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Equivalent Models and Analysis for Multi-Stage Tree Networks of Deterministic Service Time Queues

Michael J. Neely
MIT--LIDS
mjneely@mit.edu

Charles E. Rohrs
Tellabs Research Center and MIT
crohrs@trc.tellabs.com

***Abstract* -- Tree networks of single server, deterministic service time queues are often used as models for packet flow in data communication systems with Asynchronous Transfer Mode (ATM) traffic. In this paper, we present a method for analyzing packet occupancy in these systems without making any assumptions about the nature of the underlying input processes. We demonstrate how analysis of these multi-stage tree systems can be reduced to the analysis of a much simpler 2-stage equivalent model. We also develop an expression for first moments of queue occupancy in terms of first moments of a simple 1-stage equivalent model. From this, we observe an interesting phenomenon for general types of *distributable inputs*: Expected occupancy at any interior queue within a multi-stage tree network is a concave function of the multiple exogenous input rates. Expected occupancies in nodes on the edge of the network are shown to be convex.**

I. Introduction

In this paper we analyze the packet occupancy in multi-input, single output tree systems of deterministic service time queues (Fig. 1). Such systems are often used as models for packet flow in networks with ATM traffic, where packets are broken into fixed bit-length *cells*. These cells have a service time proportional to their bit-length. Assuming that the processing speed at each node is the same, the cells thus have a deterministic service time T in each node. In this paper, we shall continue to use the term "packets" rather than "cells," keeping in mind that, unless otherwise stated, the packets are fixed in length and have deterministic service time T.

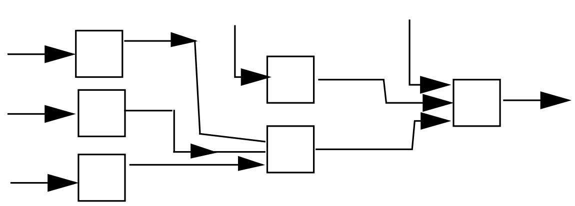

Figure 1: A multi-stage tree network of queues.

Much of the previous work in this area has concentrated on tandem chains of queues and memoryless inputs ([1]-[4]). In [5], an average queue length analysis was performed for a more general tree network using the properties of memoryless inputs. Their approach relied on a lemma from [6] that used combinatorial analysis. The approach here is applicable to more general input processes and the analysis here is self-contained and applicable to more than the mean queue length. We also work in continuous time, which enables exact analysis even when arrival streams are asynchronous.

We first examine a simple case of such a system: a 2-stage, 2-queue system with arbitrary exogenous packet arrival processes at both the first and second stages. We develop a simpler, one-queue equivalent model of this system and prove that the input-output behavior of the original system is preserved in this equivalent model. This fact was presented previously in [1] and used to analyze waiting times in tandem chains of queues with memoryless inputs. The approach used in this paper was developed independently and differs from that used in [1] as the proof presented here emphasizes the equivalence on every sample path.

We extend the use of this idea to analyze multi-stage tree systems with arbitrary input processes. We decompose the large system into "atomic" blocks of 2-stage, 2-node subsystems, and then use our equivalent model iteratively on each sub-system. The result is a simplified 2-stage system in tandem with a series of time delays or *observation windows*.

If the various input processes are all independent and stationary, the time delays can be ignored, and exact analysis of packet occupancy in a multi-stage network can be reduced to the analysis of a 2-stage equivalent model.

These models considerably reduce the complexity of the original network. They can be analyzed quite generally, and often occupancy characteristics such as means, variances, and aggregate distribution functions can be found (see [7, 8]).

Here, we use this equivalent model to develop an expression for mean occupancy at any node in a multi-stage tree system in terms of the mean occupancy of a single-stage (one queue) system whose inputs are a superposition of the original exogenous inputs. As an example, we develop explicit expressions for the average occupancy in any node when the exogenous inputs are (i) memoryless, and (ii) periodic with independent phases.

Finally, we explore a concavity phenomenon of expected occupancy. For a general class of *distributable inputs*, we show that the average queue length of any interior node in a multi-stage, deterministic service time tree system is a concave function of the exogenous input rates. Expected occupancies in nodes at the edge of the network are convex. This provides insight into buffering requirements and rate assignments in ATM tree networks. Deterministic service time queues at earlier stages smooth traffic for downstream nodes. Unbalanced loadings give the previous stage queues the best situation for smoothing the input process for the next stage.

II. THE 2-STAGE, 2-NODE SYSTEM

Here we analyze the simplest non-trivial example of a multi-stage tree system: the 2-stage, 2-node system of Fig. 2a. Exogenous input streams with arbitrary arrival patterns enter the system both at node 1 and at node 2.

