"Standing" diffusion of electromagnetic fields in superconductors with gradual resistive transitions

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Electromagnetic field diffusion in superconductors with gradual resistive transitions may exhibit a peculiar (anomalous) mode which does not exist in superconductors with sharp (ideal) resistive transitions. This is a "standing" mode. In the case of this mode, the electromagnetic field on a superconductor boundary increases with time, while the region occupied by the electromagnetic field within the superconductor does not expand. In this paper, an exact analytical solution to the appropriate nonlinear diffusion equation is derived. It is demonstrated that this solution can be physically interpreted as the standing mode. The standing mode solution is also obtained by using a "rectangular profile" approximation and these two results are compared. [S0021-8979(96)03508-1]

Models for superconducting hysteresis are based on the analytical study of electromagnetic field diffusion in hard superconductors. In the critical state model,[1–3] this study is usually performed under the assumption of ideal (sharp) resistive transition. This assumption appreciably facilitates the calculation of distribution of electric currents in superconductors of simple shapes (plane slabs, circular cross-section cylinders) and leads to the rate independent models for superconducting hysteresis.

Actual resistive transitions in superconductors are gradual, and they are customarily described by the power law:[4–6]

$$E=\left(\frac{J}{k}\right)^n, \quad (n>1). \tag{1}$$

In the above expression, E is electric field, J is current density, and k is a parameter which coordinates the dimensions in (1). The exponent "n" is a measure of the sharpness of the resistive transition and it varies in the range 7–1000.

The above power law can be used as a constitutive equation for hard superconductors. This leads to the following nonlinear diffusion equation for the current density in the case of 1D problems:

$$\frac{\partial^2 J^n}{\partial z^2}=\mu_0 k^n \frac{\partial J}{\partial t}. \tag{2}$$

Previously, the analytical self-similar solutions of Eq. (2) were derived for zero initial condition and the following boundary condition:

$$J_0(t)=J(0,t)=ct^p, \quad (t\geqslant 0,\ p\geqslant 0). \tag{3}$$

It has been found that the electromagnetic field diffusion in superconductors with gradual resistive transitions has some features which are very similar to the electromagnetic field diffusion in superconductors with ideal resistive transitions. Namely, it has been found that for $n\geqslant 7$ the actual profile of electric current density is almost rectangular. This has prompted the suggestion to approximate the actual current density profile by a rectangular one with the height equal to the instantaneous value $J_0(t)$, of the current density on the boundary of the superconductor (see Fig. 1). This "rectangular profile" approximation has led to the following equation for the zero front $z_0(t)$, of the current density:

$$\int_0^t J_0^n(\tau)\,d\tau=\frac{\mu_0 k^n}{2}\,[J_0(t)z_0^2(t)-J_0(0)z_0^2(0)], \tag{4}$$

which is valid for any monotonically increasing boundary condition $J_0(t)$.

In this paper, we intend to show that electromagnetic field diffusion in superconductors with gradual resistive transitions may exhibit a peculiar (anomalous) mode which does not exist in superconductors with ideal resistive transitions. This is a standing mode. In the case of this mode, the electromagnetic field on a superconductor boundary increases with time, while the region occupied by the electromagnetic field does not expand. We shall first find the condition for the existence of this mode by using the "rectangular profile" approximation and formula (4). Then, we shall derive the exact expressions for the standing mode through the analytical solution of nonlinear diffusion Eq. (2), that is without resorting to the rectangular profile approximation. Finally, we shall compare these two results.

To start the discussion, we turn to Eq. (4) and try to find such a monotonically increasing boundary condition $J_0(t)$ for which the zero front $z_0(t)$, stands still. To this end, we assume that $z_0(t)=z_0=$const, and, by differentiating both sides of (4), we arrive at:

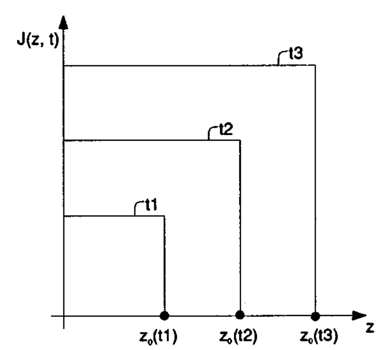

FIG. 1. Rectangular approximation of current density profiles inside the superconducting half-space.

