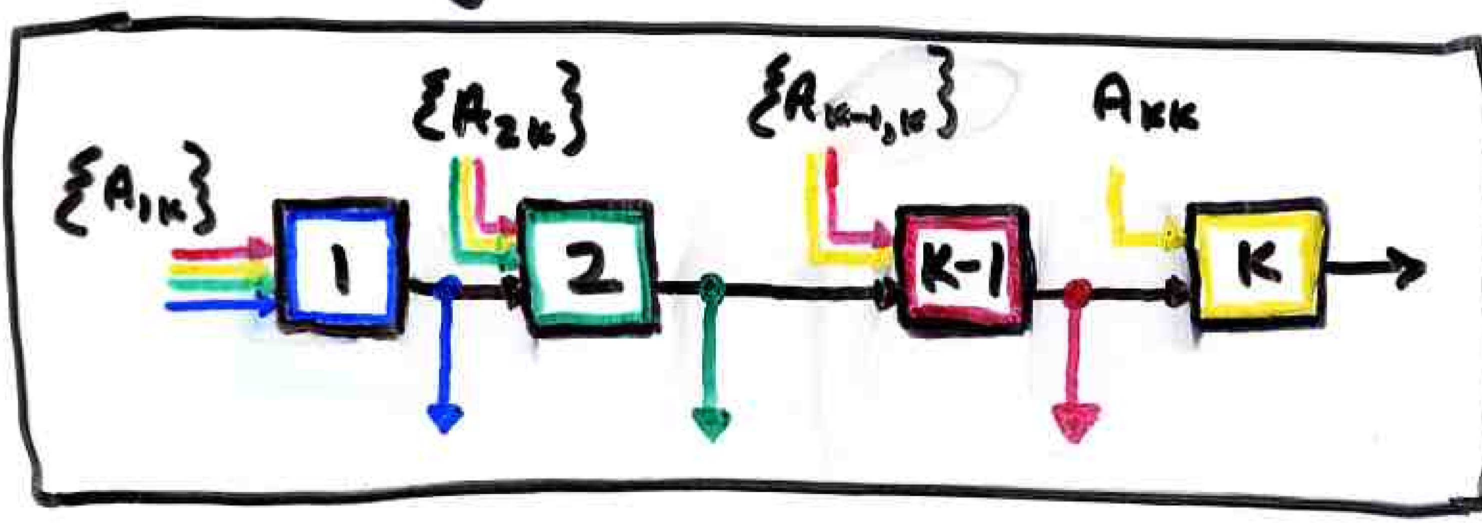
Exact Queueing Analysis of Discrete-Time Tandems with Arbitrary Arrival Processes



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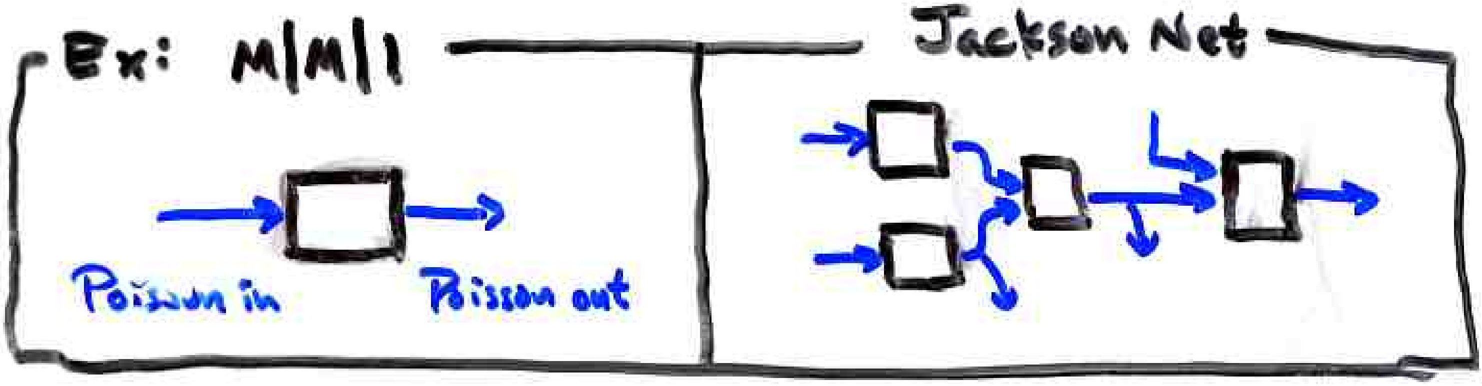
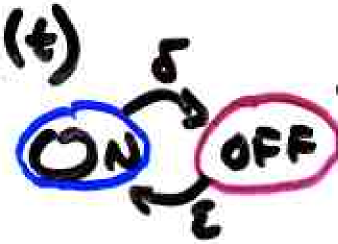
Context