In the sequel, we show that if service times of all packets in nodes 1 and 2 are deterministic with length T, then the system can be modeled as a much simpler 1-queue system in tandem with a pure delay or *observation window.* Under identical sets of inputs, the original system and its equivalent model simultaneously produce the same outputs for all time. This equivalence relationship was discovered independently and previously in [1]. Here we state and prove the theorem with greater detail and generality using sample path arguments. This sample path theory provides a means for understanding and analyzing the larger networks addressed in Section III and beyond.

Consider the system in Fig. 2a where inputs x are sent through an initial queuing stage with deterministic service time T. The departures from this first stage are then added to the external inputs y, and the superposition is sent through the queuing system S_{A2}. The nature of the x, y input arrivals is arbitrary.

Consider the same sequences of packets, x and y, as inputs to System B as shown in Fig. 2b. The queueing nodes S_{A2} and S_{B2} are identical. However, the first stage node that x packets pass through in System A is now replaced in System B by an observation window.

Description of Observation Window: Note that the "observation window" is a passive device that simply observes the number of customers which travel through it during a time interval T. Inputs x enter this window, pass through it, and exit after exactly T seconds without having their relative positions distorted in any way. Hence, the input to S_{B2} coming from the window is simply a time-delayed version of x (Fig. 3). Such a device can also be viewed as a $*/D/\infty$ queue.

The theorem below describes the equivalence of Systems A and B, assuming that both start out completely empty at time t=0.

Theorem 1 (Equivalence Theorem): If the service times within S_{A2} and S_{B2} of customers entering from the x input line are greater than or equal to T, then:

i) The final stage systems S_{A2} and S_{B2} have identical busy and idle periods.

ii) The entire system A (S_{A2} plus the first stage) is empty if and only if the entire system B is empty (S_{B2} plus the observation window).

These two statements are true even if packets from y have arbitrary service times. If, in addition, the service times within S_{A2} and S_{B2}

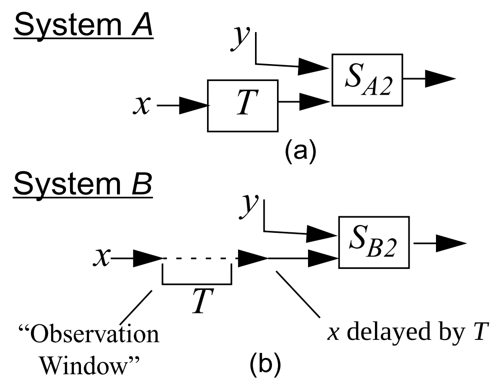

Figure 2: A 2-stage system and its equivalent model. Sub-system S_{A2} is identical to S_{B2}.

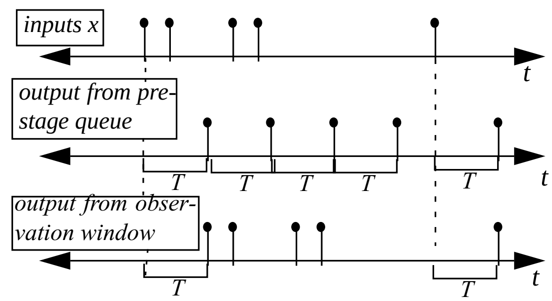

Figure 3: An example timing diagram of inputs x and the corresponding outputs in both the System A pre-stage queue and the System B observation window.

of packets from both input lines x and y are deterministically $\tilde{T}$, where $\tilde{T} \geq T$, then:

iii) The accumulated number of departures from System A and System B is the same at every instant of time, and hence the entire 2-stage System A always has exactly the same number of customers within it as the entire 2-stage System B.

Caveat: Note here that the actual ordering of the packets served may be shuffled in the equivalent model--it is only the departure time epochs that must remain the same for both systems.

Proof of Theorem 1: We start out at time t_1, and suppose that at this time:

1. The first stages of both systems are empty.
2. Up to this point, all time epochs marking the busy and idle periods of the second stage systems S_{A2} and S_{B2} have been identical,
3. The amount of unfinished work left in the final stages of both systems is identical.

We call the event that these three conditions are satisfied simultaneously a *simultaneous renewal event.* Such an event occurs at time 0 when there have been no outputs and both systems are empty. Whether or not another renewal occurs is irrelevant: we simply prove our theorem over the entire (possibly infinite) duration between simultaneous renewal events.

