Order Optimal Delay for Opportunistic Scheduling in Multi-User Wireless Uplinks and Downlinks

Michael J. Neely
University of Southern California
http://www-rcf.usc.edu/∼mjneely

***Abstract*— We consider a one-hop wireless network with independent time varying ON/OFF channels and N users, such as a multi-user uplink or downlink. We first show that general classes of scheduling algorithms that do not consider queue backlog must incur average delay that grows at least linearly with N. We then construct a dynamic queue-length aware algorithm that maximizes throughput and achieves an average delay that is independent of N. This is the first order-optimal delay result for opportunistic scheduling with asymmetric links. The delay bounds are achieved via a technique of *queue grouping* together with Lyapunov drift and statistical multiplexing concepts.**

***Index Terms*— Queueing Analysis, Stability, Stochastic Control**

I. INTRODUCTION

In this paper, we investigate the fundamental delay scaling laws in a multi-user wireless system with N time varying data links, such as a multi-user uplink or downlink. Packets arrive to the system according to independent stochastic arrival streams, with one arrival stream for each link, and are stored in separate queues to await transmission. Time is slotted, and the system can support a transmission over at most one link per timeslot. Channel conditions on each link vary independently every slot according to ON/OFF Bernoulli processes, so that a link can transmit exactly one packet during a timeslot when it is in the ON state, and cannot transmit in the OFF state. Such ON/OFF channel states might arise from channel fluctuations or fading due to user mobility. Every timeslot, a network controller views the conditions on each channel and chooses exactly one link to transmit.

This system model is central to the study of channel-aware (or "opportunistic") scheduling in wireless systems, and the model along with many generalizations have been extensively considered in the literature [2]-[24]. Landmark work by Tassiulas and Ephremides in [2] characterizes the *capacity region* of this model, consisting of the set of all arrival rate vectors the system can be configured to stably support. The work in [2] also proposes the *Longest Connected Queue* (LCQ) scheduling policy, and uses a Lyapunov drift argument to show that this policy stabilizes the system (and thus maximizes throughput) whenever input rates are interior to the capacity region. Furthermore, the work in [2] uses a stochastic coupling argument to show that, in the special case of a symmetric system with identical input rates for each user and identical channel probabilities for each link, the LCQ policy *minimizes average delay*.

This delay optimality result is generalized in [5] [11], where a delay optimal policy is developed for selecting transmission rates within the polytope capacity region associated with the Gaussian multiple access channel, and in [15] where generalizations to multi-server systems are considered. However, these delay optimality results hold only in cases when the system exhibits perfect symmetry in traffic rates and channel statistics. Indeed, these works use the stochastic coupling technique of [2], which requires this symmetry. Further, the actual average delay achieved by these strategies is unknown, even in these symmetric cases. Work in [8] computes upper bounds on the delay of stabilizing largest-queue type strategies for heterogeneous downlinks. However, these bounds grow linearly in the number of users N. Specifically, the delay bound has the form $cN/(1-\rho)$, where ρ is a parameter such that $0 < \rho < 1$ and represents the fraction the input rate vector is away from the capacity region boundary, and c is a constant that does not depend on ρ or N.[1] In simple special cases, such as when all channels are always ON and any work conserving policy is used, it can be shown that average delay can be improved to $O(1/(1-\rho))$ (and hence is independent of N), demonstrating that the bound in [8] is not always tight. However, whether or not optimal delay can grow sub-linearly with N in more general cases has remained an important open question, and is a question that we resolve in this paper.

Using the simple ON/OFF channel model, we first show that, for general classes of scheduling algorithms that use channel state information but do not consider queue backlog, average delay must grow at least linearly with N. We then construct a simple dynamic control policy called *Largest Connected Group* that uses both queue state and channel state information. We apply this policy to both symmetric and asymmetric systems (where the asymmetric cases treat large classes of systems with heterogeneous traffic rates and channel probabilities). An upper bound on average delay is derived and shown to have the form $\frac{c\log(1/(1-\rho))}{1-\rho}$, where c is a constant that is independent of ρ and N. This result can likely be extended to treat links with more than two channel states (provided that there is a finite transmission rate in the best

Manuscript received October 27, 2006; revised May 17, 2007; accepted June 8, 2007. This work was presented in part at the 44th Annual Allerton Conference on Communication, Control, and Computing, Monticello, IL, September 2006. Approved by IEEE/ACM Transactions on Networking Editor S. Borst.

This work is supported in part by one or both of the following: the National Science Foundation grant OCE 0520324, the DARPA IT-MANET program grant W911NF-07-1-0028.

The author is with the Department of Electrical Engineering, University of Southern California, Los Angeles, CA 90089 USA (web: http://www-rcf.usc.edu/ mjneely).

[1]For convenience, this paper expresses scaling laws using c to represent a generic coefficient that is independent of ρ and N. The value of c is not necessarily the same in the different expressions in which it appears.

channel state), and to treat time-correlated channel and traffic processes (possibly using the T-slot Lyapunov drift techniques of [19] [14]), although we omit this analysis for brevity.

Previous work in the area of wireless scheduling is found in [6][9][10] for systems with an infinite backlog of data, and a clearing problem in a system with N links and a fixed amount of data is treated in [12]. Stable scheduling and queueing is considered for satellite, wireless, and ad-hoc mobile systems in [2][3][4][7][8][13][14]. The work in [7] develops delay optimality results in the limit as the system loading ρ approaches 1, but does not provide asymptotic results in the number of users. Indeed, the analysis in [7] uses a fluid limit and a heavy traffic limit that may suggest each of the N queues is usually non-empty. In our analysis, we provide an average delay bound for a fixed loading factor $\rho < 1$, and obtain delay that is independent of N by scheduling to ensure that each queue is *usually empty*. This provides an advantage in the case when there are many users and ρ is a fixed fraction away from the capacity region boundary. However, while our $\frac{c\log(1/(1-\rho))}{1-\rho}$ bound in this paper has a better asymptotic in N than the previous $cN/(1-\rho)$ bound in [8], it has a slightly worse asymptotic in ρ.

Much work in the area of dynamic scheduling is developed for computer networks and switching systems, including work in [25][26][27][28] that uses Lyapunov stability theory. The work in [26] considers max-weight-match (MWM) scheduling in an $N \times N$ packet switch with i.i.d. traffic (such as Bernoulli or Poisson), and shows that average delay is no more than $cN/(1-\rho)$. Various methods of *queue groupings* are used with Lyapunov functions in [28][29][30][31] to achieve low complexity scheduling. While [28][29][30][31] does not primarily focus on delay, it is interesting to note that if an $N \times N$ switch is half loaded ($\rho < 1/2$) with independent Bernoulli or Poisson inputs, then similar queue groupings together with the Lyapunov delay technique of [26] can be used to show that average delay is $c/(1-2\rho)$ under maximal match scheduling. However, this result does not seem to extend to cases when $\rho > 1/2$. Work in [32] uses a simple frame-based algorithm for an $N \times N$ switch to show it is possible to achieve an average delay of $c\log(N)/(1-\rho)^2$, for any value $\rho < 1$. Our results in the present paper parallel our previous work in [32] for switch scheduling. However, the problem formulation and solution technique is quite different here, as the frame-based approach in [32] does not appear tractable with stochastic channel conditions. Here, we pursue a novel queue grouping approach, and show that the average delay of our wireless system can be bounded independently of the number of users N, for any value of $\rho < 1$.

We note that a different approach to showing that average delay does not grow with N is recently considered for symmetric systems in [16]. Specifically, work in [16] extends the results in [15] to show that average delay under an optimal algorithm in a system with symmetric Poisson traffic and $2N$ symmetric links is less than or equal to the corresponding average delay in a system with only N links. Our analysis uses a different technique that yields explicit delay bounds while also applying to asymmetric systems with traffic that is i.i.d. over slots but possibly non-Poisson.

In the next section, we formulate the problem and review the system capacity region from [2]. In Section III we show that a large class of backlog-unaware scheduling algorithms necessarily incur average delay that grows at least linearly with N. In Section IV we develop our backlog-aware *Largest Connected Group* algorithm and show it yields average delay that is independent of N.

II. Problem Formulation

Consider an N queue system that evolves in discrete time with integral timeslots $t \in \{0, 1, 2, \ldots\}$. Let $Q_i(t)$ represent the number of packets in queue i at the beginning of slot t (for $i \in \{1, \ldots, N\}$). Let $A_i(t)$ represent the number of new packet arrivals during slot t, and let $\mu_i(t)$ represent the transmission rate (in units of packets) during slot t. The dynamic equation for each queue $i \in \{1, \ldots, N\}$ is given by:

$$Q_i(t+1) = \max[Q_i(t) - \mu_i(t), 0] + A_i(t) \tag{1}$$

Each queue contains data that must be transmitted over a distinct link with time varying channels. Let $S_i(t) \in \{ON, OFF\}$ represent the *channel state* of link i during slot t. Assume these channel states are i.i.d. over timeslots and independent across channels, and let q_i represent the ON probability for channel i:[2]

$$q_i \triangleq Pr[S_i(t) = ON]$$

The ON/OFF channel states are assumed to be known to the network controller at the beginning of each slot. Every slot t, the network controller chooses transmission decision variables $\boldsymbol{\mu}(t) = (\mu_1(t), \ldots, \mu_N(t))$ subject to the constraints:

$$\begin{aligned} \mu_i(t) \in \{0, 1\} \quad \forall i \in \{1, \ldots, N\} \\ \mu_i(t) = 0 \text{ if } S_i(t) = OFF \\ \textstyle\sum_{i=1}^N \mu_i(t) \leq 1 \end{aligned} \tag{2}$$

The above constraints specify that at most one link can be chosen for transmission on any timeslot, and that exactly one packet can be transmitted over a given link i during a timeslot in which $S_i(t) = ON$, while no packets can be transmitted over a channel that is OFF.