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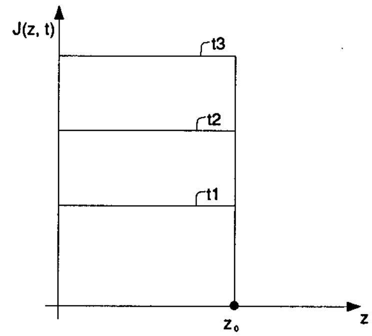

FIG. 2. Rectangular approximation of the standing mode.

$$J_0^n(t)=\frac{\mu_0 k^n}{2}\, z_0^2\, \frac{dJ_0(t)}{dt}. \tag{5}$$

The last expression can be transformed as follows:

$$\frac{2}{\mu_0 k^n z_0^2}\, dt=\frac{dJ_0(t)}{J_0^n(t)}. \tag{6}$$

By integrating both parts of (6), we obtain:

$$\frac{2}{\mu_0 k^n z_0^2}\, t=\frac{1}{n-1}\,[J_0^{1-n}(0)-J_0^{1-n}(t)]. \tag{7}$$

From (7), we derive:

$$J_0(t)=\frac{1}{\left(J_0^{1-n}(0)-\dfrac{2(n-1)}{\mu_0 k^n z_0^2}\, t\right)^{1/n-1}}. \tag{8}$$

The last expression can be represented in the form:

$$J_0(t)=\frac{c}{(t_0-t)^{1/n-1}}, \tag{9}$$

where:

$$c=\left(\frac{\mu_0 k^n z_0^2}{2(n-1)}\right)^{1/n-1}, \quad t_0=\frac{\mu_0 k^n z_0^2 J_0^{1-n}(0)}{2(n-1)}. \tag{10}$$

Thus, we have established that, if the current density on the boundary of superconducting half-space varies with time according to the expressions (9)–(10), then the zero front $z_0(t)$, of the current density stands still during the time interval $0\leq t<t_0$. In other words, during this time interval the electromagnetic field diffusion exhibits a standing mode. This mode is illustrated by Fig. 2.

It is desirable to express the boundary condition for the standing mode in terms of magnetic field $H_0(t)$ on the superconductor boundary. This can be easily accomplished by using (9) and Ampere's Law, which leads to:

$$H_0(t)=\frac{cz_0}{(t_0-t)^{1/n-1}}. \tag{11}$$

Our previous derivation has been based on the ''rectangular profile'' approximation for the electric current density. Next, we shall derive the expressions for the standing mode solution without resorting to the above approximation, but rather through an analytical solution of the nonlinear diffusion Eq. (2). It is remarkable that the standing mode solution can be obtained by using the method of separation of variables. Actually, this is the only solution which can be obtained by this method. According to the method of separation of variables, we look for the solution of Eq. (2) in the form:

$$J(z,t)=\varphi(z)\,\psi(t). \tag{12}$$

By substituting (12) into (2), after simple transformations we derive:

$$\frac{1}{\varphi(z)}\,\frac{d^2\varphi^n(z)}{dz^2}=\frac{\mu_0 k^n}{\psi^n(t)}\,\frac{d\psi(t)}{dt}. \tag{13}$$

This means that

$$\frac{\mu_0 k^n}{\psi^n(t)}\,\frac{d\psi(t)}{dt}=\lambda, \tag{14}$$

$$\frac{1}{\varphi(z)}\,\frac{d^2\varphi^n(z)}{dz^2}=\lambda, \tag{15}$$

where λ is some constant.

By integrating (14), we easily obtain:

$$\psi(t)=\left(\frac{\mu_0 k^n}{(n-1)\lambda(t_0-t)}\right)^{1/n-1}, \tag{16}$$

where t_0 is a constant of integration.

Equation (15) is more complicated than Eq. (14) and its integration is somewhat more involved. To integrate Eq. (15), we introduce the following auxiliary functions:

$$\varphi^n(z)=\theta(z), \quad R(z)=\frac{d\theta(z)}{dz}. \tag{17}$$

From (17) and (15), we derive:

$$\frac{dR}{dz}=\frac{d^2\theta}{dz^2}=\frac{d^2\varphi^n}{dz^2}=\lambda\,\varphi(z)=\lambda\,\theta^{1/n}(z). \tag{18}$$

On the other hand

$$\frac{dR}{dz}=\frac{dR}{d\theta}\cdot\frac{d\theta}{dz}=R\,\frac{dR}{d\theta}=\frac{1}{2}\,\frac{d(R^2)}{d\theta}. \tag{19}$$

By equating the right-hand sides of (18) and (19), we obtain

$$\frac{d(R^2)}{d\theta}=2\lambda\,\theta^{1/n}. \tag{20}$$

By integrating Eq. (20), we find:

$$R(z)=\sqrt{\frac{2n}{n+1}}\,\lambda[\theta(z)]^{n+1/2n}. \tag{21}$$

In (21), a constant of integration was set to zero. This can be justified on physical grounds. Indeed, the magnetic field should vanish at the zero front, that is at the same point where $J(z,t)$ vanishes. By using (12) and (17), it can be shown that the magnetic field and $J(z,t)$ are proportional to $R(z)$ and $\theta^{1/n}(z)$, respectively. This means that the above two functions should vanish simultaneously. This is only possible if the integration constant in (21) is set to zero.