Most analyzable queueing networks are reversible



This talk:

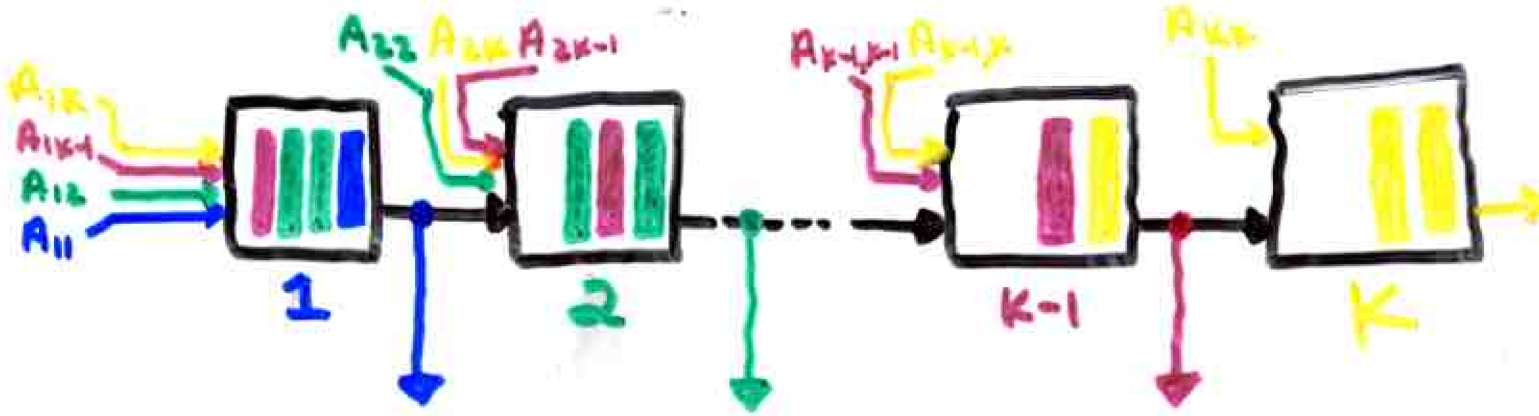
- Slotted Time
- Fixed Length Packets (service = 1 slot)
- Tandem Topology
- Arbitrary Arrival Processes $A_{ij}(t)$
  (Ex: Poisson, Markovian, ON/OFF )

Analysis: uses...

① Input/Output Relation 

② Stochastic Coupling $X \underset{st.}{\leq} Y \underset{st.}{=} Z$

Model



- $A_{ij}(t) \triangleq$ # packets arriving to node i at slot t, destined for node j.
- A packet is "class n" if destined for node n.
- $N_n^{(m)}(t) \triangleq$ # class n packets in node m @ slot t.
- $S_n^{(m)}(t) \triangleq$ # class n packets served by node m at slot t. (at most 1 service per node per timeslot)

$$\boxed{N_n^{(m)}(t+1) = \left[N_n^{(m)}(t) - S_n^{(m)}(t)\right] + A_{mn}(t) + S_n^{(m-1)}(t)}$$

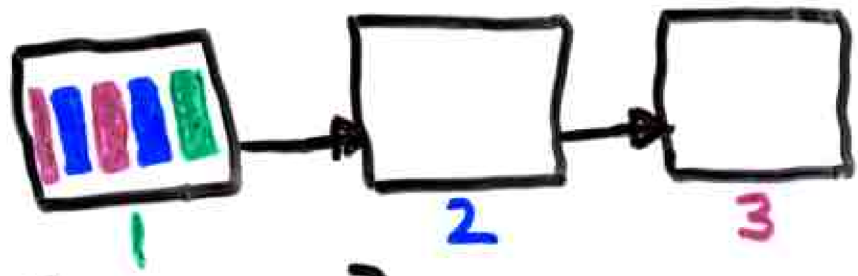
$$S_n^{(m)}(t) \in \{0,1\}, \quad S_n^{(m)}(t) = 1 \Rightarrow S_j^{(m)}(t) = 0$$

$(j \neq n)$

(Service Variables)

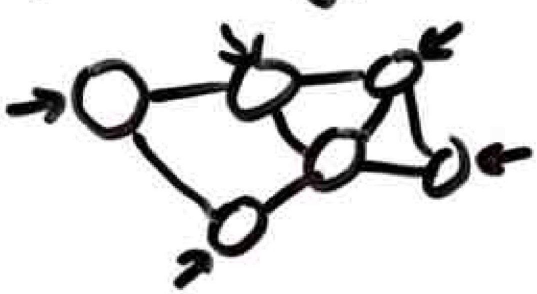
Service Policy (FIFO? LIFO? FTG? NTG?)

