The Optimality of Two Prices: Maximizing Revenue in a Stochastic Network

Longbo Huang, Michael J. Neely

***Abstract*— This paper considers the problem of pricing and transmission scheduling for an Access Point (AP) in a wireless network, where the AP provides service to a set of mobile users. The goal of the AP is to maximize its own time-average profit. We first obtain the optimum time-average profit of the AP and prove the "Optimality of Two Prices" theorem. We then develop an online scheme that jointly solves the pricing and transmission scheduling problem in a dynamic environment. The scheme uses an admission price and a business decision as tools to regulate the incoming traffic and to maximize revenue. We show the scheme can achieve any average profit that is arbitrarily close to the optimum, with a tradeoff in average delay. This holds for general Markovian dynamics for channel and user state variation, and does not require a-priori knowledge of the Markov model. The model and methodology developed in this paper are general and apply to other stochastic settings where a single party tries to maximize its time-average profit.**

***Index Terms*— Wireless Mesh Network, Pricing, Queueing, Dynamic Control, Lyapunov analysis, Optimization**

I. INTRODUCTION

In this paper, we consider the profit maximization problem of an access point (AP) in a wireless mesh network. Mobile users connect to the mesh network via the AP. The AP receives the user data and transmits it to the larger network via a wireless link. Time is slotted with integral slot boundaries $t \in \{0, 1, 2, \ldots\}$, and every timeslot the AP chooses an admission price $p(t)$ (cost per unit packet) and announces this price to all present mobile users. The users react to the current price by sending data, which is queued at the AP. While the AP gains revenue by accepting this data, it in turn has to deliver all the admitted packets by transmitting them over its wireless link. Therefore, it incurs a transmission cost for providing this service (for example, the cost might be proportional to the power consumed due to transmission). The mission of the AP is to find strategies for both packet admission and packet transmission so as to maximize its time average profit while ensuring queue stability.

We assume that the expected number of new packets sent to the AP is determined every timeslot by a *demand state variable* $M(t)$ and a *user demand function* $F(M(t), p(t))$. Specifically, the state variable $M(t)$ represents the current condition of the user population that affects its aggregate spending ability. For example, $M(t)$ can represent the integer number of users present at time t, or can be a rough estimate of the aggregate "willingness-to-pay" (such as "Low," "Medium," and "High"). The demand function $F(M(t), p(t))$ is equal to the expected number of packets that arrive on slot t under a given user condition $M(t)$ and a given price $p(t)$. We assume the AP knows the current demand state $M(t)$ and the demand function $F(M(t), p(t))$ for each slot t. However, $M(t)$ is assumed to vary according to a general finite state ergodic Markov chain, and the transition and steady state probabilities of $M(t)$ may be unknown. Similarly, the condition of the wireless channel from AP to the mesh network is potentially time varying and is determined by a Markov modulated channel state process $S(t)$. The AP is assumed to know the current channel state $S(t)$ on each timeslot t, although the transition and steady state probabilities of $S(t)$ are potentially unknown.

We develop a joint pricing and transmission scheduling algorithm (PTSA) for the AP. The PTSA algorithm has low complexity and can be viewed as making greedy decisions every timeslot. Despite its simplicity, we show that PTSA is able to dynamically react to the time varying network conditions. It yields an average net profit that can be pushed arbitrarily close to the optimum, with a corresponding tradeoff in average queueing delay.

Many existing works on revenue maximization can be found. Work in [1] [2] models the problem of maximizing revenue as a dynamic program. Work in [3] and [4] model revenue maximization as static convex optimization problems. A game theoretic perspective is considered in [5], where equilibrium results are obtained. Works [6], [7] and [8] also use game theoretic approaches with the goal of obtaining efficient strategies for both the AP and the users. The paper [9] looks at the problem from a mechanism design perspective, and [10], [11] consider profit maximization with Qos guarantees. Early work on network pricing in [12], [13], and [14] consider throughput-utility maximization rather than revenue maximization. There, prices play the role of Lagrange multipliers, and are used mainly to facilitate better utilization of the shared network resource. This is very different from the revenue maximization problem, where the service provider is only interested in its own profit. Indeed, the revenue maximization problem can be much more complex due to non-convexity issues.