Proof of (i): We start out at a simultaneous renewal time t_1, assuming that S_{A2} and S_{B2} have had identical busy/idle periods up to and including this time. Tracing the timeline of events shown in Fig. 4, we show the busy/idle periods remain synchronized until the next renewal. The next event in the timeline occurs at time t_2 when a busy period of the first stage of System A begins. This busy period consists of n packets $\{x_1, x_2, \ldots, x_n\}$ spaced closely enough together so that the first stage queue in System A does not empty until time t_3 when the n^{th} packet x_n exits the first stage and moves into system S_{A2} (clearly, $t_3 = t_2 + nT$). The packet x_1 that initiates this busy period travels through the first stages of Systems A and B in exactly the same manner, and enters the second stages of both systems at time t_2+T. Since the x, y inputs to systems S_{A2} and S_{B2} are identical during the time interval [t_1, t_2+T], we know that throughout this interval the amount of unfinished work in both S_{A2} and S_{B2} is the same as are the busy and idle periods of S_{A2} and S_{B2}.

Since the service time of x packets in system S_{A2} is at least as long as the service time T in the front end system that feeds into it, system S_{A2} cannot empty before time t_3.

Notice now that because there is no queueing in the observation window, the work delivered to S_{B2} during the interval $[t_2 + T, \tau]$ is greater than or equal to the work delivered to S_{A2} for all $\tau \in [t_2 + T, t_3]$. Thus, after time

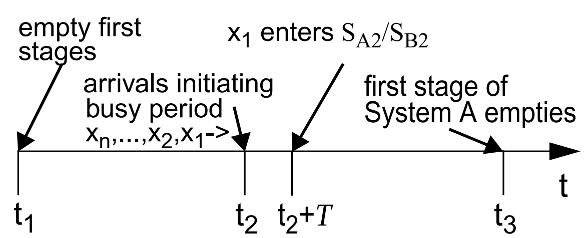

Figure 4: Events between renewal times t_1 and t_3

t_2+T, S_{B2} cannot empty before S_{A2}, and hence both S_{A2} and S_{B2} are busy on the interval $[t_2+T, t_3]$. By time t_3, both S_{A2} and S_{B2} have had the same total work delivered to them and both have had the same busy/idle periods. Thus, they both have exactly the same unfinished work at time t_3. Finally, both first stages are empty at t_3, so there is a renewal event at t_3, and (i) is proved. ❑

Proof of (ii): Suppose System B is empty. Thus, S_{B2} is empty, and hence S_{A2} must be empty by statement (i). Since the observation window of System B is empty, no packets have entered from the x input line for at least T seconds. Therefore, no packets have entered the first stage of System A for at least T seconds. Thus, if this first stage now contains packets, then these packets must have been in the system T seconds earlier. But if this is the case, then the packet that was in service at the first stage of System A at that time would have been sent to the second stage S_{A2}, and it would now be in the system S_{A2}. (Recall that service times of x packets in S_{A2} and S_{B2} are no shorter than T seconds). This contradicts the fact that S_{A2} is empty. Hence, the first and second stages of System A are empty.

The converse implication follows from a similar argument, and (ii) is proved. ❑

Notice that up to this point, we have only assumed that packets from the x input stream have a deterministic service time T within the first queuing stage of System A, and that their service times within the second stage systems S_{A2} and S_{B2} are at least as large as T. Thus, statements (i) and (ii) hold even if service times of x packets within S_{A2} and S_{B2} are variable with a minimum time of T seconds. Furthermore, the packets from the y input stream can have any type of arbitrarily distributed service times as long as they are the same for both Systems A and B. However, to prove statement (iii) we use the assumption that all packets (from both the x and y streams) have a fixed service time $\tilde{T} \geq T$ within S_{A2} and S_{B2}.

Proof of (iii): If the service times of all packets in S_{A2} and S_{B2} are identically $\tilde{T}$ seconds, then all packets are homogeneous and any rearrangement of packet orderings is irrelevant to the output processes of S_{A2} and S_{B2}. Since the final stages of Systems A and B start and stop serving packets at exactly the same times (statement (i)), all individual packet output epochs are the same and the two systems have exactly the same number of packets within them at every instant of time. ❑

III. Multi-stage tree reduction

Here we use Theorem 1 to show that all multi-stage tree systems with deterministic service times can be reduced to two-stage equivalent models. Our method is to decompose the complex tree system into its "atomic" 2-stage, 2-node sub-systems, and then to apply the equivalent model result of Theorem 1 to these sub-systems.

Consider the multi-stage system of Fig. 5, which consists of nodes {1, 2, 3, 6} on the edge of the network, and nodes {4, 5, 7, 8} in the interior. Each node has an exogenous input source. Interior nodes additionally have endogenous inputs coming from previous stages.

We represent the exogenous source inputs by their normalized rates $\rho_i = \lambda_i T$ where λ_i is the arrival rate at node i and T is the service time common to all queues. By Little's Theorem ρ_i can be thought of as the loading contribution from source i. For stability, we assume that $\sum \rho_i < 1$.

