Multi-Dimensional Integration Theorem

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1 Time Average Integration

Let $\vec{\mu}(t)$ represent a vector function of time taking values in $\mathbb{R}^N$. The sample average of $\vec{\mu}(t)$ taken at times $t_1, t_2, \ldots, t_m$ is written $\frac{1}{m}\sum_{i=1}^{m}\vec{\mu}(t_i)$. If $\vec{\mu}(t)$ takes values in a set A, then this average constitutes a convex combination of points in A, and hence is contained in the convex hull of A. Intuitively, the same result is true for time average integrals of $\vec{\mu}(t)$, because integrals can be represented as limits of finite sums. However, such a limiting argument cannot be used in general, as the set A may not contain all its limit points. The following theorem proves the result by using the *convex set separation theorem* [1], which states that a convex set and a point not in the set can be separated by a hyperplane.

Theorem 1. *(Time Average Integration) If $\vec{\mu}(t)$ is integrable and is contained within a set A for all time, then the time average integral of $\vec{\mu}(t)$ over any finite interval of size T is within the convex hull of A, i.e.:*

$$\frac{1}{T}\int_0^T \vec{\mu}(t)dt \in Conv(A)$$

Proof. Suppose the result is true when the affine hull[1] of A has dimension less than or equal to $k-1$. The result is trivially true when $k-1=0$, as this implies $\vec{\mu}(t)$ is a single point for all time. We proceed by induction on k.

Assume the affine hull of A has dimension k. By a simple change of coordinates, we can equivalently treat $\vec{\mu}(t)$ as a function taking values in $\mathbb{R}^k$. Let $\vec{p} = \frac{1}{T}\int_0^T \vec{\mu}(t)dt$. If the point $\vec{p}$ is within the set $Conv(A)$, we are done. If $\vec{p} \notin Conv(A)$, then by the convex set separation theorem there must exist a hyperplane H which separates $\vec{p}$ from $Conv(A)$, i.e., there exists a vector $\vec{z}$ and a scalar b such that

$$\begin{aligned} \vec{z}'\vec{p} &\leq b \\ \vec{z}'\vec{a} &\geq b \text{ for all } \vec{a} \in Conv(A) \end{aligned} \tag{1}$$

[1]The *affine hull* of a set A is the set $\vec{a} + X$, where $\vec{a}$ is an arbitrary element of A, and X is the smallest linear space such that $\vec{a} + X$ contains set A [1]. For example, consider a set of points within $\mathbb{R}^N$ which all lie on the same plane, or the same line. Then the affine hull is the 2-dimensional plane, or, respectively, the 1-dimensional line.

where the hyperplane H consists of all points $\vec{x} \in \mathbb{R}^k$ such that $\vec{z}'\vec{x} = b$. Thus, we have:

$$\begin{aligned} b &\geq \vec{z}'\vec{p} \\ &= \frac{1}{T}\int_0^T \vec{z}'\vec{\mu}(t)dt \end{aligned} \quad (2)$$

However, $\vec{\mu}(t) \in Conv(A)$ for all time, and hence by (1) the integrand in (2) is greater than or equal to b for all time. This implies that the set of all times $t \in [0, T]$ for which $\vec{z}'\vec{\mu}(t) > b$ must have measure zero. Hence:

$$\begin{aligned} \vec{p} &= \frac{1}{T}\int_0^T \vec{\mu}(t)dt \\ &= \frac{1}{T}\int_{\{t\in[0,T] \,|\, \vec{z}'\vec{\mu}(t)=b\}} \vec{\mu}(t)dt \end{aligned} \quad (3)$$

The integral in (3) represents the time average of a function contained in the set $A \cap H$, a set of dimension at most $k - 1$. It follows by the induction hypothesis that $\vec{p} \in Conv(A \cap H) \subset Conv(A)$, a contradiction. □

Corollary 1. *If the set A is closed, then* $\lim_{T\to\infty} \frac{1}{T}\int_0^T \vec{\mu}(t)dt \in Conv(A)$, *provided that the limit converges.*

Proof. The limit can be approached arbitrarily closely by time average integrals over finite intervals. By Theorem 1, each such time average is contained within $Conv(A)$. The limiting integral is thus a limit point of the closed set $Conv(A)$, and hence is within $Conv(A)$. □

Example: The corollary does not hold if the set A is not closed. Indeed, consider the scalar valued function $\mu(t) = 1 - 1/(t + 1)$ contained within the non-closed interval $[0, 1)$ for all $t \geq 0$. Then the time average integral of $\mu(t)$ over any finite interval is within $[0, 1)$, but the limiting average as the interval size $T \to \infty$ is equal to 1, which is not in this interval.

References

[1] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar. *Convex Analysis and Optimization.* To be published: Athena Scientific, Feb. 2003.