We consider the Furthest-to-Go (FTG) Strategy:



[Choose RED]

- FTG shown to be stable for all discrete time networks with fixed routing in:
  - Aiello, Ostrovsky, Kushilevitz, Rosen SODA 2003



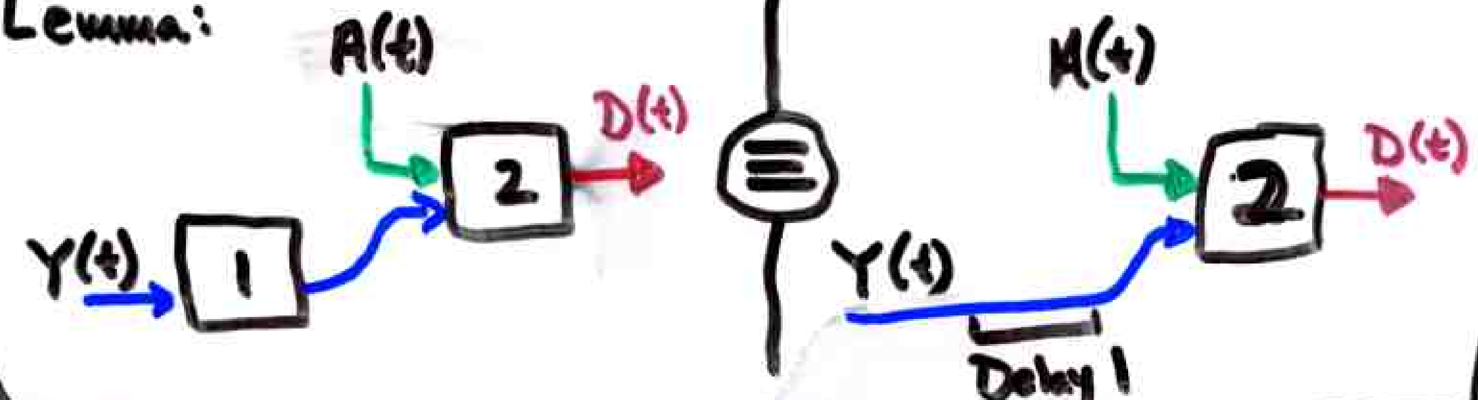
- We develop exact analysis of steady state queue occupancy for special case of tandem topology.

Equivalent Model Analysis (IN/OUT relation)

Equivalence Lemma: (developed independently in:

- Neely, Allerton 2001 (General case and proof)
- Morrison, Bell Syst. Tech. Report 1978 (Discrete Time)
- Shalmon + Kaplan, OR 1984 (Observation + Heuristic Proof)

Lemma:

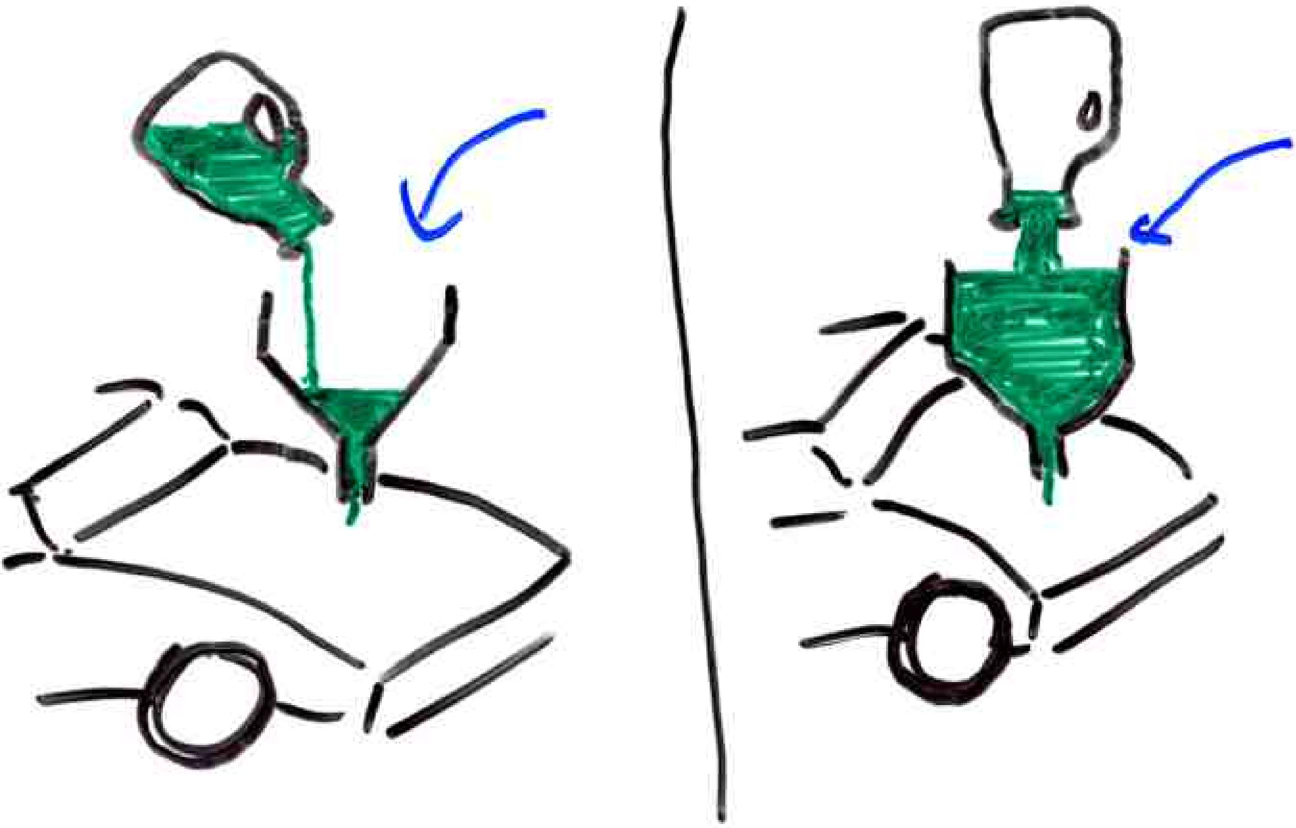


For arbitrary arrival processes $Y(t)$, $A(t)$, the departure sample path $D(t)$ is unchanged if we replace node 1 with a pure time delay.

(* Caveat: Departure ordering might change, but a packet departs system 1 at time t iff a packet departs system 2 at time t.

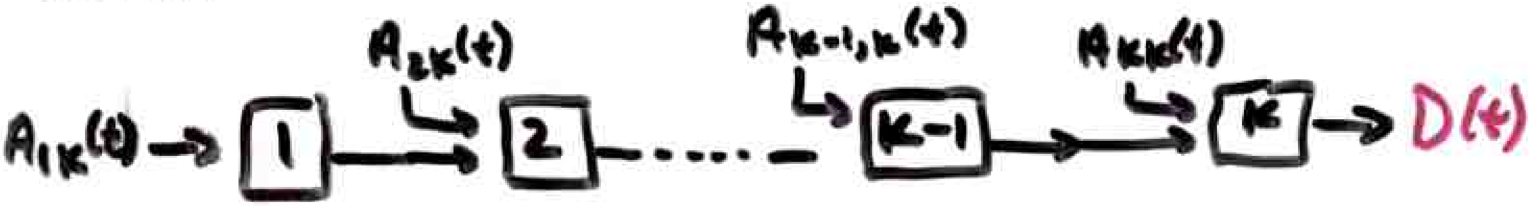
Intuition Behind Equivalence Lemma:

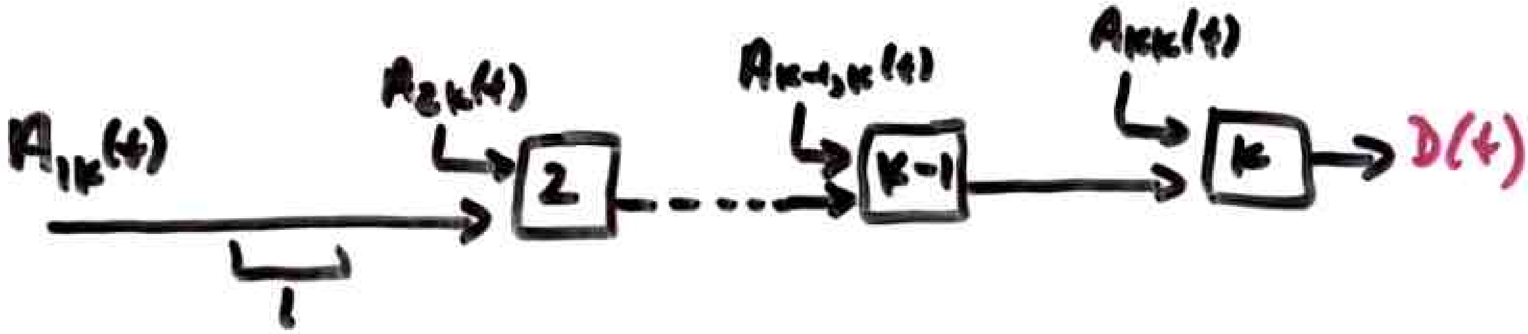
Changing your oil

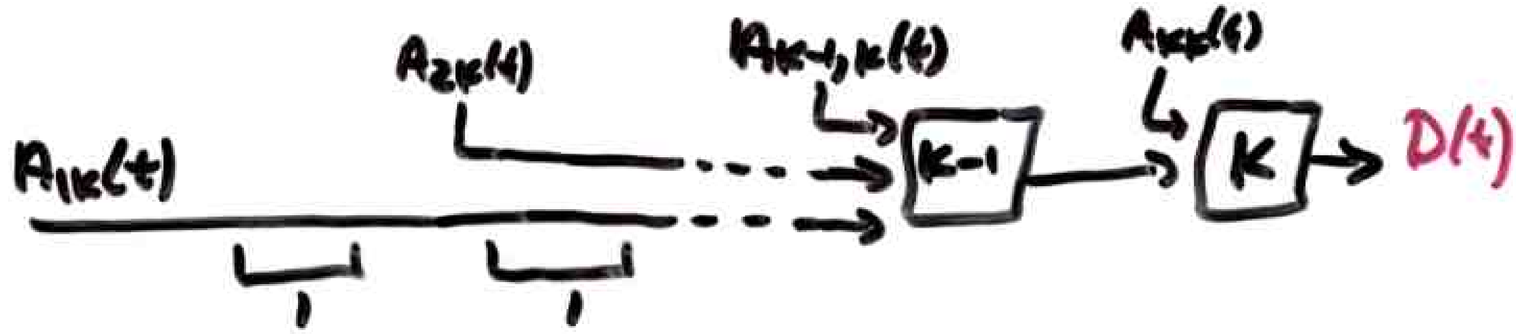


Application to our Tandem:

Consider special case of multi-input, single-output tandem:







$$\sum_{k=1}^{K} A_{kK}(t-D_k) \longrightarrow \boxed{K} \longrightarrow D(t)$$

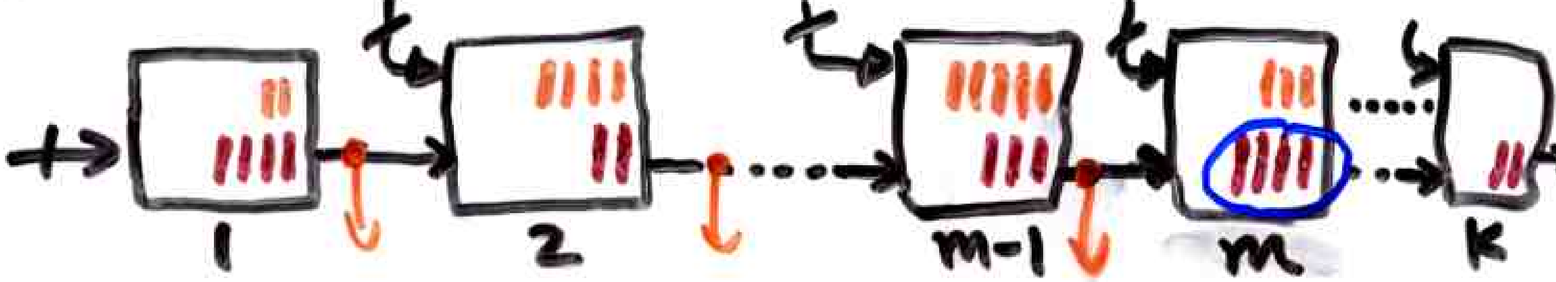
D(t) is unchanged under these transformations.

General MIMO Tandems under FTG:

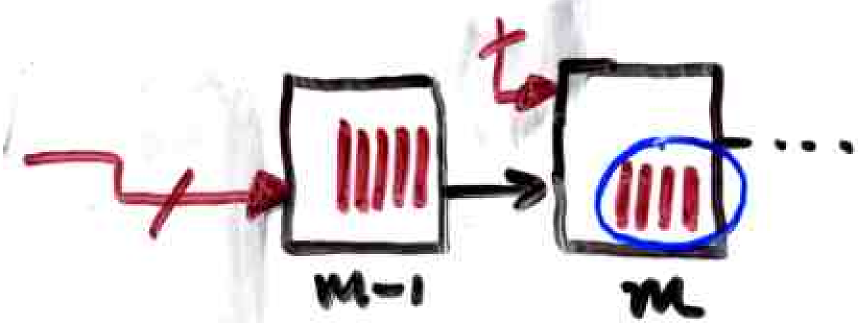
- $$Y_n^{(m)}(t) \triangleq \sum_{i=1}^{m} \sum_{j=\max(n,m)}^{K} A_{ij}(t)$$

  = collection of class n or higher packets that pass thru node m.

