly, the output operator \( C_\theta \) is given by

\[
C_n = \begin{bmatrix} C_u \\ C_v \\ C_w \end{bmatrix} = \begin{bmatrix} (ik_x/k^2_n) \partial_y & -(i \theta_n/k^2_n) I \\ I & 0 \\ (i \theta_n/k^2_n) \partial_y & (ik_x/k^2_n) I \end{bmatrix}.
\]

**D.2 Computing corrections \((k_c, \epsilon_c)\) to \((k, \epsilon)\)**

Following (9.19), we show that the effect of fluctuations around the mean velocity on corrections \( k_c \) and \( \epsilon_c \) can be obtained from the correction \( X_{\theta,c}(k_x) \) to the steady-state covariance:

\[
k_c(y) = \int_0^\infty \int_0^{\frac{\pi}{2}} \sum_{n \in \mathbb{Z}} K_k(y, k_x, \theta_n) \, d\theta \, dk_x,
\]

\[
\epsilon_c(y) = \int_0^\infty \int_0^{\frac{\pi}{2}} \sum_{n \in \mathbb{Z}} K_\epsilon(y, k_x, \theta_n) \, d\theta \, dk_x
\]

where \( K_k(y, k_x, \theta_n) \) and \( K_\epsilon(y, k_x, \theta_n) \) are obtained by taking the diagonal components of matrices \( N_k \) and \( N_\epsilon \), respectively:

\[
N_k(y, k_x, \theta_n) = \frac{1}{2} \left( C_u X_C C^*_u + C_v X_C C^*_v + C_w X_C C^*_w \right),
\]

\[
N_\epsilon(y, k_x, \theta_n) = 2 \left( k^2_x C_u X_C C^*_u + D_y C_v X_C C^*_v D^*_y + \theta^2_n C_w X_C C^*_w - ik_x D_y C_u X_C C^*_v + k_x \theta_n C_u X_C C^*_u + i \theta_n D_y C_v X_C C^*_v \right) + \theta^2_n C_w X_C C^*_w + k^2_x C_v X_C C^*_v + D_y C_w X_C C^*_w D^*_y + \theta^2_n C_u X_C C^*_u + \theta^2_n C_v X_C C^*_v + k^2_x C_w X_C C^*_w.
\]
Here, the terms on the right-hand-side of the equations are at the wavenumber pair \((k_x, \theta_n)\); \(D_y\) denotes the finite-dimensional representation of \(\partial_y\) and \(C_u\), \(C_v\), and \(C_w\) are finite-dimensional approximations of the output operators in (D.1.2) and the covariance matrix \(X_c(k_x, \theta_n)\) can be obtained as

\[
X_c(k_x, \theta_n) = X_d(k_x, \theta_n) - X_s(k_x, \theta_n),
\]

where \(X_s(k_x, \theta_n)\) and \(X_d(k_x, \theta_n)\) represent the steady-state covariance matrix in channel flow over smooth walls and the main diagonal blocks of the solution to Lyapunov equation (9.28), \(X_\theta(k_x)\), respectively. Note that these matrices have been confined to the wall-normal range \(y \in [-1, 1]\) to provide appropriate comparison between channel flows over smooth and corrugated surfaces.