This system model can be used to represent a multi-user wireless or satellite downlink, where all packets arrive to a single node that internally stores data in separate queues for transmission to the appropriate destination. Alternatively, the system can represent a multi-user wireless *uplink*, where each user has its own data that must be transmitted to a central access point. In this uplink scenario, the queues are distributed over the different users. It is assumed in this case that the access point receives queue backlog updates every slot, and determines which user transmits by sending permission signals over a dedicated control channel.

[2]Extensions to time-correllated arrival and channel processes can be treated using the T-slot Lyapunov drift techniques of [19] [14].

Definition 1: A discrete time queue $Q(t)$ with a general arrival and server rate process is *strongly stable* if:[3]

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}\left\{Q(\tau)\right\}<\infty$$

A *network* of queues is said to be strongly stable if each queue is strongly stable. Throughout this paper, we use the term "stability" to refer to strong stability. The goal is to design a scheduling algorithm that stabilizes the system while keeping time average backlog and average delay as small as possible.

A. The Capacity Region

Suppose arrivals $A_i(t)$ are i.i.d. over timeslots, and let $\lambda_i = \mathbb{E}\left\{A_i(t)\right\}$ represent the packet arrival rate of stream i (for each $i \in \{1, \ldots, N\}$). Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ represent the arrival rate vector. The *network capacity region* Λ is the closure of the set of all rate vectors $\boldsymbol{\lambda}$ for which a stabilizing algorithm exists. For a system of 2 queues ($N = 2$), the capacity region is given by all rate vectors (λ_1, λ_2) that satisfy:

$$\lambda_1 \leq q_1\ ,\ \lambda_2 \leq q_2$$
$$\lambda_1 + \lambda_2 \leq q_1 + (1-q_1)q_2$$

These inequalities are clearly *necessary* for stability, as otherwise one or both queues would have an input rate that exceeds the transmission rate capabilities of the system. It is not difficult to show that any rate vector (λ_1, λ_2) *interior* to this region can be stabilized. The capacity region for a system of N queues is shown in [2] to be the set of all rate vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ that satisfy the inequalities:

$$\sum_{i\in\mathcal{I}}\lambda_i \leq 1 - \Pi_{i\in\mathcal{I}}(1-q_i)$$

for each non-empty subset of indices $\mathcal{I} \subset \{1, \ldots, N\}$. Thus, the capacity region is described by a set of $2^N - 1$ inequality constraints. An alternate characterization of the capacity region can be given in terms of all possible expected transmission rate vectors that can be achieved by a stationary randomized scheduling policy, as shown below.

Lemma 1: (Stationary Randomized Policies [19][2]) A rate vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ is in the capacity region Λ if and only if there exists a stationary control strategy that chooses a transmission rate vector $\boldsymbol{\mu}(t) = (\mu_1(t), \ldots, \mu_N(t))$ as a (potentially random) function of the observed channel state vector $\boldsymbol{S}(t) = (S_1(t), \ldots, S_N(t))$ such that $\boldsymbol{\mu}(t)$ satisfies (2) for all t, and such that the expected transmission rate yields:

$$\mathbb{E}\left\{\mu_i(t)\right\} = \lambda_i \quad \text{for all } i \in \{1, \ldots, N\}$$

The expectation above is with respect to the stationary distribution for the channel state vector $\boldsymbol{S}(t)$ and the potentially random transmission decision that depends on $\boldsymbol{S}(t)$. □

Note that in the special case of a symmetric system where $q_i = q$ for all $i \in \{1, \ldots, N\}$, then the largest symmetric rate vector $(\lambda, \lambda, \ldots, \lambda)$ that is in the capacity region is given by the vector with $\lambda_i = r_N/N$ for all $i \in \{1, \ldots, N\}$, where r_N is the probability that at least one link is in the ON state during a timeslot: $r_N \triangleq 1 - (1-q)^N$.

B. The Single-Queue Lower Bound

A simple lower bound on the average backlog (and hence, by Little's Theorem [33], average delay), can be obtained by comparing the multi-queue system to a corresponding single-queue system with a sum arrival and channel process. Specifically, define the single-queue system to have queue backlog $Q_{single}(t)$ with dynamics:

$$Q_{single}(t+1) = \max[Q_{single}(t) - \mu_{single}(t), 0] + A_{sum}(t)$$

where $A_{sum}(t) \triangleq \sum_{i=1}^{N} A_i(t)$, and where $\mu_{single}(t) \in \{0, 1\}$, and is 1 if and only if $S_i(t) = ON$ for at least one channel $i \in \{1, \ldots, N\}$. It is easy to show that $Q_{single}(t) \leq \sum_{i=1}^{N} Q_i(t)$ for all time t, regardless of the scheduling policy used for the multi-queue system. Thus, the average backlog and average delay in the multi-queue system is lower bounded by the corresponding single queue averages. Specifically, assuming averages exist, we have:

$$\overline{W} \geq \overline{W}_{single} \quad , \quad \textstyle\sum_{i=1}^{N}\overline{Q}_i \geq \overline{Q}_{single}$$

where $\overline{W}$ is the average delay in the multi-queue system, $\overline{Q}_i$ is the average backlog in queue i of the multi-queue system, and $\overline{W}_{single}$ and $\overline{Q}_{single}$ are the delay and backlog averages in the single-queue system. Note that the single-queue system is a discrete time GI/GI/1 queue with a Bernoulli service process with service probability $\mu_{av} \triangleq 1 - \Pi_{i=1}^{N}(1-q_i)$. The average delay in such a system can be computed exactly:

$$\overline{W}_{single} \quad = \quad \frac{1 + \frac{1}{\lambda_{tot}}\mathbb{E}\left\{A_{sum}^2\right\} - 2\lambda_{tot}}{2\mu_{av}(1-\rho)} \tag{3}$$

where $\mathbb{E}\left\{A_{sum}^2\right\} \triangleq \mathbb{E}\left\{A_{sum}(t)^2\right\}$, $\lambda_{tot} \triangleq \sum_{i=1}^{N}\lambda_i$, and $\rho \triangleq \lambda_{tot}/\mu_{av}$. In the case when all inputs $A_i(t)$ are *independent* and Poisson with rates λ_i, we have: $\mathbb{E}\left\{A_{sum}^2\right\} = \lambda_{tot} + \lambda_{tot}^2$. Hence, the single-queue delay bound for Poisson traffic is given by:

$$\overline{W} \geq \overline{W}_{single} = \frac{2 - \lambda_{tot}}{2\mu_{av}(1-\rho)} \tag{4}$$

This specifies that the best possible average delay of any scheduling algorithm is $O(1/(1-\rho))$ when arrivals are independent and Poisson.

On the other hand, in the case when the inputs $A_i(t)$ are *not independent*, the lower bound can be $cN/(1-\rho)$, where c is a constant that is independent of ρ and N. Specifically, if we have $A_i(t) = A(t)$ for all $i \in \{1, \ldots, N\}$, with $A(t)$ Poisson of rate λ_{tot}/N, then queue 1 receives k packets on slot t if and only if all other queues receive k packets that slot. It follows that $\mathbb{E}\left\{A_{sum}^2\right\} = \mathbb{E}\left\{N^2 A(t)^2\right\} = N\lambda_{tot} + \lambda_{tot}^2$ and hence:

$$\overline{W}_{single} = \frac{N + 1 - \lambda_{tot}}{2\mu_{av}(1-\rho)} \text{ (correlated arrival case)} \tag{5}$$

The difference between the $c/(1-\rho)$ and $cN/(1-\rho)$ delay bounds in (4) and (5) is due to the *statistical multiplexing gains* that arise when data streams $A_i(t)$ are independent. In

[3] We note that if a queue $Q(t)$ is strongly stable and also evolves according to an ergodic Markov chain with a countably infinite state space, then the $\limsup$ on the left hand side in the stability definition above can be replaced with a regular limit that represents the steady state backlog.

this paper we focus on the case when inputs are independent. Our goal is to develop an algorithm that yields average delay close to the $c/(1-\rho)$ lower bound in (4).

III. Backlog-Unaware Scheduling

Here we show that if scheduling algorithms are restricted to a large class of policies that use channel state information but do not use queue backlog information, then average delay necessarily grows at least linearly with N. Suppose arrival processes are stationary and ergodic with rates λ_i. Let $X_i(t)$ represent the number of packets that arrive up to time t, and let $\{X_i(v)\}_{v\geq 0}$ denote the entire sample-path arrival history over time. We consider *stationary* scheduling algorithms that choose transmission rates *independent of the entire arrival history*, and hence independent of current queue backlog. Specifically, we consider the class of scheduling policies that yield transmission rates with the following property for all $i \in \{1, \ldots, N\}$:

$$\mathbb{E}\left\{\mu_i(t) \mid \{X_i(v)\}_{v\geq 0}\right\} = \mathbb{E}\left\{\mu_i(0)\right\} \triangleq \overline{\mu}_i \tag{6}$$

This is a large class of policies, including all of the stationary randomized scheduling policies used in Lemma 1. Periodic policies (such as round robin scheduling) can also be included in this class if the phase of the initial period is uniformly randomized, as in [32].