- $$Z_n^{(m)}(t) \triangleq \sum_{k \geq n} N_k^{(m)}(t)$$ = # class n or higher packets in node m.

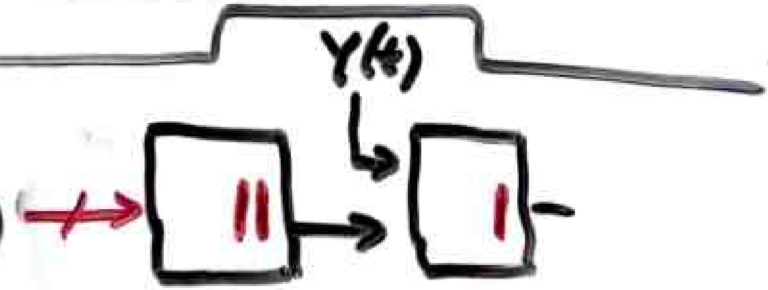


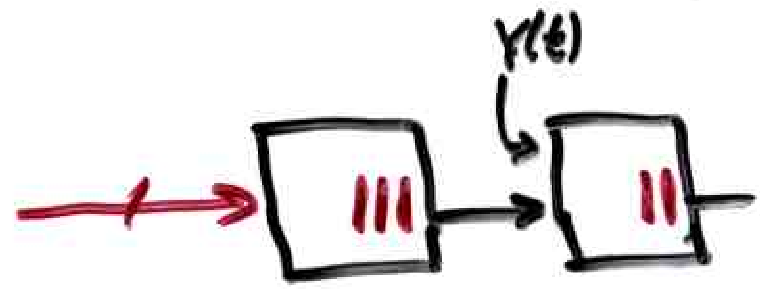
$$\sum_{i=1}^{m} \sum_{j=\max(n,m)}^{K} A_{ij}(t - D_i)$$



Now use Stochastic Coupling to Remove Delays ...

Delay Removal Theorem: If $\{A_{ij}(t)\}$ stationary and independent, then steady state # packets in Systems A and B are stochastically equivalent.

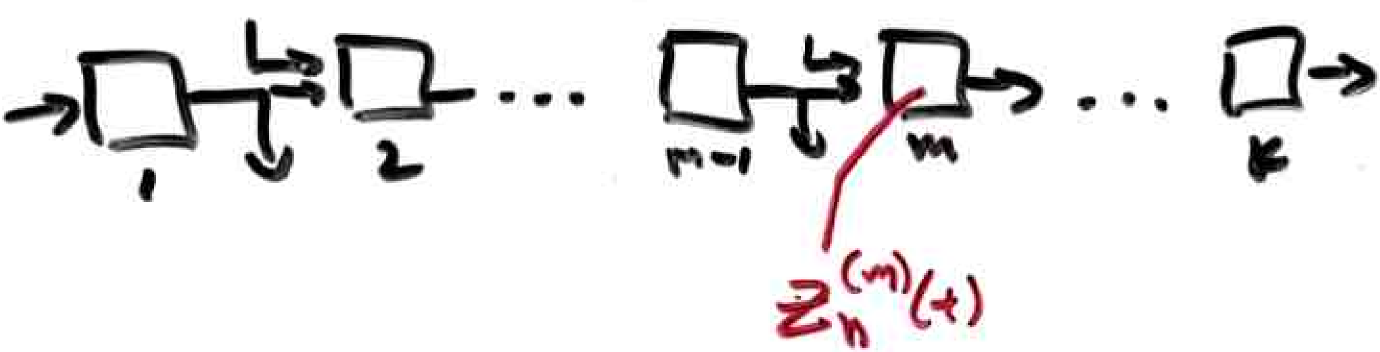
(A) $\sum_{i,j \in \Omega} A_{ij}(t - D_{ij})$ 

(B) $\sum_{i,j \in \Omega} A_{ij}(t)$ 

(Steady state exists in System A iff
steady state exists in System B)

Theorem: If inputs $\{A_{ij}(t)\}$ stationary and independent, then $Z_n^{(m)}(t)$ is steady state stochastically equivalent to:

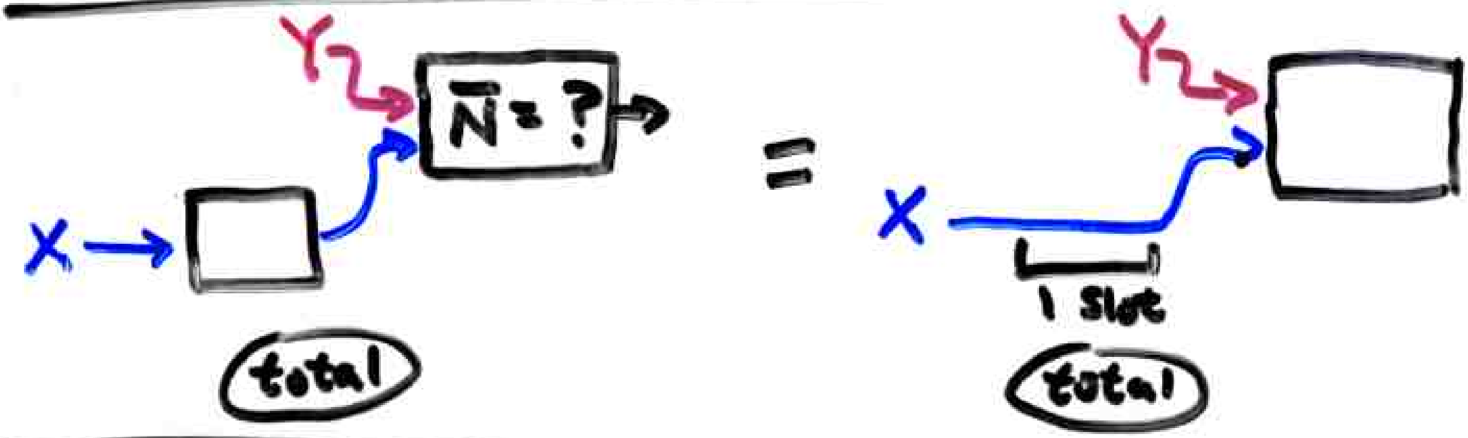
(Reduction to 2-stage system)
Y_n^m = superposition of original inputs.



Can reduce to 1-queue system if we consider only mean occupancy ...

Finding Expected class occupancy in the Canonical 2-stage system:

Let $Q(X)$ = Expected # packets in queue with input process X.



$$Q(X) + \bar{N} = \lambda_X + Q(X+Y)$$

$$\Rightarrow \bar{N} = \lambda_X + Q(X+Y) - Q(X)$$

(λ_X = rate of process X)

General formula for mean of class n in node m:

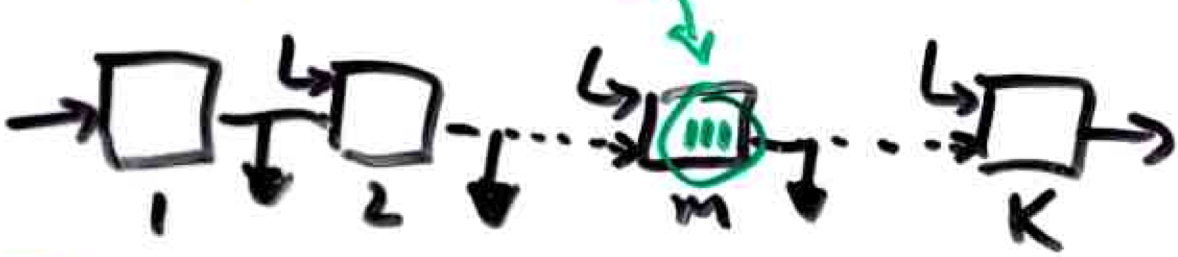
$$\bar{Z}_n^{(m)} = \gamma_n^{(m-1)} + Q\left(Y_n^{(m)}\right) - Q\left(Y_n^{(m-1)}\right)$$

where: $$Y_n^{(m)} = \sum_{i=1}^{m} \sum_{j=\max(n,m)}^{K} A_{ij}$$

$$\gamma_n^{(m)} = \sum_{i=1}^{m} \sum_{j=\max(n,m)}^{K} \lambda_{ij}$$

then:

$$\bar{N}_n^{(m)} = \bar{Z}_n^{(m)} - \bar{Z}_{n+1}^{(m)} \qquad \square$$

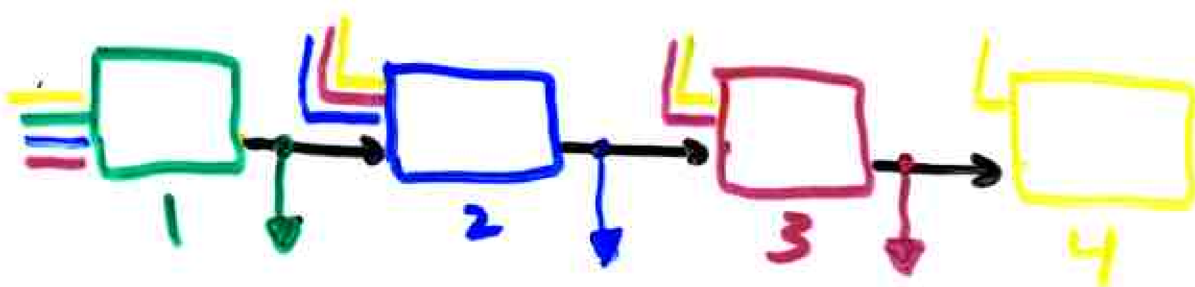


[Ex: Poisson Inputs $\Rightarrow Q(\lambda) = \frac{\lambda^2}{2(1-\lambda)} + \lambda$]

Example: 4 nodes. Poisson Inputs.

