On Taking Infimums Over Sets

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THE INFIMUM

Here we compute $\inf_{\Theta} \mathbb{E}\{X \mid \Theta\}$ for a non-negative random variable X, where the infimum is taken over all events Θ such that $Pr[\Theta] \geq \frac{1}{2}$. Let $P(x) = Pr[X \leq x]$ represent the cumulative distribution function for X. Let ω be the unique real number such that $Pr[X < \omega] \leq \frac{1}{2}$ and $Pr[X \leq \omega] \geq \frac{1}{2}$. Note that if $P(x)$ is continuous, then $Pr[X < \omega] = Pr[X \leq \omega] = \frac{1}{2}$. In general, a non-continuous distribution may have a point mass at $x = \omega$.

Lemma 1: For any non-negative random variable X, we have:

$$\inf_{\{\Theta \mid Pr[\Theta] \geq \frac{1}{2}\}} \mathbb{E}\{X \mid \Theta\} = \mathbb{E}\{X \mid X < \omega\} 2Pr[X < \omega] + \omega (1 - 2Pr[X < \omega])$$

Note that the infimum depends only on the cumulative distribution function $P(x)$. In the special case when $P(x)$ is continuous at $x = \omega$, then $Pr[X < \omega] = Pr[X \leq \omega] = \frac{1}{2}$, and hence the lemma implies that the infimum is equal to $\mathbb{E}\{X \mid X \leq \omega\}$.

Proof: To prove the lemma, let $p(x) \triangleq \frac{dP(x)}{dx}$ represent the generalized density function of X (which may contain impulses if $P(x)$ is not continuous). Consider any event Θ such that $Pr[\Theta] \geq \frac{1}{2}$. Define the conditional probability distribution $f(x) \triangleq p_{X \mid \Theta}(x \mid \Theta)$. Note that $p(x) = p_{X \mid \Theta}(x \mid \Theta) Pr[\Theta] + p_{X \mid \Theta^c}(x \mid \Theta^c) Pr[\Theta^c]$ (where Θ^c represents the complement of the event Θ). Hence, $p_{X \mid \Theta}(x \mid \Theta) \leq p(x)/Pr[\Theta] \leq p(x)/\frac{1}{2}$. That is:

$$f(x) \leq 2p(x) \quad \text{for all } x \tag{1}$$

Note also that $f(x)$ is a probability distribution for a non-negative variable, so that $\int_0^\infty f(x)dx = 1$. We have:

$$\begin{aligned}
\mathbb{E}\{X \mid \Theta\} &= \int_0^{\omega^-} x f(x) dx + \int_{\omega^-}^{\infty} x f(x) dx \\
&= \int_0^{\omega^-} x 2p(x) dx + \int_0^{\omega^-} x[f(x) - 2p(x)]dx + \int_{\omega^-}^{\infty} x f(x) dx \\
&\geq \int_0^{\omega^-} x 2p(x) dx + \omega \int_0^{\omega^-} [f(x) - 2p(x)]dx + \omega \int_{\omega^-}^{\infty} f(x) dx \qquad (2) \\
&= \mathbb{E}\{X \mid X < \omega\} 2Pr[X < \omega] + \omega - \omega \int_0^{\omega^-} 2p(x) dx \qquad (3) \\
&= \mathbb{E}\{X \mid X < \omega\} 2Pr[X < \omega] + \omega (1 - 2Pr[X < \omega]) \qquad (4)
\end{aligned}$$

where (2) follows because (1) implies the integrand of the second integral is non-positive for all x (so that $\int_0^{\omega^-} x[f(x) - 2p(x)]dx \geq \omega \int_0^{\omega^-} [f(x) - 2p(x)]dx$), and (3) follows because $\int_0^{\omega^-} f(x)dx + \int_{\omega^-}^{\infty} f(x)dx = 1$.

The lower bound (4) holds for all events Θ such that $Pr[\Theta] \geq 1/2$, and hence:

$$\inf_{\{\Theta \mid Pr[\Theta] \geq \frac{1}{2}\}} \mathbb{E}\left\{X \mid \Theta\right\} \geq \mathbb{E}\left\{X \mid X < \omega\right\} 2Pr[X < \omega] + \omega\left(1 - 2Pr[X < \omega]\right)$$

We now show that the reverse inequality is also true. Let A be the outcome of a biased coin flip that is independent of X. Specifically, let $Pr[A = 1] = q, Pr[A = 0] = 1 - q$, where q is the value such that $qPr[X = \omega] = (\frac{1}{2} - Pr[X < \omega])$. Note that $0 \leq q \leq 1$ because $Pr[X = \omega] + Pr[X < \omega] \geq \frac{1}{2}$ but $Pr[X < \omega] \leq \frac{1}{2}$.

Consider the particular event Θ^* defined as follows:

$$\Theta^* \triangleq \{\{X < \omega\} \ \cup \ \{\{X = \omega\} \cap \{A = 1\}\}\} \tag{5}$$

That is, Θ^* represents the event that either $X < \omega$, or both $X = \omega$ and $A = 1$. Note that $Pr[\Theta^*] = 1/2$, because $Pr[\Theta^*] = Pr[X < \omega] + qPr[X = \omega]$. We then have:

$$\begin{aligned} \mathbb{E}\left\{X \mid \Theta^*\right\} &= \mathbb{E}\left\{X \mid X < \omega\right\} \frac{Pr[X < \omega]}{Pr[\Theta^*]} + \omega \frac{qPr[X = \omega]}{Pr[\Theta^*]} \\ &= \mathbb{E}\left\{X \mid X < \omega\right\} 2Pr[X < \omega] + \omega\left(1 - 2Pr[X < \omega]\right) \end{aligned}$$

Thus, the particular event Θ^* allows the conditional expectation to meet the lower bound of (4). Thus, Θ^* is the minimizing event, and its resulting expectation is equal to the infimum, proving the lemma. □

We note that there is nothing special about the number $1/2$. Indeed, a similar statement can be proven for sets Θ such that $Pr[\Theta] \geq p$, where p is any nonzero probability. However, it is important that $p > 0$. As an example of the crazy things that can happen when conditioning on a probability zero event, consider a random variable X that is distributed uniformly between 0 and 1. Define a new random variable Y such that $Y = X$ if $X > 0$, and $Y = -10$ if $X = 0$. Because X and Y are the same variable with probability 1, we have $Pr[X > x] = Pr[Y > x]$ for all x, and hence X is stochastically greater than Y, and likewise Y is stochastically greater than X. Note that $\inf_{\Theta} \mathbb{E}\left\{X \mid \Theta\right\} = 0$, where the infimum is taken over all possible events. The minimizing event is equal to the event $\Theta^* \triangleq \{X = 0\}$, which is an event of zero probability. However, $\mathbb{E}\left\{Y \mid X = 0\right\} = -10 < 0$.