Theorem 1: (Backlog Unaware Scheduling) Consider any scheduling algorithm that satisfies (6) and stabilizes the system with finite average backlogs $\overline{Q}_i$ and average delay $\overline{W}$. Then:

(a) For all t, we have:

$$\mathbb{E}\left\{Q_i(t)\right\} \geq \mathbb{E}\left\{U_i(t)\right\}$$

where $U_i(t)$ represents the "unfinished work" (or fractional packets) at time t in a continuous time queueing system with the same arrivals $A_i(t)$ but with a constant transmission rate $\overline{\mu}_i$ (and hence deterministic service times $1/\overline{\mu}_i$).

(b) Suppose there are symmetric channel probabilities $q_i = q$ and symmetric rates $\lambda_i = \lambda_{tot}/N$ for all $i \in \{1, \ldots, N\}$. Assume $\lambda_{tot} \leq r_N$ (where $r_N \triangleq 1-(1-q)^N$ is the maximum system output rate). If the arrival streams are continuous time Poisson processes, then average delay necessarily satisfies:

$$\overline{W} \quad \geq \quad \frac{N}{2r_N(1-\rho)}$$

where $\rho \triangleq \lambda_{tot}/r_N$.

(c) For asymmetric systems, let r_{max} represent the maximum possible sum output rate:

$$r_{max} \triangleq 1 - \Pi_{i=1}^{N}(1-q_i)$$

Let γ_1 and γ_2 be positive constants less than 1. If there are at least $\gamma_1 N$ arrival processes with transmission rates at least $\gamma_2\lambda_{tot}/N$, then average delay is at least $\gamma_1^2\gamma_2 N/(2r_{max})$, and hence grows at least linearly with N.

Proof: See Appendix A. □

Note that the assumptions on γ_1 and γ_2 in part (c) of the theorem precludes the case of a fixed number of users dominating the total arrival rate. In that case, it is possible for total average delay to be independent of N because the majority of packets can have small delay averages.

IV. The Queue Grouping Algorithm

Here we develop a dynamic algorithm that involves *queue grouping*, and show that the algorithm has average delay that is independent of N. We first review the delay result from [8] that provides a (loose) upper bound on the average delay of the LCQ policy from [2]. Recall that the LCQ policy chooses to transmit over the ON link with the largest queue backlog (breaking ties randomly and uniformly), and is shown in [2] to stabilize the system whenever input rates are inside the capacity region Λ, and to minimize average delay in the special case of a symmetric system.

Assume channel states are independent with probabilities q_i for $i \in \{1, \ldots, N\}$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ be the rate vector, and suppose that there exists a value $\epsilon > 0$ such that $\boldsymbol{\lambda}+\boldsymbol{\epsilon} \in \Lambda$ (where $\boldsymbol{\epsilon} \triangleq (\epsilon, \epsilon, \ldots, \epsilon)$). Thus, we assume $\boldsymbol{\lambda}$ is strictly interior to the capacity region, and that a positive value ϵ can be added to each component to yield another vector that is within the capacity region.

Lemma 2: (Delay of LCQ [8])[4] Suppose arrival vectors $\boldsymbol{A}(t)$ are i.i.d. over timeslots, and that $\boldsymbol{\lambda} + \boldsymbol{\epsilon} \in \Lambda$. Then:

(a) The LCQ policy stabilizes the system and yields average delay that is upper bounded as follows:

$$\overline{W} \leq \frac{\lambda_{tot} + \sum_{i=1}^{N}\mathbb{E}\left\{A_i^2\right\} - 2\sum_{i=1}^{N}\lambda_i^2}{2\lambda_{tot}\epsilon} \tag{7}$$

where $\mathbb{E}\left\{A_i^2\right\} \triangleq \mathbb{E}\left\{A_i(t)^2\right\}$, and $\lambda_{tot} \triangleq \sum_{i=1}^{N}\lambda_i$.

(b) If arrival streams $A_i(t)$ have symmetric rates $\lambda_i = \lambda_{tot}/N$ for $i \in \{1, \ldots, N\}$ (possibly with different variances), if $q_i = q$ for all $i \in \{1, \ldots, N\}$, and if $\lambda_{tot} = \rho r_N$ for some value ρ such that $0 < \rho < 1$ (where $r_N \triangleq 1-(1-q)^N$), then average delay satisfies:

$$\overline{W} \leq \frac{N\left[1 + \frac{1}{\lambda_{tot}}\sum_{i=1}^{N}\sigma_i^2\right] - \lambda_{tot}}{2r_N(1-\rho)} \tag{8}$$

where $\sigma_i^2 \triangleq \mathbb{E}\left\{A_i^2\right\} - \lambda_i^2$ is the variance of $A_i(t)$.

Note that part (b) follows immediately from part (a) by using $\epsilon = r_N/N - \lambda_{tot}/N$, so that: $\epsilon = r_N(1-\rho)/N$. The upper bound on average delay has the form $cN/(1-\rho)$. The lemma holds for arrival vectors with components that are *arbitrarily correlated*, and hence in this sense the asymptotic with N is tight (recall the single-queue bound (5) in the case of correlated arrivals). Specifically, note that for Poisson traffic we have $\sigma_i^2 = \lambda_i$ for all i, so that $\frac{1}{\lambda_{tot}}\sum_{i=1}^{N}\sigma_i^2 = 1$. Hence, from (8) we have:

$$\overline{W} \leq \frac{2N - \lambda_{tot}}{2r_N(1-\rho)}$$

This is roughly a factor of 2 larger than the single queue lower bound in (5) for the case of correlated Poisson arrivals. Further note that for independent arrivals, if N is treated as a fixed constant (that does not scale) but ρ scales to 1, this LCQ policy achieves the optimal $O(1/(1-\rho))$ scaling with respect to ρ (from the lower bound (4)).

[4] The derivation in [8] considers a more general system with variable transmission rates that can be any real number, and obtains a slightly different bound in this case, but still with the $cN/(1-\rho)$ structure. The expression (7) follows as a special case of Theorem 2 in the case $N = K$.

A. *Intuition for Queue Grouping*

Here we assume arrival streams $A_i(t)$ are i.i.d. over timeslots and are also independent of each other. To provide intuition on the advantages of queue grouping, define $q_{min} = \min_{i\in\{1,\ldots,N\}} q_i$, and compare the system of N parallel queues (with channel probabilities $q_i \geq q_{min}$ for all $i \in \{1,\ldots,N\}$) to a *single queue system* with a Bernoulli server with rate q_{min} and with an arrival process given by $A_{sum}(t)$, the sum of the individual $A_i(t)$ arrival processes. It can be shown that if the N queue system schedules according to *any work conserving scheduling policy* (i.e., a policy that always serves a non-empty ON queue if one is available), the resulting backlog is stochastically less than the backlog in the single queue system (this result is not required in our analysis below, but is of independent interest and is proven in Appendix B). It follows that if $\lambda_{tot} < q_{min}$, then the average delay in the multi-queue system is no more than the average delay in the single queue system. In particular, if the input processes $A_i(t)$ are independent and Poisson, then we have:

$$\overline{W} \leq \frac{2-\lambda_{tot}}{2(q_{min}-\lambda_{tot})}$$

Therefore, delay in this case does not grow linearly with N. Further, this result holds whenever the input rate vector is within a factor ρ of the capacity region boundary, for any value ρ such that $0 < \rho < \gamma$, where $\gamma \triangleq q_{min}/r_{max}$. To see this, note that this result holds for any rates such that $\sum_i \lambda_i < q_{min}$, and let Λ^* denote the closure of this region. It follows that:

$$\begin{aligned}\gamma\Lambda \quad &\subset \quad \gamma\left\{\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \geq \mathbf{0}, \sum_i \lambda_i \leq r_{max}\right\} \\ &= \quad \left\{\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \geq \mathbf{0}, \sum_i \lambda_i \leq \gamma r_{max}\right\} = \Lambda^*\end{aligned}$$

where the first inclusion follows because $\sum_i \lambda_i \leq r_{max}$ is a necessary condition for $\boldsymbol{\lambda} \in \Lambda$ (it is not necessarily sufficient).

Thus, Λ^* contains the set $\gamma\Lambda$. However, this single-queue comparison does not apply when $\gamma \leq \rho < 1$. To achieve a larger fraction of the capacity region, we can assemble each of the N queues of the system into K distinct groups. Intuitively speaking, each single group can be compared to a corresponding single queue system with a Bernoulli transmission rate of q_{min}. The advantage is that now we only require the sum of transmission rates *within each group* to be less than q_{min} (so that larger input rate vectors can generally be supported). Each group is then treated as a single queue, and the LCQ algorithm is applied to that system of K "queues," yielding an $O(K)$ delay result via Lemma 2. In the next section we make this intuition precise.

