Nonlinear diffusion in anisotropic superconductors

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This article presents an analytical study of nonlinear diffusion of electromagnetic fields in anisotropic superconducting media. The case of anisotropic media is treated as a perturbation of isotropic media and analytical expressions for nonlinear diffusion of circularly polarized electromagnetic fields are derived.
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Nonlinear diffusion of electromagnetic fields in superconductors has been a topic of increasing interest lately. However, only the case of **isotropic** superconducting media has been discussed. In this article, an attempt is made to study nonlinear diffusion of circularly polarized electromagnetic fields in **anisotropic** media. This problem for isotropic media was solved in Ref. 1 in the case of ideal resistive transitions and in Ref. 2 for gradual resistive transitions described by the ''power law.'' The power law has been observed in numerous experiments, and it has been extensively used in recent studies of nonlinear diffusion of electromagnetic fields in superconductors albeit only for linear polarization of electric field (see, for instance, Ref. 3 and references therein).

In our discussions, the following constitutive relations for anisotropic superconducting media with gradual resistive transitions will be used:

$$J_x(E_x,E_y)=(1+\epsilon)kE_x\left(\sqrt{(1+\epsilon)E_x^2+(1-\epsilon)E_y^2}\right)^{1/n-1}, \tag{1}$$

$$J_y(E_x,E_y)=(1-\epsilon)kE_y\left(\sqrt{(1+\epsilon)E_x^2+(1-\epsilon)E_y^2}\right)^{1/n-1}, \tag{2}$$

where k is a parameter that coordinates the dimensions of both sides in Eqs. (1) and (2), while ϵ is some relatively small parameter which accounts for the anisotropicity of the media. It is clear that the properties of superconductor enter into Eqs. (1) and (2) through parameters n, ϵ, and k.

In the limiting case of $\epsilon=0$, expressions (1) and (2) are reduced to

$$J_x^{(0)}(E_x,E_y)=kE_x\left(\sqrt{E_x^2+E_y^2}\right)^{1/n-1}=kE^{1/n-1}E_x, \tag{3}$$

$$J_y^{(0)}(E_x,E_y)=kE_y\left(\sqrt{E_x^2+E_y^2}\right)^{1/n-1}=kE^{1/n-1}E_y, \tag{4}$$

which are constitutive relations for isotropic superconducting media with gradual resistive transitions described by the power law: $E=(J/k)^n$, $(n>1)$.

Thus, the anisotropic media with constitutive relations (1) and (2) can be mathematically treated as perturbations of isotropic media described by the power law. This suggests that the perturbation technique can be very instrumental in the mathematical analysis of nonlinear diffusion in anisotropic media with constitutive relations (1) and (2). In the limiting case of $n=\infty$, expressions (1) and (2) describe ideal (''sharp'') resistive transitions with critical currents $J_x^c=(1+\epsilon)k$ and $J_y^c=(1-\epsilon)k$. It is also important to note that the Jacobian matrix for $\mathbf{J}(\mathbf{E})$ defined by Eqs. (1) and (2) is symmetric. This guarantees the absence of local cyclic (hysteretic-type) losses.

Now, consider a plane circularly polarized electromagnetic wave penetrating superconducting half-space $z>0$. The magnetic field on the boundary of this half space is specified as follows:

$$H_x(0,t)=H_m\cos(\omega t+\gamma),\quad H_y(0,t)=H_m\sin(\omega t+\gamma). \tag{5}$$