Construction of an Equivalent System: Suppose we wish to understand the packet dynamics in node 8 of Fig. 5. We can reduce the complexity of the problem by applying Theorem 1 to all nodes on the edge of the network. For example, consider edge node 1 and interior node 4 as a 2-node sub-unit of the larger network. Treating the exogenous input ρ_1 as the "x" stream from Theorem 1, and the remaining inputs to node 4 as a collective "y" stream, we replace node 1 with an observation window (Fig. 6). This creates an equivalent system in which the queueing dynamics of all nodes beyond node 4 are unchanged.

Theorem 2 (Multi-Stage Tree Reduction): Suppose all queues in a multi-stage tree have deterministic service time T, and all exoge-

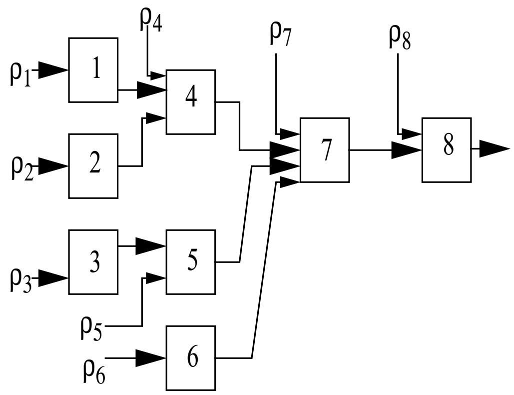

Figure 5: A 4-layer tree network.

nous input lines are stationary and independent of one another. Then, the steady-state queue length distribution for an arbitrary node of the tree is identical to the distribution in an equivalent 2-stage model.

The equivalent model is formed by removing all nodes more than 1-stage beyond the node in question, and placing their corresponding exogenous inputs directly into the remaining queues.

<u>Example</u>: Consider node 7 of Fig. 5. The equivalent 2-stage system model is given in Fig. 7.

<u>Proof of Theorem 2</u>: Consider a single node within a multi-stage tree. In order to ensure equivalent representation of this node in our simpler model, we isolate this node and keep the input processes on each line feeding into it the same. The endogenous inputs to the node in question are the departure processes from the previous stage nodes. Using Theorem 1 iteratively, we find that these departure processes are unchanged when we use observation windows to replace all nodes which are more than one stage behind the node in question. An example is shown in Fig. 6. Now, by the stationarity and independence of the lines feeding into the pre-stages, the time delays introduced by the observation windows do not effect the arrival statistics. Therefore, we can ignore all observation windows and treat these inputs as if they were directly applied to the corresponding queues of the 2-stage model. ❑

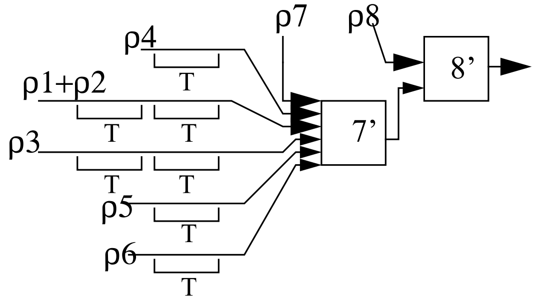

Figure 6: Using Theorem 1 iterativley to obtain an equivalent system for node 8 of Fig. 5.

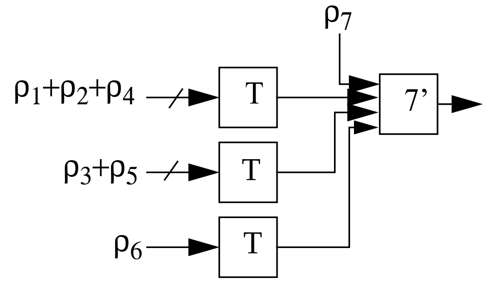

Figure 7: An equivalent model for node 7 of Fig. 5.

Theorem 2 demonstrates that solving for the occupancy statistics in a general 2-stage system (shown in Fig. 8a) provides a solution that characterizes arbitrary multi-stage configurations. For this reason, it suffices to restrict our attention to 2-stage systems.

Note that our Theorem 1 is general enough to cover the case when the deterministic service times in the downstream nodes are not identical, but are the monotonically increasing (i.e., $T_1 \le T_2 \le \ldots \le T_n$). Considering this case, we find that all of our analysis works in the exact same manner, and hence our reduction result of Theorem 2 also holds for the case when service times are monotonically increasing with each stage. However, our equivalence model fails when the deterministic service times $\{T_i\}$ decrease at some stage. Finding a good model to handle this scenario remains an open problem.