The above prior work does not directly solve the profit maximization problem for APs in a wireless network for one or more of the following reasons: (1) Most works consider time-invariant systems, i.e. the network condition does not change with time. (2) Works that model the problem as an optimization problem rely heavily on the assumption that the

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user utility function or the demand function is concave. (3) Many of the prior works adopt the flow rate allocation model, where a single fixed operating point is obtained and used for all time. However, in a wireless network, the network condition can easily change due to channel fading and/or node mobility, so that a fixed resource allocation decision may not be efficient. Also, as has been pointed out in [15], the user utility function does not always have the concavity property. Indeed, profit maximization problems are often non-convex in nature. Hence, they are generally hard to solve, even in the static case where the channel condition, user condition, and demand function is fixed for all time. It is also common to look for single-price solutions in these static network problems. Our results show that single-price solutions are not always optimal, and that even for static problems the AP can only maximize time average profit by providing a "regular" price some fraction of the time, and a "reduced price" at other times.

Moreover, most network pricing work considers flow allocation that neglects the packet-based nature of the traffic, and neglects issues of queueing delay. An exception is the recent work in [16] that considers a packet-based model for a free market wireless network. However, [16] focuses on network-wide efficiency and on guarantees of non-negative profit to all participants, and does not consider the very different problem of maximizing revenue for a single AP.

In order to enable the AP to better react to the varying network condition, to overcome the difficulty of solving non-convex/non-concave optimization problems, and to better operate in a packet-based setting, we propose a novel joint pricing and transmission scheduling algorithm (PTSA). PTSA has the same nature as the schemes proposed in [16], which are "state-dependent" [12], although it solves a very different problem. As we will see later, PTSA bypasses the non-concavity/non-convexity difficulty by turning the static optimization problem into a stochastic optimization problem. Our analysis of the performance of PTSA uses the Lyapunov techniques and general utility-optimization framework developed in [17] [18] [19]. We show that PTSA can achieve a time-average profit that is arbitrarily close to optimum, and obtain an explicit tradeoff between profit and queuing delay.

In the next section we describe the network model. In Section III we characterize the optimal time average profit and prove the "Optimality of Two Prices" theorem. The PTSA algorithm is presented in Section IV, where performance optimality is proven. Preliminary simulation results are provided in Section V.

II. Network Model

We consider the network as shown in Fig 1. The network is assumed to operate in slotted time, i.e. $t \in \{0, 1, 2, ...\}$.

A. Arrival Model: The Demand Function

We first describe the packet arrival model. Let $M(t)$ be the demand state at time t. $M(t)$ might be the number of present mobile users, or could represent the current demand situation, such as the demand being "High", "Medium" or "Low." We assume that $M(t)$ evolves according to a finite state ergodic Markov chain with state space $\mathcal{M}$. Let π_m represent the steady state probability that $M(t) = m$. The value of $M(t)$ is assumed known at the beginning of each slot t, although the transition and steady state probabilities are potentially unknown.

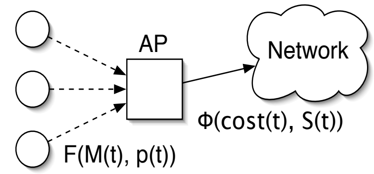

Fig. 1. An Access Point (AP) that connects mobile users to a larger network.