By using the Maxwell equations, it is easy to find that the distribution of electric field in half-space $z>0$ satisfies the following coupled nonlinear partial differential equations:

$$\frac{\partial^2 E_x}{\partial z^2}=\mu_0\frac{\partial J_x(E_x,E_y)}{\partial t},\quad \frac{\partial^2 E_y}{\partial z^2}=\mu_0\frac{\partial J_y(E_x,E_y)}{\partial t} \tag{6}$$

subject to the boundary conditions

$$\begin{aligned}&\frac{\partial E_x}{\partial z}(0,t)=\mu_0\omega H_m\cos(\omega t+\gamma),\\ &\frac{\partial E_y}{\partial z}(0,t)=-\mu_0\omega H_m\sin(\omega t+\gamma),\end{aligned} \tag{7}$$

$$E_x(\infty)=E_y(\infty)=0. \tag{8}$$

Next, by using the perturbation technique, we shall look for the solution of the boundary value problem (6)–(8) in the form

$$E_x(z,t)=E_x^0(z,t)+\epsilon e_x(z,t), \tag{9}$$

$$E_y(z,t)=E_y^0(z,t)+\epsilon e_y(z,t), \tag{10}$$

We shall also use the following ϵ-expansions for constitutive relations (1) and (2):

$$J_x(E_x,E_y)=J_x^0(E_x,E_y)+\epsilon J_x^0(E_x,E_y)\left[1+\frac{1-n}{2n}\cdot\frac{E_x^2-E_y^2}{E^2}\right]+\cdots, \tag{11}$$

$$J_y(E_x,E_y)=J_y^0(E_x,E_y)-\epsilon J_y^0(E_x,E_y)\left[1-\frac{1-n}{2n}\cdot\frac{E_x^2-E_y^2}{E^2}\right]\cdots, \tag{12}$$

where $J_x^0(E_x,E_y)$ and $J_y^0(E_x,E_y)$ are defined by expressions (3) and (4), respectively, while $E=\sqrt{E_x^2+E_y^2}$. By substituting

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expressions (9)–(12) into Eqs. (6) and boundary conditions (7) and (8), and equating the terms of like powers of ϵ, we end up with the following boundary value problems for E_x^0, E_y^0 and e_x, e_y:

$$\frac{\partial^2 E_x^0}{\partial z^2} = \mu_0 \frac{\partial J_x^0(E_x^0, E_y^0)}{\partial t}, \quad \frac{\partial^2 E_y^0}{\partial z^2} = \mu_0 \frac{\partial J_y^0(E_x^0, E_y^0)}{\partial t}, \tag{13}$$

$$\frac{\partial E_x^0}{\partial z}(0,t) = \omega \mu_0 H_m \cos(\omega t + \gamma),$$

$$\frac{\partial E_y^0}{\partial z} = -\omega \mu_0 H_m \sin(\omega t + \gamma), \tag{14}$$

$$E_x^0(\infty) = E_y^0(\infty) = 0 \tag{15}$$

and

$$\frac{\partial^2 e_x}{\partial z^2} - \mu_0 \frac{\partial}{\partial t}\left(\frac{\partial J_x^0}{\partial E_x}(E_x^0, E_y^0)e_x + \frac{\partial J_x^0}{\partial Ey}(E_x^0, E_y^0)e_y\right)$$
$$= \mu_0 \frac{\partial}{\partial t}\left[J_x^0(E_x^0, E_y^0)\left(1 + \frac{1-n}{2n} \cdot \frac{(E_x^0)^2 - (E_y^0)^2}{(E^0)^2}\right)\right], \tag{16}$$

$$\frac{\partial^2 e_y}{\partial z^2} - \mu_0 \frac{\partial}{\partial t}\left(\frac{\partial J_y^0}{\partial E_x}(E_x^0, E_y^0)e_x + \frac{\partial J_y^0}{\partial E_y}(E_x^0, E_y^0)e_y\right)$$
$$= -\mu_0 \frac{\partial}{\partial t}\left[J_y^0(E_x^0, E_y^0)\left(1 - \frac{1-n}{2n} \cdot \frac{(E_x^0)^2 - (E_y^0)^2}{(E^0)^2}\right)\right], \tag{17}$$

$$\frac{\partial e_x}{\partial z}(0,t) = \frac{\partial e_y}{\partial z}(0,t) = 0, \quad e_x(\infty, t) = e_y(\infty, t) = 0. \tag{18}$$