B. *The Largest Connected Group (LCG) Algorithm*

Below we specify the queue grouping algorithm for a general set of groups. We then discuss intelligent ways to form the groups for both symmetric and asymmetric systems. Let $\{\mathcal{G}_1,\ldots,\mathcal{G}_K\}$ represent any general grouping of the queue indices $i \in \{1,\ldots,N\}$ into disjoint sets, where K is the number of groups. We assume each group $\mathcal{G}_k$ is a non-empty subset of $\{1,\ldots,N\}$, groups are disjoint, and the union of all K groups is equal to the set of all queue indices $\{1,\ldots,N\}$. For each group index $k \in \{1,\ldots,K\}$, define:

$$\begin{aligned}A_{sum,k}(t) &\triangleq \sum_{i\in\mathcal{G}_k} A_i(t) \\ Q_{sum,k}(t) &\triangleq \sum_{i\in\mathcal{G}_k} Q_i(t) \\ \lambda_{sum,k}(t) &\triangleq \sum_{i\in\mathcal{G}_k} \lambda_i\end{aligned}$$

Further define the indicator function $1_k(t)$ to take the value 1 if group $\mathcal{G}_k$ has at least one index i that corresponds to a non-empty queue with an ON channel state on slot t, so that $Q_i(t) > 0$ and $S_i(t) = ON$.

The Largest Connected Group (LCG) Algorithm: Every timeslot t, the network controller observes the queue backlogs and current channel states, and selects the group index $k \in \{1,\ldots,K\}$ that maximizes $Q_{sum,k}(t)1_k(t)$, breaking ties arbitrarily. It then chooses to transmit over any link $i \in \mathcal{G}_k$ that corresponds to a non-empty queue with a channel that is ON, i.e., any non-empty connected queue of the selected group. If there are no such queues for slot t, remain idle.

For all $k \in \{1,\ldots,K\}$, define:

$$q_{min,k} \triangleq \min_{i\in\mathcal{G}_k} q_i$$

Now define Λ_K as the K *dimensional capacity region* of a system with K queues with Bernoulli ON probabilities $q_{min,k}$ for $k \in \{1,\ldots,K\}$. That is, Λ_K is the set of all non-negative rate vectors $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_K)$ such that

$$\sum_{k\in\mathcal{I}} \omega_k \leq 1 - \Pi_{k\in\mathcal{I}}(1-q_{min,k})$$

for all subsets $\mathcal{I} \subset \{1,\ldots,K\}$.

Theorem 2: (LCG Performance for General Groups) Suppose channels are independent with ON probabilities q_i for $i \in \{1,\ldots,N\}$, and arrival vectors $\boldsymbol{A}(t)$ are i.i.d. over slots with rate vector $\boldsymbol{\lambda}$. If there exists a value $\epsilon > 0$ such that:

$$(\lambda_{sum,1}+\epsilon, \lambda_{sum,2}+\epsilon, \ldots, \lambda_{sum,K}+\epsilon) \in \Lambda_K$$

then the system is stable, and:

$$\overline{\sum_i Q_i} \leq \frac{\left[\lambda_{tot} + \sum_{k=1}^K \mathbb{E}\left\{A_{sum,k}^2\right\} - 2\sum_{k=1}^K \lambda_{sum,k}^2\right]}{2\epsilon}$$

where:

$$\overline{\sum_i Q_i} \triangleq \limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1}\sum_{i=1}^N \mathbb{E}\left\{Q_i(\tau)\right\}$$

Further, if arrival processes $A_i(t)$ are independent of each other, then:

$$\overline{\sum_i Q_i} \leq \frac{\left[\lambda_{tot} + \sum_{i=1}^N \sigma_i^2 - \sum_{k=1}^K \lambda_{sum,k}^2\right]}{2\epsilon} \quad (9)$$

where $\sigma_i^2 \triangleq \mathbb{E}\left\{A_i^2\right\} - \lambda_i^2$ for $i \in \{1,\ldots,N\}$.

Proof: The first part of the theorem is proven in the next section using a Lyapunov drift argument. We note that this argument uses a novel comparison between the drift of LCG and the drift of another queue-length dependent algorithm. Inequality (9) then follows immediately by noting that if arrivals from different streams are independent, then:

$$\mathbb{E}\left\{A_{sum,k}^2\right\} = (\lambda_{sum,k})^2 + \sum_{i\in\mathcal{G}_k} \sigma_i^2$$

□

Note that the LCG algorithm breaks ties arbitrarily. However, intuition from the LCQ algorithm in [2] suggests that serving larger queues tends to yield better delay performance. Thus, an intuitively good tie breaking rule is to *serve the queue with the largest backlog* among all ties under LCG. If there are further ties under this rule, then break the ties randomly and uniformly over all groups.

This tie breaking rule also ensures the vector queueing process $\boldsymbol{Q}(t)$ evolves according to a *discrete time Markov chain*, in which case Foster's criterion [34] can be used to ensure the chain has a valid steady state with steady state queue occupancies $\overline{Q}_i$ and hence an average delay $\overline{W} = \frac{1}{\lambda_{tot}}\sum_{i=1}^N \overline{Q}_i$. To simplify notation, for the remainder of this paper we assume that such steady state limits exist whenever the system is stable.

C. Choosing Groups for Symmetric Systems

Consider a symmetric system such that $q_i = q$ for all $i \in \{1,\ldots,N\}$, and define a loading parameter ρ such that $0 < \rho < 1$. Define the group size K as:

$$K = \left\lceil \frac{\log(2/(1-\rho))}{\log(1/(1-q))} \right\rceil \tag{10}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Note that K is chosen independently of the number of queues N. For simplicity, assume that N is a multiple of K, so that we form distinct groups $\mathcal{G}_1,\ldots,\mathcal{G}_K$, each with N/K elements. Suppose that all input rates are identical, so that $\lambda_i = \lambda_{tot}/N$ for all $i \in \{1,\ldots,N\}$. Assume that $\lambda_{tot} \leq \rho r_N$ (where $r_N = 1-(1-q)^N$), so that the rate vector is at least a factor of ρ away from the capacity region boundary.

Theorem 3: (Symmetric Performance) Consider a uniformly loaded symmetric system as described above, with a group size K given by (10). If N is a multiple of K, if $A_i(t)$ is i.i.d. over slots for all $i \in \{1,\ldots,N\}$, and if all input streams are independent of each other, then the LCG algorithm stabilizes the system and yields:

$$\sum_i \overline{Q}_i \leq \frac{K\left[\lambda_{tot} + \sum_{i=1}^N \sigma_i^2\right] - \lambda_{tot}^2}{r_N(1-\rho)}$$

Therefore, average delay satisfies:

$$\begin{aligned}\overline{W} &\leq \frac{K\left[1+\frac{1}{\lambda_{tot}}\sum_{i=1}^N\sigma_i^2\right]-\lambda_{tot}}{r_N(1-\rho)} \qquad (11)\\ &\leq \frac{\log(2/(1-\rho))\left[1+\frac{1}{\lambda_{tot}}\sum_{i=1}^N\sigma_i^2\right]}{r_N(1-\rho)\log(1/(1-q))}\\ &\quad+\frac{1+\frac{1}{\lambda_{tot}}\sum_{i=1}^N\sigma_i^2-\lambda_{tot}}{r_N(1-\rho)}\end{aligned}$$

The term $\frac{1}{\lambda_{tot}}\sum_{i=1}^N \sigma_i^2$ is typically $O(1)$. Indeed, for Poisson traffic it is exactly equal to 1, and for any traffic that satisfies $A_i(t) \leq A_{max}$ for all i (for some finite bound A_{max}) it is less than A_{max}. Thus, the above result demonstrates that average delay satisfies $\overline{W} \leq \frac{c\log(1/(1-\rho))}{1-\rho}$, where c is a constant independent of ρ and N. This demonstrates that average delay does not grow with N. Recall that the single-queue lower bound of (4) implies that all algorithms must have average delay at least $c/(1-\rho)$. Hence, the LCG algorithm scales optimally with N, and differs from the optimal scaling in ρ by at most a logarithmic factor $\log(1/(1-\rho))$.

Proof: (Theorem 3) Note that Λ_K in this case is the capacity region associated with a symmetric system of K queues with independent Bernoulli channels, each with ON probability q. Define r_K as the largest sum rate from this K queue system. It follows that the symmetric rate vector $\boldsymbol{\omega} = (r_K/K,\ldots,r_K/K)$ is contained in Λ_K. Further note that $\lambda_{sum,k} = \lambda_{tot}/K$ for all $k \in \{1,\ldots,K\}$. To ensure that the conditions of Theorem 2 hold, we desire to find a value $\epsilon > 0$ such that:

$$(\lambda_{sum,1}+\epsilon,\ldots,\lambda_{sum,K}+\epsilon) \in \Lambda_K$$

It suffices to show that $\lambda_{sum,k} + \epsilon \leq r_K/K$, which is equivalent to showing:

$$\lambda_{tot} + \epsilon K \leq r_K \tag{12}$$

To this end, note by (10) that

$$K\log(1/(1-q)) \geq \log(2/(1-\rho))$$

and hence:

$$(1-q)^K \leq (1-\rho)/2$$

It follows that:

$$r_K \triangleq 1-(1-q)^K \geq (1+\rho)/2 \geq (1+\rho)r_N/2$$

Therefore (using the fact that $\lambda_{tot} \leq \rho r_N$):

$$\begin{aligned}r_K - \lambda_{tot} \geq r_K - \rho r_N &\geq (1+\rho)r_N/2 - \rho r_N\\ &= r_N(1-\rho)/2\end{aligned}$$

It follows that choosing $\epsilon \triangleq r_N(1-\rho)/(2K)$ ensures that (12) is satisfied. The result follows by applying inequality (9) from Theorem 2. □

D. Asymmetric Systems

Consider a general asymmetric system with N queues and independent channels with ON probabilities $\{q_i\}$ for $i \in \{1,\ldots,N\}$. Define $q_{min} = \min_{[i\in\{1,\ldots,N\}]} q_i$. Define a loading parameter ρ such that $0 < \rho < 1$, and choose the group size K as follows:

$$K = \left\lceil \frac{\log(2/(1-\rho))}{\log(1/(1-q_{min}))} \right\rceil \tag{13}$$

Further define:

$$\begin{aligned} r_a &\triangleq 1-(1-q_{min})^K\\ r_{max} &\triangleq 1-\Pi_{i=1}^N(1-q_i)\end{aligned}$$

Note that r_a is the maximum output rate in a system of K queues with independent Bernoulli channels with probability q_{min}, and r_{max} is the maximum output rate of the asymmetric system of N queues. We assume that $N \geq K$.