Every timeslot, the AP first makes a *business decision* by deciding whether or not to allow new data (this decision can be based on knowledge of the current $M(t)$ state). Let $Z(t)$ be a $0/1$ variable for this decision, defined as:[1]

$$Z(t) = \begin{cases} 1 & \text{if the AP allows new data on slot } t \\ 0 & \text{else} \end{cases} \quad (1)$$

If the AP chooses $Z(t) = 1$, it then chooses a per-unit price $p(t)$ for incoming data and advertises this price to the mobile users. We assume that price is restricted to a compact set of price options $\mathcal{P}$, so that $p(t) \in \mathcal{P}$ for all t. We assume the set $\mathcal{P}$ includes the constraint that prices are non-negative and bounded by some finite maximum price p_{max}. Let $R(t)$ be the total number of packets that are sent by the mobile users in reaction to this price. The income earned by the AP on slot t is thus $Z(t)R(t)p(t)$.

The arrival $R(t)$ is a random variable that depends on the demand state $M(t)$ and the current price $p(t)$ via a demand function $F(M(t), p(t))$:

$$F : (M(t), p(t)) \mapsto \mathbb{E}\{R(t)\} \quad (2)$$

Specifically, the demand function maps $M(t)$ and $p(t)$ into the *expected value* of arrivals $\mathbb{E}\{R(t)\}$. We further assume that there is a maximum value R_{max}, so that $R(t) \leq R_{max}$ for all t, regardless of $M(t)$ and $p(t)$. The higher order statistics for $R(t)$ (beyond its expectation and its maximum value) are arbitrary. The random variable $R(t)$ is assumed to be conditionally independent of past history given the current $M(t)$ and $p(t)$. The demand function $F(m, p)$ is only assumed to be continuous and to satisfy $0 \leq F(m, p) \leq R_{max}$ for all $m \in \mathcal{M}$ and all $p \in \mathcal{P}$. The set $\mathcal{P}$ is assumed only to be compact (i.e., closed and bounded), and may consist of a finite discrete set of prices.

Example: In the case when $M(t)$ represents the number of mobile users in range of the AP at time t, a useful example model for $F(M(t), p(t))$ is:

$$F(M(t), p(t)) = M(t)\hat{F}(p(t))$$

where $\hat{F}(p)$ is the expected number of packets sent by a single user in reaction to price p, a curve that is possibly obtained

[1]The $Z(t)$ decisions are introduced to allow stability even in the possible situation where user demand is so high that incoming traffic would exceed transmission capabilities, even if price were set to its maximum value p_{max}.

via empirical data. In this case, we assume that the number of users is bounded by some value M_{max} and the maximum number of packets sent by any single user is bounded by some value R_{max}^{single}, so that $R_{max} = M_{max} R_{max}^{single}$.

In Section IV, we show that this type of demand function (i.e, $F(m,p) = m\hat{F}(p)$) leads to an interesting situation where the AP can make "*demand state blind*" pricing decisions, where prices are chosen without knowledge of $M(t)$.

B. Transmission Model: The Rate-Cost Function

Let $S(t)$ represent the channel condition of the wireless link from AP to the mesh network on slot t. We assume that the channel state process $S(t)$ is a finite state ergodic Markov chain with state space $\mathcal{S}$. Let π_s represent the steady state probability that $S(t) = s$. The transition and steady state probabilities of $S(t)$ are potentially unknown to the AP, although we assume the AP knows the current $S(t)$ value at the beginning of each slot t.

We assume that the transmission rate of the AP's outgoing link is determined every timeslot by a resource allocation decision (such as power) and by the current channel state $S(t)$. We model this decision completely by its *cost* to the AP, and define $cost(t)$ as the cost of the transmission decision on slot t. We assume that $cost(t)$ is chosen within some compact set of costs $\mathcal{C}$, and that $\mathcal{C}$ includes the constraint $0 \leq cost \leq C_{max}$ for some finite maximum cost C_{max}. The transmission rate is then given by *rate-cost*[2] function $\mu(t) = \Phi\left(cost(t), S(t)\right)$. In our problem, we assume that $\Phi\left(cost, S(t)\right)$ is continuous in the variable $cost$ for every given $S(t)$, and that $\Phi(0, S(t)) = 0$ for all $S(t)$. Further, we assume there is a finite maximum transmission rate, so that:

$$\Phi\left(cost(t), S(t)\right) \leq \mu_{max} \quad \text{for all } cost(t), S(t), t \tag{3}$$