IV. Expected Occupancy

Theorems 1 and 2 enable exact analysis of tree networks with general types of input sources, and can be used to find queueing means, variances, and smoothing properties (see [7,8]). Here, we illustrate a simple method of calculating the expected number of packets in any node of a multi-stage tree. We assume that all service times are deterministically T seconds, and that the exogenous arrivals are independent and stationary. Using Theorem 2, we can reduce any such multi-stage tree system to a general 2-stage equivalent model with G nodes at the first stage, as shown in Fig. 8a. The input processes $\{x_i\}$ feeding into the first stage nodes of Fig. 8a are composed of superpositions of the original exogenous inputs.

Let λ_i be the arrival rate from process x_i, and let $\rho_i = \lambda_i T$ be the normalized rate. Let us suppose that the mean occupancy for a single stage queue with a superposition of these inputs can be determined. Specifically, let $Q(\rho_0 + \rho_1 + \ldots + \rho_G)$ represent the expected number of packets in a single-stage */D/1 system with inputs $x_0, x_1, \ldots x_G$. Notice that the Q function here is not simply a function of the sum total of loadings $\rho_0 + \rho_1 + \ldots + \rho_n$, but it is also a function of the actual types of processes that generate these loadings. To be more precise, we could write $Q_{X0, X1, \ldots, XG}(\rho_0 + \ldots + \rho_G)$. However, we will continue to use the more compact notation above, keeping in mind that

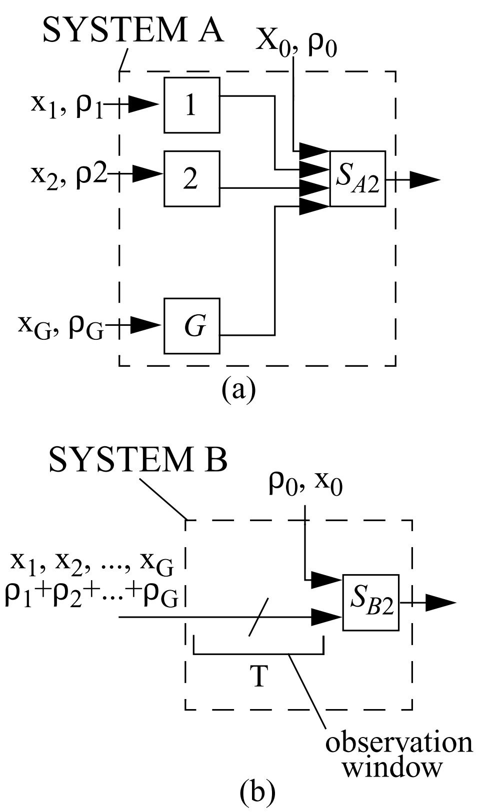

Figure 8: The general 2-stage network and its equivalent model.

there are G+1 underlying input processes.

Also define $\rho = \rho_0 + \rho_1 + \ldots + \rho_G$ to be the total normalized rate into the final stage queue. By Little's Theorem, ρ also represents the total loading on the final queue, i.e., the probability that the server in the final node is busy. We make the stability assumption $\rho < 1$.

Using Theorem 1 iteratively on System A of Fig. 8a, we obtain its equivalent model. This model is illustrated as System B in Fig. 8b. Observing the diagrams, we have:

$$E(\text{\# packets in System } A) = Q(\rho_1) + \ldots + Q(\rho_G) + E(\text{\# in } S_{A2}) \quad (1)$$

$$E(\text{\# packets System } B) = E(\text{\# in window}) + Q(\rho_0 + \rho_1 + \ldots + \rho_G) = (\rho - \rho_0) + Q(\rho_0 + \rho_1 + \ldots + \rho_G) \quad (2)$$

Statement (iii) of Theorem 1 indicates that the total number of packets in both Systems A and B will be the same at every instant of time. We can thus equate (1) and (2), which leads to:

$$E(\text{\# in } S_{A2}) = (\rho - \rho_0) + Q(\rho_0 + \rho_1 + \ldots + \rho_G) - (Q(\rho_1) + \ldots + Q(\rho_G)) \quad (3)$$

Equation (3) expresses the formula for expected occupancy at any node of a multi-stage tree network (with deterministic service times T) in terms of expected occupancies of single stage systems. The inputs to these single stage systems are a superposition of the exogenous inputs to the original tree network. Below we provide explicit formulas for the cases when the exogenous inputs are memoryless and when they are superpositions of independent periodic sources with period P (as in constant bit rate voice streams).