Consider heterogeneous input rates $(\lambda_1, \ldots, \lambda_N)$. Define λ_{tot} as the sum of all rates, and assume that $\lambda_{tot} = \rho r_{max}$. Define $\tilde{\lambda} \triangleq \max_{i \in \{1,\ldots,N\}} \lambda_i$. Form K groups $\mathcal{G}_1, \ldots, \mathcal{G}_K$ by packing indices to groups in any manner that ensures all groups are disjoint, non-empty, and that:

$$\lambda_{sum,k} \leq \lambda_{tot}/K + \tilde{\lambda} \quad \text{for all } k \in \{1, \ldots, K\} \tag{14}$$

This is easily accomplished as follows: Place the first K indices individually into each of the K groups (so that all groups have at least one index), and then sequentially place the remaining indices into any group for which the current sum of rates in that group does not yet exceed λ_{tot}/K.

To proceed, we make the following additional assumption concerning the size of the largest input rate $\tilde{\lambda}$:

$$\tilde{\lambda} \leq (1-\rho) r_{max}/(3K) \tag{15}$$

Note that the average size of each input is given by $\lambda_{tot}/N = \rho r_{max}/N$. Because N can be much larger than K, this additional assumption (15) states that the largest input is upper bounded by a number much larger than the average.[5]

Theorem 4: (Asymmetric Performance) Consider an asymmetric system as described above, and assume the group size K satisfies (13). Assume that $N \geq K$, and that the largest input rate $\tilde{\lambda}$ satisfies (15). If inputs are independent of each other, then the LCG algorithm stabilizes the system and yields:

$$\sum_i \overline{Q}_i \leq \frac{3K\left[\lambda_{tot} + \sum_{i=1}^N \sigma_i^2 - \sum_{k=1}^K \lambda_{sum,k}^2\right]}{r_{max}(1-\rho)}$$

Hence, average delay satisfies:

$$\overline{W} \leq \frac{3K\left[1 + \frac{1}{\lambda_{tot}}\sum_{i=1}^N \sigma_i^2 - \frac{1}{\lambda_{tot}}\sum_{k=1}^K \lambda_{sum,k}^2\right]}{r_{max}(1-\rho)}$$

Because K satisfies (13), we again see that average delay is $\frac{c\log(1/(1-\rho))}{1-\rho}$, and so is independent of N.

Proof: (Theorem 4) Similar to the proof of the symmetric case, the inequality (13) can be used to show:

$$r_a \triangleq 1 - (1-q_{min})^K \geq r_{max}(1+\rho)/2$$

and hence (using (14), (15) and the fact that $\lambda_{tot} = \rho r_{max}$):

$$\frac{r_a}{K} - \lambda_{sum,k} \geq \frac{r_{max}(1-\rho)}{6K} \quad \text{for all } k \in \{1, \ldots, K\} \tag{16}$$

However, note that the capacity region associated with K queues, each with independent Bernoulli channels with probabilities q_{min}, is a subset of Λ_K (this is because the set Λ_K has queues with probabilities $q_{min,k} \geq q_{min}$ for all $k \in \{1, \ldots, K\}$). Therefore, the vector $\boldsymbol{\omega} = (r_a/K, \ldots, r_a/K)$ is contained in the set Λ_K. It follows from (16) that we can define ϵ as follows: $\epsilon = r_{max}(1-\rho)/(6K)$. The result follows by plugging this value of ϵ into (9) of Theorem 2. □

[5]The proof of Theorem 4 is unchanged if the condition (15) is replaced by the weaker condition that $\delta \leq (1-\rho)r_{max}/(3K)$, where δ is defined as the smallest value such that $\lambda_{sum,k} \leq \lambda_{tot}/K + \delta$ for all $k \in \{1, \ldots, K\}$. Hence, it is desirable to pack the K groups as evenly as possible.

V. Lyapunov Analysis

Here we use Lyapunov drift theory to prove Theorem 2 of the previous section. We begin with a simple but important Lyapunov drift result from [19] [35].

A. Lyapunov Drift

Let $\boldsymbol{Q}(t)$ represent a vector process of discrete time queues that evolves according to some probability law. Let $L(\boldsymbol{Q})$ be a non-negative function of the queue vector. Define the *conditional Lyapunov drift* $\Delta(\boldsymbol{Q}(t))$ as follows:[6]

$$\Delta(\boldsymbol{Q}(t)) \triangleq \mathbb{E}\left\{L(\boldsymbol{Q}(t+1)) - L(\boldsymbol{Q}(t)) \mid \boldsymbol{Q}(t)\right\} \tag{17}$$

Lemma 3: (Lyapunov Drift [19] [35]) Suppose there is a non-negative function $L(\boldsymbol{Q})$, a value $\epsilon > 0$, and two processes $B(t)$ and $h(t)$ such that for all time t and all possible $\boldsymbol{Q}(t)$, we have:

$$\Delta(\boldsymbol{Q}(t)) \leq \mathbb{E}\left\{B(t) - \epsilon h(t) \mid \boldsymbol{Q}(t)\right\}$$

Then:

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\left\{h(\tau)\right\} \leq \limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \frac{\mathbb{E}\left\{B(\tau)\right\}}{\epsilon} \quad \square$$

B. Proof of Theorem 2

Define the Lyapunov function:

$$L(\boldsymbol{Q}) \triangleq \frac{1}{2}\sum_{k=1}^K \left(\sum_{i\in\mathcal{G}_k} Q_i\right)^2 = \frac{1}{2}\sum_{k=1}^K (Q_{sum,k}(t))^2$$

Thus, $L(\boldsymbol{Q}(t))$ is the sum of squares of the total backlog associated with each group $k \in \{1, \ldots, K\}$. To compute $\Delta(\boldsymbol{Q}(t))$, define for each $k \in \{1, \ldots, K\}$

$$\mu_{sum,k}(t) \triangleq \sum_{i\in\mathcal{G}_k} \mu_i(t)$$

Because the sum transmission rate is no more than 1, $\mu_{sum,k}(t)$ represents the transmission rate offered to group k during slot t. Define $\tilde{\mu}_{sum,k}(t)$ to be the actual number of packets transmitted by group k during this slot (so that $\tilde{\mu}_{sum,k}(t) \in \{0,1\}$ and can only be 1 if the group has a non-empty ON queue during slot t). For each group k, we have:

$$Q_{sum,k}(t+1) = Q_{sum,k}(t) - \tilde{\mu}_{sum,k}(t) + A_{sum,k}(t)$$

Squaring both sides of the above equality and using the fact that $\tilde{\mu}_{sum,k}(t)^2 = \tilde{\mu}_{sum,k}(t) \in \{0,1\}$ yields:

$$\frac{Q_{sum,k}(t+1)^2}{2} = \frac{Q_{sum,k}(t)^2}{2} + B_k(t) + Q_{sum,k}(t)A_{sum,k}(t) - Q_{sum,k}(t)\tilde{\mu}_{sum,k}(t)$$

where

$$B_k(t) \triangleq \left[\frac{\tilde{\mu}_{sum,k}(t) + A_{sum,k}(t)^2 - 2A_{sum,k}(t)\tilde{\mu}_{sum,k}(t)}{2}\right]$$

[6]Strictly speaking, the conditional drift should use notation $\Delta(\boldsymbol{Q}(t), t)$ as a general drift may also depend on t, but we use the simpler notation $\Delta(\boldsymbol{Q}(t))$ to formally represent the right hand side of (17).

Taking conditional expectations and summing over all k yields:

$$\Delta(\boldsymbol{Q}(t)) = \mathbb{E}\{B(t) \mid \boldsymbol{Q}(t)\} + \sum_{k=1}^{K} Q_{sum,k}(t)\lambda_{sum,k} - \sum_{k=1}^{K} Q_{sum,k}(t)\mathbb{E}\{\tilde{\mu}_{sum,k}(t) \mid \boldsymbol{Q}(t)\} \quad (18)$$

where $B(t) \triangleq \sum_{k=1}^{K} B_k(t)$, and where we have used the fact that arrivals are i.i.d. over slots and hence have expected values that are independent of the current queue state.