We assume that packets can be continuously split, so that $\mu(t) = \Phi\left(cost(t), S(t)\right)$ determines the portion of packets that can be sent over the link from AP to the network on slot t (for this reason, the rate function can also be viewed as taking units of *bits*). Of course, the set $\mathcal{C}$ can be restricted to a finite set of costs that correspond to integral units for $\Phi(cost(t), S(t))$ in systems where packets cannot be split.

C. Queueing Dynamics and other Notations

Let $U(t)$ be the queue backlog of the AP at time t, in units of packets.[3] Note that this is a *single commodity* problem as we do not distinguish packets from different users.[4] We assume the following queueing dynamics for $U(t)$:

$$U\left(t+1\right) = \max\left[U(t) - \mu(t), 0\right] + Z(t)R(t) \tag{4}$$

where $\mu(t) = \Phi(cost(t), S(t))$. Throughout the paper, we adopt the following notion of queue stability:

$$\mathbb{E}\left\{U\right\} \triangleq \limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\left\{U(\tau)\right\} < \infty \tag{5}$$

[2]This is essentially the same as the rate-power curve in [18].

[3]The packet units can be fractional. Alternatively, the backlog could be expressed in units of *bits*.

[4]Our analysis can be extended to treat multi-commodity models, although that is omitted for brevity.

III. Characterizing The Maximum Profit

In this section, we characterize the optimal average profit that is achievable over the class of all possible control polices that stabilize the queue at the AP. We show that it suffices for the AP to use only *two prices* for every demand state $M(t)$ to maximize its profit.

A. The Maximum Profit

To describe the maximum average profit, we use an analysis that is similar to the analysis of the minimum average power for stability problem in [18]. Note that the theorem in [18] considers the problem of using minimum average power to serve the given incoming traffic, while in our case, the AP needs to balance between the profit from data admission and the cost for packet transmission. The following theorem shows that optimality can be achieved over the class of stationary randomized pricing and transmission scheduling strategies with the following structure: Every slot the AP observes $M(t) = m$, and makes a business decision $Z(t)$ by independently and randomly choosing $Z(t) = 1$ with probability $\phi^{(m)}$ (for some $\phi^{(m)}$ values defined for each $m \in \mathcal{M}$). If $Z(t) = 1$, then the AP allocates a price randomly from a countable *collection of prices* $\{p_1^{(m)}, p_2^{(m)}, p_3^{(m)}, \ldots\}$, with probabilities $\{\alpha_k^{(m)}\}_{k=1}^{\infty}$. Similarly, the AP observes $S(t) = s$ and makes a transmission decision by choosing $cost(t)$ randomly from a set of costs $\{cost_k^{(s)}\}_{k=1}^{\infty}$ with probabilities $\{\beta_k^{(s)}\}_{k=1}^{\infty}$.

Theorem 1: (Maximum Profit with Stability) The optimal average profit for the AP, with its queue being stable, is given by $Profit_{av}^{opt}$, where $Profit_{av}^{opt}$ is the solution to the following optimization problem:

$$\max \;\; Profit_{av} = Income_{av} - Cost_{av} \tag{6}$$

$$\text{s.t.} \;\; Income_{av} = \mathbb{E}_m\left\{\phi^{(m)}\sum_{k=1}^{\infty}\alpha_k^{(m)}F(m, p_k^{(m)})p_k^{(m)}\right\} \tag{7}$$

$$Cost_{av} = \mathbb{E}_s\left\{\sum_{k=1}^{\infty}\beta_k^{(s)}cost_k^{(s)}\right\} \tag{8}$$