The boundary value problem (13)–(15) describes the penetration of circularly polarized electromagnetic wave in isotropic superconducting half-space $z > 0$. The solution to this problem has been found in Ref. 2. For the case when the initial phase γ in Eq. (11) is such that the initial phase of $\mathbf{E}^0$ on the boundary ($z = 0$) is equal to zero, this solution can be written as follows:

$$E_x^0(z,t) = E_m\left(1 - \frac{z}{z_0}\right)^{\alpha'} \cos[\omega t + \theta(z)], \tag{19}$$

$$E_y^0(z,t) = E_m\left(1 - \frac{z}{z_0}\right)^{\alpha'} \sin[\omega t + \theta(z)], \tag{20}$$

$$z_0 = \frac{\sqrt[4]{2n(n+1)(3n+1)^2}}{(n-1)\sqrt{\omega \mu_0 \sigma_m}}, \quad \sigma_m = k E_m^{1/n-1}, \tag{21}$$

$$\theta(z) = \alpha'' \ln\left(1 - \frac{z}{z_0}\right), \tag{22}$$

$$\alpha' = \frac{2n}{n-1}, \quad \alpha'' = \frac{\sqrt{2n(n+1)}}{n-1} \tag{23}$$

and E_m is determined from the equation

$$H_m = \frac{|\alpha' + i\alpha''|}{\omega \mu_0 z_0} E_m. \tag{24}$$

By substituting (19) and (20) into Eqs. (16) and (17) and by using expressions (3) and (4), after straightforward but somewhat lengthy transformations we derive the following equations for e_z and e_y:

$$\frac{\partial^2 e_x}{\partial z^2} - \mu_0 \sigma_m\left(1 - \frac{z}{z_0}\right)^{-2} \frac{\partial}{\partial t}\left[e_x\left(\frac{1+n}{2n} + \frac{1-n}{2n} \times \cos 2[\omega t + \theta(z)]\right) + e_y \frac{1-n}{2n} \sin 2[\omega t + \theta(z)]\right]$$
$$= \mu_0 \sigma_m E_m\left(1 - \frac{z}{z_0}\right)^{2/(n-1)} \frac{\partial}{\partial t}\left\{\frac{3n+1}{4n} \cos[\omega t + \theta(z)] + \frac{1-n}{4n} \cos 3[\omega t + \theta(z)]\right\}, \tag{25}$$

$$\frac{\partial^2 e_y}{\partial z^2} - \mu_0 \sigma_m\left(1 - \frac{z}{z_0}\right)^{-2} \frac{\partial}{\partial t}\left[e_x \frac{1-n}{2n} \sin 2[\omega t + \theta(z)] + e_y\left(\frac{1+n}{2n} - \frac{1-n}{2n} \cos 2[\omega t + \theta(z)]\right)\right]$$
$$= -\mu_0 \sigma_m E_m\left(1 - \frac{z}{z_0}\right)^{2/(n-1)} \frac{\partial}{\partial t}\left(\frac{3n+1}{4n} \sin[\omega t + \theta(z)] - \frac{1-n}{4n} \sin 3[\omega t + \theta(z)]\right). \tag{26}$$

To simplify the above equations, we introduced new state variables

$$\varphi(z,t) = e_x(z,t) + i e_y(z,t), \tag{27}$$

$$\psi(z,t) = e_x(z,t) - i e_y(z,t). \tag{28}$$

By looking for the solution in terms of Fourier series

$$\varphi(z,t) = \sum_{k=-\infty}^{\infty} \varphi_{2k+1}(z) e^{i(2k+1)\omega t}, \tag{29}$$

$$\psi(z,t) = \sum_{k=-\infty}^{\infty} \psi_{2k+1}(z) e^{i(2k+1)\omega t} \tag{30}$$