Given $\boldsymbol{Q}(t)$ and the channel states, the LCG algorithm is designed to choose transmission rates that maximize the expression $\sum_k Q_{sum,k}(t)\tilde{\mu}_{sum,k}(t)$ over all possible transmission decisions during slot t that are subject to the constraints:

$$\tilde{\mu}_{sum,k}(t) \in \{0,1\} \text{ for all } k \in \{1,\ldots,K\} \quad (19)$$

$$\sum_{k=1}^{K} \tilde{\mu}_{sum,k}(t) \leq 1 \quad (20)$$

$$\tilde{\mu}_{sum,k}(t) \leq 1_k(t) \text{ for all } k \in \{1,\ldots,K\} \quad (21)$$

Hence, it also maximizes the conditional expectation of this expression given $\boldsymbol{Q}(t)$. It follows that the LCG algorithm *minimizes the final term in the drift expression (18) over all feasible transmission rate decisions that satisfy the constraints (19)-(21) during slot t*. Therefore, we have:

$$\Delta(\boldsymbol{Q}(t)) \leq \mathbb{E}\{B(t) \mid \boldsymbol{Q}(t)\} + \sum_{k=1}^{K} Q_{sum,k}(t)\lambda_{sum,k} - \sum_{k=1}^{K} Q_{sum,k}(t)\mathbb{E}\{\mu_k^*(t) \mid \boldsymbol{Q}(t)\} \quad (22)$$

where $(\mu_1^*(t),\ldots,\mu_K^*(t))$ represents any transmission rate decision vector that satisfies (19)-(21).

Now recall that, according to the conditions of Theorem 2, we have:

$$(\lambda_{sum,1}+\epsilon,\ldots,\lambda_{sum,K}+\epsilon) \in \Lambda_K$$

where Λ_K is the capacity region of a virtual system with K independent queues with channel probabilities $q_{min,k}$ for $k \in \{1,\ldots,K\}$. Let $\boldsymbol{S}^v(t)$ represent the channel states of this virtual system (having independent entries with $Pr[S_k^v(t) = ON] = q_{min,k}$ for all $k \in \{1,\ldots,K\}$). By Lemma 1, we know there exists a stationary randomized control policy that makes transmission decisions $(\mu_1^v(t),\ldots,\mu_K^v(t))$ as a (potentially random) function of $\boldsymbol{S}^v(t)$, such that:

$$\mathbb{E}\{\mu_k^v(t)\} = \lambda_{sum,k} + \epsilon \text{ for all } k \in \{1,\ldots,K\} \quad (23)$$

Now, for each group $\mathcal{G}_k$ ($k \in \{1,\ldots,K\}$), we define an index $i^*(k) \in \mathcal{G}_k$ as follows: If $Q_{sum,k}(t) = 0$, then choose any queue $i \in \mathcal{G}_k$ and label this choice $i^*(k)$. If $Q_{sum,k}(t) > 0$, choose any queue $i \in \mathcal{G}_k$ such that $Q_i(t) > 0$, and define this queue as $i^*(k)$.

For each $k \in \{1,\ldots,K\}$, let H_k be an independent Bernoulli variable with $Pr[H_k = 1] = q_{min,k}/q_{i^*(k)}$. Note that this is a valid probability because $q_{min,k} \leq q_{i^*(k)}$. Now define *virtual channel states* $\boldsymbol{S}^v(t) = (S_1^v(t),\ldots,S_K^v(t))$ as follows:

$$S_k^v(t) = \begin{cases} ON & \text{if } S_{i^*(k)}(t) = ON \text{ and } H_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

It follows that the virtual channels $\boldsymbol{S}^v(t)$ are independent Bernoulli channels with $Pr[S_k^v(t) = ON] = q_{min,k}$ for all $k \in \{1,\ldots,K\}$ (regardless of $\boldsymbol{Q}(t)$), which is exactly the right distribution to correspond with the virtual system for the capacity region Λ_K. Furthermore, $\{S_k^v(t) = ON\}$ implies that $\{S_{i^*(k)}(t) = ON\}$. Now define a virtual transmission rate vector $\boldsymbol{\mu}^v(t) = (\mu_1^v(t),\ldots,\mu_K^v(t))$ according to the stationary randomized control policy that chooses $\boldsymbol{\mu}^v(t)$ based only on $\boldsymbol{S}^v(t)$, and yields (23). It follows that the virtual transmission rates $\mu_k^v(t)$ yield (23) regardless of $\boldsymbol{Q}(t)$. Further, this virtual rate vector is *feasible* for the virtual system, and so it has at most one non-zero entry, and for each entry $k \in \{1,\ldots,K\}$ it satisfies $\mu_k^v(t) = 0$ if $S_k^v(t) = OFF$.

Now choose *actual* transmission rates $\mu_k^*(t) = \mu_k^v(t)$ if $Q_{sum,k}(t) > 0$, and $\mu_k^*(t) = 0$ if $Q_{sum,k}(t) = 0$. It follows that the $(\mu_1^*(t),\ldots,\mu_K^*(t))$ vector satisfies the constraints (19)-(21). Indeed, it inherits the constraints (19)-(20) from the $(\mu_1^v(t),\ldots,\mu_K^v(t))$ vector. Constraint (21) is satisfied because if $\mu_k^*(t) = 1$, then $Q_{sum,k}(t) > 0$ and $S_k^v(t) = ON$ (so that $S_{i^*(k)}(t) = ON$), implying that there is at least one non-empty connected queue in group $\mathcal{G}_k$.

Furthermore, for any $k \in \{1,\ldots,K\}$ such that $Q_{sum,k}(t) > 0$, we have:

$$\begin{aligned} \mathbb{E}\{\mu_k^*(t) \mid \boldsymbol{Q}(t)\} &= \mathbb{E}\{\mu_k^v(t) \mid \boldsymbol{Q}(t)\} \\ &= \mathbb{E}\{\mu_k^v(t)\} \quad (24) \\ &= \lambda_{sum,k} + \epsilon \quad (25) \end{aligned}$$

where (24) follows because the distribution of the virtual transmission vector $\boldsymbol{\mu}^v(t)$ does not depend on the queue state $\boldsymbol{Q}(t)$, and (25) follows from (23). Plugging the expression for for $\mathbb{E}\{\mu_k^*(t) \mid \boldsymbol{Q}(t)\}$ from (25) into the final term on the right hand side of (22) yields:

$$\Delta(\boldsymbol{Q}(t)) \leq \mathbb{E}\{B(t) \mid \boldsymbol{Q}(t)\} + \sum_{k=1}^{K} Q_{sum,k}(t)\lambda_{sum,k} - \sum_{k=1}^{K} Q_{sum,k}(t)(\lambda_{sum,k} + \epsilon)$$

and thus:

$$\Delta(\boldsymbol{Q}(t)) \leq \mathbb{E}\{B(t) \mid \boldsymbol{Q}(t)\} - \epsilon \sum_{k=1}^{K} Q_{sum,k}(t) \quad (26)$$

The inequality (26) is in the exact form for application of the Lyapunov drift lemma (Lemma 3) with $h(t) \triangleq \sum_k Q_{sum,k}(t)$, and hence:

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1}\sum_{k=1}^{K} \mathbb{E}\{Q_{sum,k}(\tau)\} \leq \limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \frac{\mathbb{E}\{B(\tau)\}}{\epsilon}$$

Because $\mathbb{E}\{A_{sum,k}(t)\} = \lambda_{sum,k}$, $\mathbb{E}\left\{A_{sum,k}^2(t)\right\} = \mathbb{E}\left\{A_{sum,k}^2\right\}$, and $\mathbb{E}\{\tilde{\mu}_{sum,k}(t)\} \leq 1$ for all t, the process $B(t)$ satisfies $\mathbb{E}\{B(t)\} \leq B$ for all t (where B is a finite constant). It follows that the queueing network is strongly stable and hence $\mathbb{E}\{Q_i(t)/t\} \to 0$ [18]. Thus:

$$\lim_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\{\tilde{\mu}_{sum,k}(\tau)\} = \lambda_{sum,k}$$

It follows that:

$$\limsup_{t\to\infty} \tfrac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\{B(\tau)\} = \tfrac{1}{2}\left[\lambda_{tot} + \sum_k \mathbb{E}\left\{A_{sum,k}^2\right\} - 2\sum_k \lambda_{sum,k}^2\right]$$

which completes the proof of Theorem 2.

APPENDIX A — PROOF OF THEOREM 1

The proof closely follows our previous work in [32].