$$\lambda_{av} = \mathbb{E}_m\left\{\phi^{(m)}\sum_{k=1}^{\infty}\alpha_k^{(m)}F(m, p_k^{(m)})\right\} \tag{9}$$

$$\mu_{av} = \mathbb{E}_s\left\{\sum_{k=1}^{\infty}\beta_k^{(s)}\Phi\left(cost_k^{(s)}, s\right)\right\} \tag{10}$$

$$0 \leq \phi^{(m)} \leq 1 \qquad \forall m \in \mathcal{M} \tag{11}$$

$$\mu_{av} \geq \lambda_{av} \tag{12}$$

$$p_k^{(m)} \in \mathcal{P} \qquad \forall k, \forall m \in \mathcal{M} \tag{13}$$

$$cost_k^{(s)} \in \mathcal{C}, \qquad \forall k, \forall s \in \mathcal{S} \tag{14}$$

$$\sum_{k=1}^{\infty}\alpha_k^{(m)} = 1 \qquad \forall m \in \mathcal{M} \tag{15}$$

$$\sum_{k=1}^{\infty}\beta_k^{(s)} = 1 \qquad \forall s \in \mathcal{S} \tag{16}$$

where $\mathbb{E}_s$ and $\mathbb{E}_m$ denote the expectation over the steady state distribution for $S(t)$ and $M(t)$, respectively, and $\phi^{(m)}$, $\alpha_k^{(m)}$, $p_k^{(m)}$, $\beta_k^{(s)}$, and $cost_k^{(s)}$ are auxiliary variables with the interpretation given in the text preceding Theorem 1.

The proof of Theorem 1 contains two parts. Part I shows that no algorithm that stabilizes the AP can achieve an average profit that is larger than the optimal solution of the problem (6)-(16). Part II shows that we can achieve a profit of at least $\rho Profit_{av}^{opt}$ (for any ρ such that $0 < \rho < 1$) with a particular stationary randomized algorithm that also yields average arrival and transmission rates λ_{av} and μ_{av} that satisfy $\lambda_{av} < \mu_{av}$. The formal proof is given in Appendix A.

Because the sets $\mathcal{P}, \mathcal{C}$ are compact and the functions $F(m,p)$ and $\Phi(cost, s)$ are continuous for all $m \in \mathcal{M}$ and $s \in \mathcal{S}$ (where $\mathcal{M}$ and $\mathcal{S}$ have finite state space), it can be shown that $Profit_{av}^{opt}$ can be achieved by a particular stationary randomized algorithm.[5] The following important corollary to Theorem 1 is somewhat simpler and is useful for analysis of the online algorithm described in Section IV.

***Corollary** 1:* There exists a control algorithm $STAT^*$ that makes stationary and randomized business and pricing decisions $Z^*(t)$ and $p^*(t)$ depending only on the current demand state $M(t)$ (and independent of queue backlog), and makes stationary randomized transmission decisions $cost^*(t)$ depending only on the current channel state $S(t)$ (and independent of queue backlog) such that:

$$\mathbb{E}\{Z^*(t)R^*(t)\} \leq \mathbb{E}\{\mu^*(t)\} \tag{17}$$

$$\mathbb{E}\{Z^*(t)p^*(t)F(M(t),p^*(t))\} - \mathbb{E}\{cost^*(t)\} = Profit_{av}^{opt} \tag{18}$$

where $Profit_{av}^{opt}$ is the optimal time average profit, and where $\mu^*(t) = \Phi(cost^*(t), S(t))$. The above expectations are taken with respect to the steady state distributions for $M(t)$ and $S(t)$. Specifically:

$$\begin{aligned} \mathbb{E}\{Z^*(t)R^*(t)\} &= \mathbb{E}_m\{Z^*(t)F(m,p^*(t))\} \\ \mathbb{E}\{\mu^*(t)\} &= \mathbb{E}_s\{\Phi(cost^*(t),s)\} \quad \square \end{aligned}$$

B. The Optimality of Two Prices

The following two theorems show that instead of considering a countably infinite collection of prices $\{p_1^{(m)}, p_2^{(m)}, \ldots\}$ for the stationary algorithm of Corollary 1, it suffices to consider only *two* price options for each distinct demand state $M(t) \in \mathcal{M}$.