Example--Memoryless Inputs: The $Q(\rho)$ function for memoryless inputs simply becomes the expected occupancy of an M/D/1 queue. This can be obtained using the well known P-K formula (see [9, 11]), and the result is:

$$Q(\rho) = \rho + \frac{\rho^2}{2(1-\rho)} \quad (4)$$

Hence, for the system of Fig. 8a, we use (4) in (3) to find:

$$E(\# \text{ in } S_{A2}) = \rho + \frac{\rho^2}{2(1-\rho)} - \sum_{i=1}^{G} \frac{\rho_i^2}{2(1-\rho_i)} \quad (5)$$

Notice that if identical loadings $\rho_i = \rho/G$ are used on each input line and the number of first stage queues G is increased to infinity, then the expected occupancy in the second stage converges to $Q(\rho)$. This illustrates the intuitive notion that the superposition of a large number of independent and identical low rate streams looks Poisson to a queue.

Example--Periodic (CBR) Inputs: Here we consider continuous bit rate input streams with periodic arrivals and independent phases. This type of traffic model is often used for voice data or for fixed bit rate video. There are M total flows, and each flow is a P-periodic arrival process. Stationarity holds because we assume that the first arrival of each flow is uniformly distributed over the first period interval [0, P). We have $(M - M_0)$ of the iid flows distributed over the G first stage nodes, where M_i flows feed into node i, and $M_1 + M_2 + \ldots + M_G = (M - M_0)$. The remaining M_0 flows feed directly into the final queue.

For this type of input process, the exact complementary distribution function $C_M[n]$ for single stage systems with M independent, periodic inputs of period P ($\rho = MT/P < 1$) is obtained in [10]. We state their expression: For $n \le M$, $C_M[n] =$

$$\sum_{k=1}^{M-n} \binom{M}{k+n} \left(\frac{kT}{P}\right)^{k+n} \left(1 - \frac{kT}{P}\right)^{M-k-n} \left(\frac{P/T - M + n}{P/T - k}\right) \quad (6)$$

Using this, the expected occupancy for a single stage system can be written as:

$$Q(\rho) = Q_{P/T}[M] = \sum_{n=0}^{M-1} C_M[n] \quad (7)$$

where we use the $Q_{P/T}[M]$ notation as a more convenient way to express Q(ρ) for this type of input process. From (3) we have:

$$E(\# \text{ in } S_{A2}) = (M - M_0)(T/P) + \sum_{n=0}^{M-1} C_M[n] - \left[\sum_{n=0}^{M1-1} C_{M1}[n] + \ldots + \sum_{n=0}^{MG-1} C_{MG}[n]\right] \quad (8)$$

These examples demonstrate the power that Theorems 1 and 2 provide for exactly analyzing complex networks: By these theorems, we find that node occupancy in multi-stage systems can be understood in terms of node occupancy of simpler systems.

V. A Concavity Result

Theorem 2 indicates that multi-stage systems can be reduced to simpler 2-stage equivalent models. We thus keep our attention on 2-stage systems and explore an interesting concavity result for expected occupancy. This result provides a starting point for addressing questions concerning the optimal loading distributions on the input lines of multi-stage tree networks of queues. The result gives an important precise statement to the notion that a deterministic service time queue smooths the arrival process for downstream nodes. It shows that balanced loading minimizes the effect of this smoothing function.

Recall that $Q_{X_i}(\rho_i)$ represents the expected occupancy in a single stage queueing system with input process X_i and loading ρ_i. We wish to show a result about the convexity of $Q(\rho)$ as a function of ρ. In order for this type of statement to make sense, we must parameterize an input stream by its loading value. The idea is to consider only input processes that can be represented as a sum of iid subflow processes. The loading is then determined by the number of subflow components used.

Definition: A process X with loading ρ is M-*distributable* iff:

$$X = \sum_{i=1}^{M} X_i \quad (9)$$

where each of the component processes X_i is iid with loading $\rho_i = \rho/M$.

A process is *infinitely distributable* if it is M-distributable for all M.

The Poisson process is the canonical example an infinitely distributable process although it is not the only one.

For an M-distributable input process we can now consider the function $Q(k\Delta)$ for

$\Delta = \rho/M$ and $k = 0, 1, \ldots, M$. Note that such a $Q(k\Delta)$ function is monotonically increasing in k as adding another subflow to the input flow can only increase the number in the queueing system. Finally, we note that we must define what we mean by a convex function with a discrete domain.

<u>Definition</u>: The function $Q(k\Delta)$ is *convex* in k iff:

$$Q((k+1)\Delta) - Q(k\Delta) \geq Q(k\Delta) - Q((k-1)\Delta)$$

$$\text{for} \quad k = 1, 2, \ldots, M-1.$$

This definition of convexity preserves the usual notions of convexity generally known for functions of a continuous variable or continuous multivariables.