Proof: (Theorem 1 part (a)) Consider a particular queue i, and assume that $Q_i(0) = 0$. Consider the system viewed in continuous time, where $\mu_i(t)$ is viewed as a continuous time process that is constant on unit intervals, so that $\mu_i(t) = \mu_i(\lfloor t \rfloor)$ for all real times t. Let $X_i(t)$ represent the total number of packets that have arrived from stream i up to time t. Let $\tilde{Q}_i(t)$ represent the fractional packets in this system with the same arrivals but operating without the timeslot structure. It is not difficult to show that:

$$Q_i(t) \geq \tilde{Q}_i(t) \text{ for all real time } t \tag{27}$$

and hence $\mathbb{E}\{Q_i(t)\} \geq \mathbb{E}\left\{\tilde{Q}_i(t)\right\}$ for all t. Further, the value of $\tilde{Q}_i(t)$ is given by:

$$\tilde{Q}_i(t) = \sup_{\tau \geq 0}\left[X_i(t) - X_i(t-\tau) - \int_{t-\tau}^{t} \mu_i(v)dv\right]$$

Taking expectations of both sides with respect to the stochastic arrival process X_i (describing $X_i(u)$ for all u such that $0 \leq u \leq t$) yields:

$$\begin{aligned}
&\mathbb{E}\left\{\tilde{Q}_i(t)\right\} \\
&= \mathbb{E}_{X_i}\mathbb{E}_{\mu_i \mid X_i}\left\{\sup_{\tau \geq 0}\left[X_i(t) - X_i(t-\tau) - \int_{t-\tau}^{t} \mu_i(v)dv\right]\right\} \\
&\geq \mathbb{E}_{X_i}\left\{\sup_{\tau \geq 0}\left[X_i(t) - X_i(t-\tau) - \int_{t-\tau}^{t} \mathbb{E}\{\mu_i(v) \mid X_i\}\,dv\right]\right\} \\
&= \mathbb{E}_{X_i}\left\{\sup_{\tau \geq 0}\left[X_i(t) - X_i(t-\tau) - \int_{t-\tau}^{t} \overline{\mu}_i dv\right]\right\}
\end{aligned}$$

where the first inequality follows by Jensen's inequality together with the fact that the $\sup(\cdot)$ operator is convex. The final equality follows because (from property (6)), the expected transmission rate does not depend on the arrival history and is equal to $\overline{\mu}_i$ for all time. However, note that the final expression on the right hand side is equal to $\mathbb{E}_{X_i}\{U_i(t)\}$, where $U_i(t)$ is the unfinished work in a continuous time queueing system with the same inputs but with a constant server rate $\overline{\mu}_i$ for all time. Therefore, we obtain the lower bound:

$$\mathbb{E}\{Q_i(t)\} \geq \mathbb{E}\left\{\tilde{Q}_i(t)\right\} \geq \mathbb{E}\{U_i(t)\} \quad \text{for all } t$$

completing the proof of part (a) of Theorem 1. □

Proof: (Theorem 1 part (b)) Suppose the system is symmetric so that $q_i = q$ and $\lambda_i = \lambda_{tot}/N$ for all $i \in \{1, \ldots, N\}$, and that inputs are Poisson. By part (a), we know that $\mathbb{E}\{Q_i(t)\} \geq \mathbb{E}\{U_i(t)\}$, where $U_i(t)$ is the unfinished work in an $M/D/1$ queue with constant service time $1/\overline{\mu}_i$. Taking t to infinity yields the steady state value, and hence: $\overline{Q}_i \geq \overline{U}_i$. The steady state unfinished work in an $M/D/1$ queue with arrival rate λ_i and constant service time $1/\overline{\mu}_i$ is equal to $\frac{\lambda_i}{2(\overline{\mu}_i - \lambda_i)}$, which can be computed by adding $\lambda_i/(2\overline{\mu}_i)$, the average portion of a packet remaining in the server, to the expression for the average number of packets in the buffer of an $M/D/1$ queue [33]. Because $\lambda_i = \lambda_{tot}/N$, we have:

$$\sum_{i=1}^{N} \overline{Q}_i \geq \sum_{i=1}^{N} \frac{\lambda_{tot}/N}{2(\overline{\mu}_i - \lambda_{tot}/N)}$$

Note that $\sum_{i=1}^{N} \overline{\mu}_i \leq r_N$ (as the sum transmission rate cannot exceed $1 - (1-q)^N$). Therefore, the right hand side in the above inequality is greater than or equal to the solution to:

$$\begin{aligned}
\text{Minimize:} \quad & \textstyle\sum_{i=1}^{N} \frac{\lambda_{tot}/N}{2(\mu_i - \lambda_{tot}/N)} \\
\text{Subject to:} \quad & \textstyle\sum_{i=1}^{N} \mu_i \leq r_N
\end{aligned}$$

The above optimization seeks to minimize a convex symmetric function of $(\mu_1, \ldots, \mu_N)$ over the simplex constraint, and is minimized at the symmetric point $\mu_i = r_N/N$ for all $i \in \{1, \ldots, N\}$. Therefore:

$$\sum_{i=1}^{N} \overline{Q}_i \geq \frac{\lambda_{tot}}{2(r_N/N - \lambda_{tot}/N)} = \frac{N\lambda_{tot}}{2r_N(1-\rho)}$$

where $\rho \triangleq \lambda_{tot}/r_N$. Dividing both sides by λ_{tot} and using Little's Theorem proves the result. □

Proof: (Theorem 1 part (c)) Again from part (a), we have that $\mathbb{E}\{Q_i(t)\} \geq \mathbb{E}\{U_i(t)\}$, where $U_i(t)$ is the unfinished work in a queue with a packet arrival process of rate λ_i and a constant server queue of rate $\overline{\mu}_i$. By Little's Theorem , the steady state expected number of packets in the server is equal to $\lambda_i/\overline{\mu}_i$, and hence the expected unfinished work in the server is equal to $\lambda_i/(2\overline{\mu}_i)$. This is certainly a lower bound on the expected total unfinished work in the system, and hence:

$$\begin{aligned}
\sum_{i=1}^{N} \overline{Q}_i \geq \sum_{i=1}^{N} \frac{\lambda_i}{2\overline{\mu}_i} \quad &\geq \quad \inf_{[(\mu_i) \mid \sum_i \mu_i \leq r_{max}]} \sum_{i=1}^{N} \frac{\lambda_i}{2\mu_i} &(28) \\
&= \quad \frac{1}{2r_{max}}\left(\sum_{i=1}^{N} \sqrt{\lambda_i}\right)^2 &(29)
\end{aligned}$$

where (28) follows because we again have $\sum_i \overline{\mu}_i \leq r_{max}$, and (29) holds because the solution to the convex optimization problem in the previous line is given by $\mu_i = r_{max}\sqrt{\lambda_i}/(\sum_j \sqrt{\lambda_j})$, which can be proven with a simple Lagrange multiplier argument. Because there are at least $\gamma_1 N$ values of λ_i that are greater than or equal to $\gamma_2 \lambda_{tot}/N$, the right hand side of (29) is greater than or equal to $\gamma_1^2 \gamma_2 N \lambda_{tot}/(2r_{max})$. Dividing by λ_{tot} bounds the average delay and proves the result. □

APPENDIX B — STOCHASTIC INEQUALITIES

Here we derive the stochastic comparison result stated in Section IV-A. We first review basic stochastic inequality facts for any random variables X and Y (see [36]).

Definition 2: A random variable X is said to be *stochastically less than* a random variable Y (written $X \underset{st.}{\leq} Y$) if:

$$Pr[X > \omega] \leq Pr[Y > \omega] \text{ for all real values } \omega$$

Lemma 4: (Stochastic Coupling [36]) $X \underset{st.}{\leq} Y$ if and only if there exists a third random variable $\hat{X}$ (that lies on the same probability space as X), such that $X \leq \hat{X}$, and $\hat{X}$ has the same probability distribution as Y.

Now let $A_i(t)$ denote the number of packets that arrive to queue i of the multi-queue system during slot t. We view $\{A_i(t)\}_{i=1}^{N}$ as a general set of discrete time arrival processes, possibly correlated over timeslots and across queues. Let $S_i(t)$

represent the ON/OFF state of channel i during slot t, assumed to be independent over timeslots and across queues with $Pr[S_i(t) = ON] = q_i$ for all t and for each $i \in \{1, \ldots, N\}$. Consider any arbitrary server allocation policy, and let $Q_i(t)$ represent the resulting number of packets in queue i during slot t under this policy. Define the total sum backlog as follows: $Q_{sum}(t) \triangleq \sum_{i=1}^{N} Q_i(t)$.

Define $q_{min} \triangleq \min_{i \in \{1,\ldots,N\}} q_i$. Consider a single discrete time queue with i.i.d. Bernoulli service opportunities with rate q_{min}, and with arrival process: $A_{sum}(t) \triangleq \sum_{i=1}^{N} A_i(t)$. That is, the arrivals to this single queue are identical to the sum arrival process of the multi-queue system. Let $Q_{single}(t)$ represent the number of packets in the single queue system at time t. Assume that both the single and multi-queue systems are initially empty, so that $Q_{sum}(0) = Q_{single}(0) = 0$.

Theorem 5: (Stochastic Inequality) If the multi-queue system uses any *work conserving* scheduling policy, i.e., any policy that always places a server to a non-empty ON queue if there is one available, then for all $t \in \{0, 1, 2, \ldots\}$:

$$Q_{sum}(t) \underset{st.}{\leq} Q_{single}(t)$$

Proof: Consider a *new* multi-queue system with queues $\hat{Q}_i(t)$, $i \in \{1, \ldots, N\}$. The new queueing system is initially empty and has exactly the same arrival processes $A_i(t)$ and channel state processes $S_i(t)$ as the original multi-queue system. However, it has a different server allocation rule, described as follows: At any slot t, if $Q_{sum}(t) = 0$ (that is, if the original multi-queue system is empty), then define i^* as the smallest index such that $\hat{Q}_{i^*}(t) > 0$ (define $i^* \triangleq 1$ if all queues of the new system are also empty). The new multi-queue system independently allocates a server to queue i^* with probability q_{min}/q_{i^*}, and else remains idle. Note that this server is allocated independent of the channel state $S_{i^*}(t)$. If this channel is ON, the new multi-queue system has a service opportunity. It follows that this service opportunity arises independently with probability q_{min}.