it can be shown that only φ_3, φ_{-1}, ψ_1, and ψ_{-3} are not equal to zero. For φ_3 and ψ_1 the following coupled ordinary differential equations (ODEs) can be derived:

$$\left(1 - \frac{z}{z_0}\right)^2 \frac{d^2 \varphi_3}{dz^2} - i\chi_3\left[a\varphi_3 + \left(1 - \frac{z}{z_0}\right)^{i2\alpha''} \psi_1\right]$$
$$= i\zeta_3 E_m\left(1 - \frac{z}{z_0}\right)^{2n/(n-1) + i3\alpha''} \tag{31}$$

$$\left(1 - \frac{z}{z_0}\right)^2 \frac{d^2 \psi_1}{dz^2} - i\chi_1\left[a\psi_1 + \left(1 - \frac{z}{z_0}\right)^{-i2\alpha''} \varphi_3\right]$$
$$= i\nu_1 E_m\left(1 - \frac{z}{z_0}\right)^{2n/(n-1) + i\alpha''}, \tag{32}$$

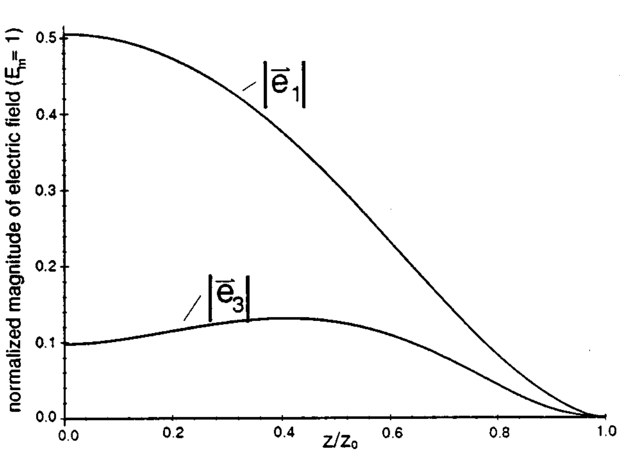

FIG. 1. Magnitude of perturbations $\mathbf{e}_1(z/z_0)$ and $\mathbf{e}_3(z/z_0)$ at $n=\infty$.

where

$$\chi_{2k+1}=(2k+1)\omega\mu_0\sigma_m\,\frac{1-n}{2n},\quad a=\frac{1+n}{1-n},\tag{33}$$

$$\zeta_3=3\omega\mu_0\sigma_m\,\frac{1-n}{4-n},\quad \nu_1=\omega\mu_0\sigma_m\,\frac{3n+1}{4n}.\tag{34}$$

The solution of Eqs. (31) and (32) should be subject to the boundary conditions

$$\frac{d\varphi_3}{dz}(0)=\frac{d\psi_1}{dz}(0)=0,\quad \varphi_3(\infty)=\psi_1(\infty)=0.\tag{35}$$

Similar ODEs can be derived for φ_{-1} and ψ_{-3}. However, this can be avoided because φ_{-1} and ψ_1 as well as φ_{-3} and φ_3 are complex conjugate.

The particular solution of ODEs (31) and (32) has the form

$$\varphi_3^{(p)}(z)=C_3\left(1-\frac{z}{z_0}\right)^{\lambda_3},\quad \psi_1^{(p)}=C_1\left(1-\frac{z}{z_0}\right)^{\lambda_1},\tag{36}$$

where

$$\lambda_3=\frac{2n}{n-1}+i3\alpha'',\quad \lambda_1=\frac{2n}{n-1}+i\alpha''.\tag{37}$$

Coefficients C_3 and C_1 satisfy the following simultaneous equations:

$$[\lambda_3(\lambda_3-1)-\chi_3az_0^2]C_3-i\chi_3z_0^2C_1=i\zeta_3z_0^2E_m,\tag{38}$$

$$-i\chi_1z_0^2C_3+[\lambda_1(\lambda_1-1)-i\chi_1az_0^2]C_1=i\nu_1z_0^2E_m.\tag{39}$$

It is clear from Eqs. (21), and (33), (34), and (37) that the coefficients in Eqs. (38), and (39) depend only on n. This opens the opportunity to compute the ratios C_1/E_m and C_3/E_m as functions of n.