All packets uniformly dist. destinations $\in \{1, \dots, 4\}$

Inputs loaded s.t. all queues have same aggregate rate λ:



We push λ from $0 \to 1$ and plot average network delay:

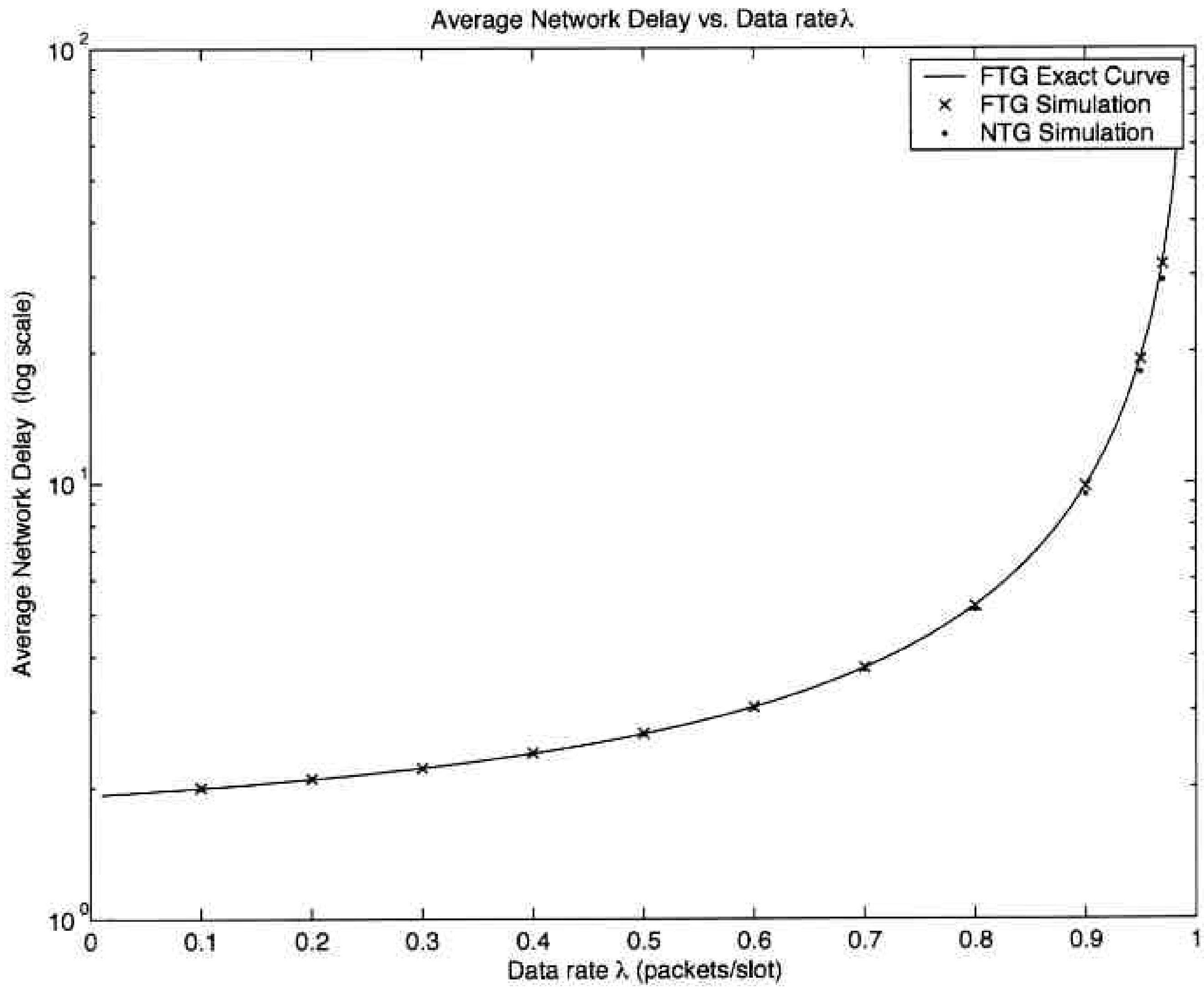
FTG exact

FTG simulation

NTG simulation

(NTG = "nearest-to-go")

Concluding Slide



We have presented a simple and exact calculus for discrete time tandems using sample path theory and stochastic coupling.