Similarly, at any slot t in which $Q_{sum}(t) > 0$, define $\Theta(t)$ to be the set of non-empty queue indices of the original multi-queue system. Define $q(t)$ as the probability that at least one of these non-empty queues has an ON channel state:

$$q(t) \triangleq 1 - \Pi_{i \in \Theta(t)} (1 - q_i)$$

Note that $q_{min} \leq q(t)$ for all t such that $Q_{sum}(t) > 0$. If the original multi-queue system does not serve any packet during this slot, then the new multi-queue system also remains idle on this slot. If the original multi-queue system serves a packet from a queue i during this slot, then independently with probability $q_{min}/q(t)$ the new multi-queue system allocates a server to queue i. Else, the new multi-queue system remains idle. It follows that this service opportunity in the new system also arises independently with probability q_{min}.

Claim 1: $Q_i(t) \leq \hat{Q}_i(t)$ for all $t \in \{0, 1, 2, \ldots\}$ and all $i \in \{1, \ldots, N\}$.

This claim follows because whenever $Q_i(t)$ is non-empty but no packet is served from this queue, then no packet is served from queue $\hat{Q}_i(t)$. It is not difficult to show this implies that $Q_i(t) \leq \hat{Q}_i(t)$ for all t, proving Claim 1.

Now define $\hat{Q}_{sum}(t) \triangleq \sum_{i=1}^{N} \hat{Q}_i(t)$, and note by Claim 1 that for all $t \in \{0, 1, 2, \ldots\}$:

$$Q_{sum}(t) \leq \hat{Q}_{sum}(t) \tag{30}$$

Claim 2: For all timeslots $t \in \{0, 1, 2, \ldots, \}$, $\hat{Q}_{sum}(t)$ has the same probability distribution as $Q_{single}(t)$.

This claim follows because the single queue system is initially empty, has arrival process $A_{sum}(t)$, and whenever it is non-empty it independently serves a packet with probability q_{min}. Likewise, the new multi-queue system is initially empty and has arrival process $A_{sum}(t)$. Further, whenever it is non-empty, it independently serves a packet with probability q_{min}. Thus, the two processes are stochastically equivalent.

Claim 2 and (30) together prove the result. □

REFERENCES

[1] M. J. Neely. Order optimal delay for opportunistic scheduling in multi-user wireless uplinks and downlinks. *Proc. of Allerton Conf. on Communication, Control, and Computing (invited paper)*, Sept. 2006.

[2] L. Tassiulas and A. Ephremides. Dynamic server allocation to parallel queues with randomly varying connectivity. *IEEE Transactions on Information Theory*, vol. 39, pp. 466-478, March 1993.

[3] N. Kahale and P. E. Wright. Dynamic global packet routing in wireless networks. *Proc. IEEE INFOCOM*, 1997.

[4] M. Andrews, K. Kumaran, K. Ramanan, A. Stolyar, and P. Whiting. Providing quality of service over a shared wireless link. *IEEE Communications Magazine*, vol. 39, no.2, pp.150-154, 2001.

[5] E. M. Yeh. *Multiaccess and Fading in Communication Networks*. PhD thesis, Massachusetts Institute of Technology, Laboratory for Information and Decision Systems (LIDS), 2001.

[6] L. Li and A. Goldsmith. Capacity and optimal resource allocation for fading broadcast channels: Part i: Ergodic capacity. *IEEE Trans. Inform. Theory*, pp. 1083-1102, March 2001.

[7] S. Shakkottai, R. Srikant, and A. Stolyar. Pathwise optimality of the exponential scheduling rule for wireless channels. *Advances in Applied Probability*, vol. 36, no. 4, pp. 1021-1045, Dec. 2004.

[8] M. J. Neely, E. Modiano, and C. E. Rohrs. Power allocation and routing in multi-beam satellites with time varying channels. *IEEE Transactions on Networking*, vol. 11, no. 1, pp. 138-152, Feb. 2003.

[9] X. Liu, E. K. P. Chong, and N. B. Shroff. A framework for opportunistic scheduling in wireless networks. *Computer Networks*, vol. 41, no. 4, pp. 451-474, March 2003.

[10] N. Jindal and A. Goldsmith. Capacity and optimal power allocation for fading broadcast channels with minimum rates. *IEEE Transactions on Information Theory*, vol. 49, no. 11, Nov. 2003.

[11] E. M. Yeh and A. S. Cohen. Throughput and delay optimal resource allocation in multiaccess fading channels. *Proc. Int. Symp. on Information Theory (ISIT)*, May 2003.

[12] M. Sharif and B. Hassibi. A delay analysis for opportunistic transmission in fading broadcast channels. *Proc. IEEE INFOCOM*, 2005.

[13] L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Transacations on Automatic Control*, vol. 37, no. 12, pp. 1936-1949, Dec. 1992.

[14] M. J. Neely, E. Modiano, and C E Rohrs. Dynamic power allocation and routing for time varying wireless networks. *IEEE Journal on Selected Areas in Communications*, vol. 23, no. 1, pp. 89-103, January 2005.

[15] A. Ganti, E. Modiano, and J. N. Tsitsiklis. Transmission scheduling for multi-channel satellite and wireless networks. *Proceedings of the 40th Annual Allerton Conf. on Communication, Control, and Computing*, Oct. 2002.

[16] A. Ganti, E. Modiano, and J. N. Tsitsiklis. Optimal transmission scheduling in symmetric communication models with intermittent connectivity. *IEEE Transactions on Information Theory*, vol. 53, no. 3, March 2007.

[17] M. J. Neely, E. Modiano, and C. Li. Fairness and optimal stochastic control for heterogeneous networks. *Proc. IEEE INFOCOM*, March 2005.

[18] M. J. Neely. Energy optimal control for time varying wireless networks. *IEEE Transactions on Information Theory*, vol. 52, no. 7, pp. 2915-2934, July 2006.

[19] L. Georgiadis, M. J. Neely, and L. Tassiulas. Resource allocation and cross-layer control in wireless networks. *Foundations and Trends in Networking*, vol. 1, no. 1, pp. 1-149, 2006.

[20] J. W. Lee, R. R. Mazumdar, and N. B. Shroff. Opportunistic power scheduling for dynamic multiserver wireless systems. *IEEE Transactions on Wireless Communications*, vol. 5, no.6, pp. 1506-1515, June 2006.

[21] A. Eryilmaz and R. Srikant. Fair resource allocation in wireless networks using queue-length-based scheduling and congestion control. *Proc. IEEE INFOCOM*, March 2005.

[22] A. Stolyar. Maximizing queueing network utility subject to stability: Greedy primal-dual algorithm. *Queueing Systems*, vol. 50, pp. 401-457, 2005.

[23] M. J. Neely. Optimal energy and delay tradeoffs for multi-user wireless downlinks. *Proc. IEEE INFOCOM*, April 2006.

[24] M. J. Neely. Super-fast delay tradeoffs for utility optimal fair scheduling in wireless networks. *IEEE Journal on Selected Areas in Communications, Special Issue on Nonlinear Optimization of Communication Systems*, vol. 24, no. 8, pp. 1489-1501, Aug. 2006.

[25] N. McKeown, V. Anantharam, and J. Walrand. Achieving 100% throughput in an input-queued switch. *Proc. IEEE INFOCOM*, 1996.

[26] E. Leonardi, M. Mellia, F. Neri, and M. Ajmone Marsan. Bounds on average delays and queue size averages and variances in input-queued cell-based switches. *Proc. IEEE INFOCOM*, 2001.

[27] P. R. Kumar and S. P. Meyn. Stability of queueing networks and scheduling policies. *IEEE Trans. on Automatic Control*, vol.40,.n.2, pp.251-260, Feb. 1995.

[28] A. Mekkittikul and N. McKeown. A practical scheduling algorithm to achieve 100% throughput in input-queued switches. *Proc. IEEE INFOCOM*, 1998.

[29] J. G. Dai and B. Prabhakar. The throughput of data switches with and without speedup. *Proc. IEEE INFOCOM*, 2000.

[30] D. Shah. Maximal matching scheduling is good enough. *Proc. IEEE Globecom*, Dec. 2003.

[31] X. Wu and R. Srikant. Bounds on the capacity region of multi-hop wireless networks under distributed greedy scheduling. *Proc. IEEE INFOCOM*, April 2006.

[32] M. J. Neely, E. Modiano, and Y.-S. Cheng. Logarithmic delay for $n \times n$ packet switches under the crossbar constraint. *IEEE Transactions on Networking*, vol. 15, no. 3, pp. 657-668, June 2007.

[33] D. P. Bertsekas and R. Gallager. *Data Networks*. New Jersey: Prentice-Hall, Inc., 1992.

[34] Søren Asmussen. *Applied Probability and Queues, Second Edition*. New York: Spring-Verlag, 2003.

[35] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Springer-Verlag, London, 1993.

[36] S. Ross. *Stochastic Processes*. John Wiley & Sons, Inc., New York, 1996.