Next, by using (17) in (21), we find:

$$\frac{d\theta(z)}{dz}=\sqrt{\frac{2n}{n+1}}\,\lambda[\theta(z)]^{n+1/2n}. \tag{22}$$

By integrating (22), we derive:

$$[\theta(z)]^{n-1/n}=\frac{(n-1)^2}{2(n+1)n}\,\lambda(z_0-z)^2, \tag{23}$$

where z_0 is a constant of integration.

From (17) and (23), we obtain:

$$\varphi(z)=\left[\frac{(n-1)^2\lambda}{2(n+1)n}\,(z_0-z)^2\right]^{1/n-1}. \tag{24}$$

Now, by substituting (16) and (24) into (12), we find the following analytical (and *exact*) solution of nonlinear diffusion Eq. (2):

$$J(z,t)=\left[\frac{(n-1)\mu_0k^n(z_0-z)^2}{2(n+1)n(t_0-t)}\right]^{1/n-1}. \tag{25}$$

It is remarkable that, as a result of substitution, the constant λ cancels out, and the solution (25) does not depend on λ at all.

The obtained solution (25) can be physically interpreted as follows. Suppose that at time $t=0$ the electric current density satisfies the following initial condition:

$$J(z,0)=\begin{cases}\left[\dfrac{(n-1)\mu_0k^n(z_0-z)^2}{2(n+1)nt_0}\right]^{1/n-1}, & \text{if } 0\leqslant z\leqslant z_0\\ 0, \quad \text{if } 0\leqslant z\leqslant z_0\end{cases}. \tag{26}$$

Suppose also that the current density satisfies the following boundary condition during the time interval $0\leqslant t<t_0$:

$$J_0(t)=J(0,t)=\left[\frac{(n-1)\mu_0k^nz_0^2}{2(n+1)n(t_0-t)}\right]^{1/n-1} \tag{27}$$

Then, according to (25), the exact solution to the initial-boundary value problem (26)–(27) for the nonlinear diffusion Eq. (2) can be written as follows:

$$J(z,t)=\begin{cases}\left[\dfrac{(n-1)\mu_0k^n(z_0-z)^2}{2(n+1)n(t_0-t)}\right]^{1/n-1}, & \text{if } 0\leqslant z\leqslant z_0\\ 0, \quad \text{if } z>z_0.\end{cases} \tag{28}$$

This solution is illustrated by Fig. 3 and it is apparent that it has the physical meaning of the standing mode. It is also clear from formula (28) (and Fig. 3) that the above solution has the following self-similarity property: the profiles of electric current density for different instants of time can be obtained from one another by dilation (or contraction) along the *J*-axis. In other words, these profiles remain similar to one another. This suggests that solution (28) can be derived by using dimensional analysis. However, we shall not delve further into this matter.

From the practical point of view, it is desirable to express the boundary condition (27) for the standing mode in terms of magnetic field $H_0(t)$ on the superconductor boundary. According to Ampere's Law, we have:

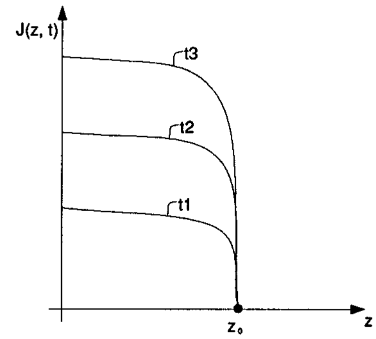

FIG. 3. Current density profiles corresponding to the exact analytical solution of the diffusion equation.

$$H_0(t)=\int_0^{z_0}J(z,t)dz. \tag{29}$$

By substituting (28) into (29) and performing the integration, we obtain:

$$H_0(t)=\frac{n-1}{n+1}\,z_0\left[\frac{(n-1)\mu_0k^nz_0^2}{2(n+1)n(t_0-t)}\right]^{1/n-1}. \tag{30}$$

It is also instructive to compare the above exact standing mode solution with the standing mode expressions derived on the basis of the rectangular profile approximation. First, it is clear from formula (28) (and Fig. 3) that, for sufficiently large "n", the actual current density profiles for the standing mode are almost rectangular. Second, it is apparent that the boundary condition (27) can be written in the form (9) with "c" and "t_0" defined as follows:

$$c=\left[\frac{(n-1)\mu_0k^nz_0^2}{2(n+1)n}\right]^{1/n-1},$$

$$t_0=\frac{(n-1)\mu_0k^nz_0^2J_0^{1-n}(0)}{2(n+1)n}. \tag{31}$$

By comparing (31) with (10), it can be concluded that for sufficiently large "n" these expressions are practically identical. Thus, the rectangular profile approximation is fairly accurate as far as the prediction of the standing mode diffusion is concerned.

The origin of the standing mode can be elucidated on physical grounds as follows. Under the boundary condition (27), the electromagnetic energy entering the superconducting material at any instant of time is just enough to affect the almost uniform increase in electric current density in the region ($0\leqslant z\leqslant z_0$) already occupied by the field, but insufficient to affect the further diffusion of the field into the material.

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