<u>Theorem 3</u> (Convexity of average queue length): For an M-distributable input process entering a deterministic service time queue, the function giving the expected number of packets in the queueing system, $Q(k\Delta)$, is convex.

<u>Proof of Theorem 3</u>: Consider the deterministic service time queues represented in Fig. 9. A switch has been added to the top input as we will compare the situation in system S_2 when the top input is present and when it is not. Let $\bar{n}_{ON}(S_2)$ ($\bar{n}_{OFF}(S_2)$) be the average number of packets in the S_2 queueing system when the top source is ON (OFF). Using Theorem 1 we can write: (eqs. 10,11)

$$\bar{n}_{ON}(S_2) + Q(k\Delta) + Q(\Delta) = (k+1)\Delta + Q((k+1)\Delta)$$

$$\bar{n}_{OFF}(S_2) + Q((k-1)\Delta) + Q(\Delta) = k\Delta + Q(k\Delta)$$

which lead to eqs. (12, 13) below:

$$\bar{n}_{ON}(S_2) - (k+1)\Delta = Q((k+1)\Delta) - Q(k\Delta) - Q(\Delta)$$

$$\bar{n}_{OFF}(S_2) - k\Delta = Q(k\Delta) - Q((k-1)\Delta) - Q(\Delta).$$

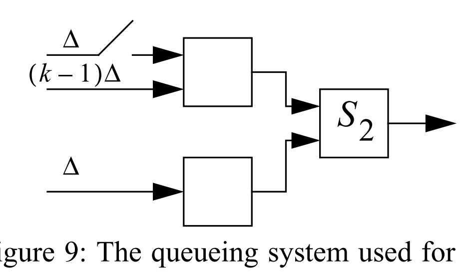

Figure 9: The queueing system used for the convexity proof, with an ON/OFF switch.

Consider the queueing system S_2 when the top source is ON. While the expected number in the S_2 system is $\bar{n}_{ON}(S_2)$, the expected number in the S_2 server alone is equal to the loading of the S_2 system, $(k+1)\Delta$. The left side of (12) represents the expected number in the *queue* of system S_2 with the top source ON while the left side of (13) represents the expected number in the *queue* of system S_2 with the top source OFF. A sample function argument similar to that used in the proof of Theorem 1 shows formally the not surprising fact that the average number of packets in the *queue* of system S_2 is never less when the top source is ON than when the top source is OFF:

$$\bar{n}_{ON}(S_2) - (k+1)\Delta \geq \bar{n}_{OFF}(S_2) - k\Delta \quad (14)$$

So,

$$Q((k+1)\Delta) - Q(k\Delta) \geq Q(k\Delta) - Q((k-1)\Delta) \quad (15)$$

❑

This theorem also holds for infinitely-distributable inputs. Analysis shows that as $M \to \infty$, the $Q(\rho)$ curve becomes a continuous, monotonically increasing convex function of a continuous variable. Memoryless inputs are infinitely-distributable, and we clearly see that the corresponding $Q(\rho)$ function given in (4) is continuous, monotonically increasing, and convex.

<u>Two Stage Systems</u>: We now consider the two stage system with distributable inputs shown in Fig. 10. We are interested in finding an "optimal" distribution for the input loadings. Should all loadings be distributed over the pre-stage units equally? Will an imbalance be helpful or hurtful? The following theorem provides insight into this question. We get the proof of the theorem as a straightforward application of Theorem 3.

<u>Theorem 4</u> (Convexity and concavity of two stage systems): Assume the input process $(\rho_1, \ldots, \rho_G)$ in Fig. 10 is distributable with fixed total loading $\rho = \rho_1 + \ldots + \rho_G$. Then:

(i) The expected occupancy in the first stage system of Fig. 10 is a convex symmetric function of the input loadings $\rho_1, ..., \rho_G$. The occupancy at this stage is minimized when all loadings are distributed uniformly.

(ii) The expected occupancy in the final stage of the system in Fig. 10 is a concave symmetric function of the input loadings $\rho_1,..., \rho_G$. The occupancy at this stage is maximized when all loadings are distributed uniformly.