It can be shown[2] that the solution of homogeneous ODEs corresponding to Eqs. (31) and (32) has the form

$$\varphi_3^{(h)}(z)=A\left(1-\frac{z}{z_0}\right)^{\beta},\quad \psi_1^{(h)}(z)=B\left(1-\frac{z}{z_0}\right)^{\beta-i2\alpha''},\tag{40}$$

where β is the solution of the following characteristic equation:

$$(\beta^2-\beta-i\chi_3az_0^2)[(\beta-i2\alpha'')^2-(\beta-i2\alpha'')-i\chi_1az_0^2]+\chi_3\chi_1z_0^4=0.\tag{41}$$

It can be shown that the above characteristic equation has two roots, β_1 and β_2, with positive real parts. By using these roots and expressions (36) and (40), the solution of Eqs. (31), and (32) can be written as follows:

$$\varphi_3(z)=A_1\left(1-\frac{z}{z_0}\right)^{\beta_1}+A_2\left(1-\frac{z}{z_0}\right)^{\beta_2}+C_2\left(1-\frac{z}{z_0}\right)^{\lambda_3},\tag{42}$$

$$\psi_1(z)=B_1\left(1-\frac{z}{z_0}\right)^{\beta_1-i2\alpha''}+B_2\left(1-\frac{z}{z_0}\right)^{\beta_2-i2\alpha''}+C_1\left(1-\frac{z}{z_0}\right)^{\lambda_1}.\tag{43}$$

The unknown coefficients A_2, A_2, B_1, and B_2 can be found from the boundary conditions (35) at $z=0$ and from the fact that expressions (40) should satisfy homogeneous ODEs corresponding to Eqs. (31) and (32). This yields the following simultaneous equations for the above coefficients:

$$\beta_1A_1+\beta_2A_2=-\lambda_3C_3,\tag{44}$$

$$(\beta_1-i2\alpha'')B_1+(\beta_2-i2\alpha'')B_2=-\lambda_1C_1,\tag{45}$$

$$(\beta_1^2-\beta_1-i\chi_3az_0^2)A_1-i\chi_3z_0^2B_1=0,\tag{46}$$

$$(\beta_2^2-\beta_2-i\chi_3az_0^2)A_2-i\chi_3z_0^2B_2=0.\tag{47}$$

Again, it is easy to see that the coefficients of characteristic Eq. (41) as well as the coefficients of simultaneous Eqs. (44)–(47) depend only on n. This allows one to compute the roots β_1 and β_2 as well as the ratios A_1/E_m, A_2/E_m, B_1/E_m, and B_2/E_m as functions of n. In the limiting case of $n=\infty$ (ideal resistive transition-critical state model), one can compute specific numerical values of the above quantities. These values are as follows: $\beta_1=2+i\sqrt{2}$, $\beta_2=1.921+i3.699$, $C_1/E_m=\frac{3}{2}-i9\sqrt{2}/16$, $C_3/E_m=i9\sqrt{2}/16$, $A_1/E_m=-0.129+i0.116$, $A_2/E_m=0.071-i0.990$, $B_1/E_m=-0.043+i0.039$, $B_2/E_m=-1.899+i0.513$. By using these values, all desired quantities can be found. For instance, the magnitudes of the first and third harmonics $\mathbf{e}_1$ and $\mathbf{e}_3$ of the perturbation can be computed as the functions of z. The results of these computations are shown in Fig. 1. For gradual resistive transitions (finite n), the roots β_1 and β_2, as well as all the mentioned coefficients, have been computed as functions of n. The lack of space prohibits us from presenting these computations.

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