Proof of Theorem 4:

Proof of (i): The expected occupancy at the first stage is just the sum of the expected occupancies in the individual first stage queues: $Q(\rho_1) + Q(\rho_2) + \ldots + Q(\rho_G)$. Each of the individual terms of the sum are convex functions by Theorem 3. Hence, the sum will also be convex. Clearly, the expected occupancy at the first stage is a symmetric function of the input loadings. The minimum of this convex symmetric function is thus achieved when all input loadings are distributed equally. ❑

Proof of (ii): Using Theorem 1, we can use an equivalent model of the system in Fig. 10 to find that the expected occupancy in the whole system is $\rho + Q(\rho)$, where $\rho = \rho_1 + \ldots + \rho_G$. Note that this value depends only on the total loading, not on the particular loading assignments $(\rho_1, ..., \rho_G)$. Hence, the expected occupancy in the second stage is just the total occupancy minus the first stage occupancy:

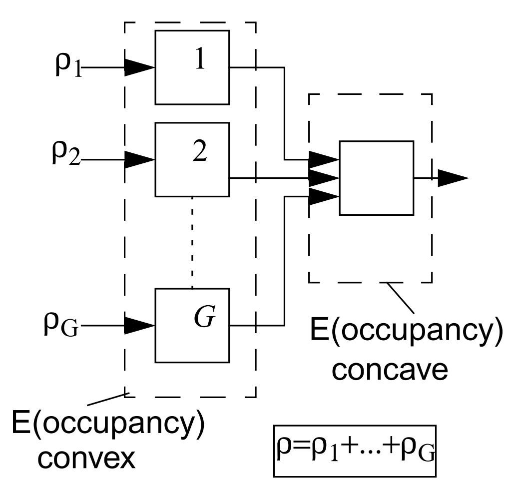

Figure 10: A 2-stage system with distributable inputs.

E(Occupancy at second stage)

$$= \rho + Q(\rho) - [Q(\rho_1) + \ldots + Q(\rho_G)] \quad (16)$$

For a fixed total loading ρ, the above is just a constant minus a convex symmetric function. Hence, the expected second stage occupancy is a concave symmetric function, whose maximum is achieved when all individual loadings ρ_i are equal ($\rho_i = \rho/G$). ❑

Notice that adding an exogenous input ρ_0 to the second stage in Fig. 10 (and redefining the total loading to be $\rho = \rho_0 + \rho_1 + \ldots + \rho_G$) does not change the concavity property of the expected occupancy in this node as a function of $(\rho_0, \rho_1, \ldots, \rho_G)$. However, it is clear that the symmetry is then only with respect to modifications of the original variables $\rho_1, \ldots, \rho_G$.

Theorem 4 provides insight into how changing the input loading distribution will affect the overall system of Fig. 10. In particular, the theorem gives a statement of how deterministic queues smooth the arrival process for future stages. When most of the traffic goes through only a few first stage queues, there is more smoothing than when the traffic is evenly spread over many first stage queues. Our theorem is in terms of expected occupancies, but note that these values are directly related to the amount of buffer slots we require for a given packet loss threshold ε. Smoothing properties of these networks are further explored in [8].

VI. Conclusions

Packet occupancy in multi-stage tree networks of identical deterministic service time queues can be tractably analyzed using the equivalent model results of Theorems 1 and 2. We have shown that if the exogenous inputs to such networks are independent and stationary, then multi-stage analysis can be reduced to analysis of a simpler 2-stage system. The inputs to the 2-stage system are simply superpositions of the original exogenous input processes.

Using this theory, we have developed an expression for the mean occupancy in any node of a multi-stage tree network in terms of the mean occupancy of a single stage (one queue) system with an arrival process consisting of a superposition of the original exogenous inputs. As examples, this expression was written explicitly for the cases when the exogenous inputs are (i) memoryless, and (ii) periodic.

Focusing our attention on 2-stage systems, we have further explored convexity behavior of expected node occupancies. We have created a definition of a "distributable input," and have shown that simple, comparative observations about 2-stage systems and their equivalent models lead to several convexity/concavity results about expected packet occupancy in the first and second stages of our system. Specifically, we have shown that the expected occupancy function $Q(\rho)$ for a single queue is a convex, monotonically increasing function of the loading ρ. This implies that the number of packets in the first stage is a convex symmetric function of the exogenous input rates. Likewise, the second stage occupancy is a concave symmetric function of the exogenous input rates. For the second stage, the maximum of this concave symmetric function is achieved by the uniform rate distribution.

Expanding the 2-stage result via Theorem 2 implies that expected occupancy is concave in the rate assignment for any interior node of a multi-stage tree system. Nodes on the edge of the network have convex expected occupancies. These results provide much insight into tree systems and the smoothing properties of deterministic service time queues. It is interesting to pursue extensions of this theory towards the development of buffer requirements and optimal loading assignments in ATM networks with various types of data traffic.

Notice that our results are stated in terms of fixed length packets flowing through the network, and these packets have identical deterministic service times T in all network nodes. The exact same results apply to deterministic service time queues with monotonically increasing service times $T_1 \leq T_2 \leq \ldots \leq T_n$. However, decreasing service times pose a formidable challenge. It would be interesting to develop models to cover this type of network.

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