***Theorem** 2:* Let $(\lambda^{(m)*}, Income^{(m)*})$ represent any rate-income tuple formed by a stationary randomized algorithm that chooses $Z(t) \in \{0,1\}$ and $p(t) \in \mathcal{P}$, so that:

$$\begin{aligned} \mathbb{E}\{Z(t)F(M(t),p(t)) \mid M(t) = m\} &= \lambda^{(m)*} \\ \mathbb{E}\{Z(t)p(t)F(M(t),p(t)) \mid M(t) = m\} &= Income^{(m)*} \end{aligned}$$

Then:

a) $(\lambda^{(m)*}, Income^{(m)*})$ can be expressed as a convex combination of at most three points in the set $\Omega^{(m)}$, defined:

$$\Omega^{(m)} \triangleq \{(ZF(m,p), ZpF(m,p)) \mid Z \in \{0,1\}, p \in \mathcal{P}\}$$

b) If $(\lambda^{(m)*}, Income^{(m)*})$ is on the boundary of the convex hull of $\Omega^{(m)}$, then it can be expressed as a convex combination of at most two elements of $\Omega^{(m)}$, corresponding to at most two business-price tuples (Z_1, p_1), (Z_2, p_2).

c) If the demand function $F(m,p)$ is continuous in p for each $m \in \mathcal{M}$, and if the set of price options $\mathcal{P}$ is *connected*, then any $(\lambda^{(m)*}, Income^{(m)*})$ point (possibly not on the boundary of the convex hull of $\Omega^{(m)}$) can be expressed as a convex combination of at most two elements of $\Omega^{(m)}$.

Proof: Part (a): It is known that for any vector random variable $\vec{X}$ that takes values within a set Ω, the expected value $\mathbb{E}\{\vec{X}\}$ is in the convex hull of Ω (see, for example, Appendix 4.B in [17]). Therefore, the 2-dimensional point $(\lambda^{(m)*}; Income^{(m)*})$ is in the convex hull of the set $\Omega^{(m)}$. By Caratheodory's theorem (see, for example, [20]), any point in the convex hull of the 2-dimensional set $\Omega^{(m)}$ can be achieved by a convex combination of at most three elements of $\Omega^{(m)}$.

Part (b): We know from part (a) that $(\lambda^{(m)*}, Income^{(m)*})$ can be expressed as a convex combination of at most three elements of $\Omega^{(m)}$ (say, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_3$). Suppose these elements are distinct. Because $(\lambda^{(m)*}, Income^{(m)*})$ is on the boundary of the convex hull of $\Omega^{(m)}$, it cannot be in the interior of the triangle formed by $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_3$. Hence, it must be on an edge of the triangle, so that it can be reduced to a convex combination of two or fewer of the $\boldsymbol{\omega}_i$ points.

Part (c): We know from part (a) that $(\lambda^{(m)*}, Income^{(m)*})$ is in the convex hull of the 2-dimensional set $\Omega^{(m)}$. An extension to Caratheodory's theorem in [21] shows that any such point can be expressed as a convex combination of at most *two* points in $\Omega^{(m)}$ if $\Omega^{(m)}$ is the union of at most two connected components. The set $\Omega^{(m)}$ can clearly be written:

$$\Omega^{(m)} = \{(0;0)\} \cup \{(F(m,p); pF(m,p)) \mid p \in \mathcal{P}\}$$

which corresponds to the cases $Z = 0$ and $Z = 1$. Let $\hat{\Omega}^{(m)}$ represent the set on the right hand side of the above union, so that $\Omega^{(m)} = \{(0;0)\} \cup \hat{\Omega}^{(m)}$. Because the $F(m,p)$ function is continuous in p for each $m \in \mathcal{M}$, the set $\hat{\Omega}^{(m)}$ is the image of the connected set $\mathcal{P}$ through the continuous function $(F(m,p), pF(m,p))$, and hence is itself connected [22]. Thus, $\Omega^{(m)}$ is the union of at most two connected components. It follows that $(\lambda^{(m)*}; Income^{(m)*})$ can be achieved via a convex combination of at most two elements in $\Omega^{(m)}$. $\square$

***Theorem** 3:* (Optimality of Two Prices) Let $(\lambda^*, Income^*)$ represent the rate-income tuple corresponding to any stationary randomized policy $Z^*(t)$, $p^*(t)$, $cost^*(t)$, possibly being the policy of Corollary 1 that achieves an optimal profit $Profit_{av}^{opt}$. Specifically, assume the algorithm yields an average profit $Profit_{av}^*$ (defined by the left hand side of (18)), and that:

$$\begin{aligned} \lambda^* &= \mathbb{E}_m\{Z^*(t)F(m,p^*(t))\} \\ Income^* &= \mathbb{E}_m\{Z^*(t)p^*(t)F(m,p^*(t))\} \end{aligned}$$

Then for each $m \in \mathcal{M}$, there exists two business-price tuples $(Z_1^{(m)}, p_1^{(m)})$ and $(Z_2^{(m)}, p_2^{(m)})$ and two probabilities

[5] The same theorem can be shown for discontinuous functions and non-compact sets, but the optimum is then a *supremum* over all stationary algorithms that satisfy the constraints of Theorem 1. This generalization would not affect the results or analysis of our PTSA algorithm.

$q_1^{(m)}, q_2^{(m)}$ (where $q_1^{(m)} + q_2^{(m)} = 1$) such that:

$$\lambda^* = \sum_{m\in\mathcal{M}} \pi_m \sum_{i=1}^{2} \left[q_i^{(m)} Z_i^{(m)} F(m, p_i^{(m)})\right]$$
$$Income^* \leq \sum_{m\in\mathcal{M}} \pi_m \sum_{i=1}^{2} \left[q_i^{(m)} Z_i^{(m)} p_i^{(m)} F(m, p_i^{(m)})\right]$$

That is, a new stationary randomized pricing policy can be constructed that yields the same average arrival rate λ^* and an average income that is greater than or equal to $Income^*$, but which uses at most two prices for each user demand state $m \in \mathcal{M}$.[6]

Proof: For the stationary randomized policy $Z^*(t)$ and $p^*(t)$, define:

$$\lambda^{(m)*} \triangleq \mathbb{E}\left\{Z^*(t)F(m, p^*(t)) \mid M(t) = m\right\}$$
$$Income^{(m)*} \triangleq \mathbb{E}\left\{Z^*(t)p^*(t)F(m, p^*(t)) \mid M(t) = m\right\}$$

Note that the point $(\lambda^{(m)*}, Income^{(m)*})$ can be expressed as a convex combination of at most three points $\boldsymbol{\omega}_1^{(m)}$, $\boldsymbol{\omega}_2^{(m)}$, $\boldsymbol{\omega}_3^{(m)}$ in $\Omega^{(m)}$ (from Theorem 2 part (a)). Then $(\lambda^{(m)*}, Income^{(m)*})$ is inside (or on an edge of) the triangle formed by $\boldsymbol{\omega}_1^{(m)}$, $\boldsymbol{\omega}_2^{(m)}$, $\boldsymbol{\omega}_3^{(m)}$. Thus, for some value $\delta \geq 0$ the point $(\lambda^{(m)*}, Income^{(m)*} + \delta)$ is on an *edge* of the triangle. Hence, the point $(\lambda^{(m)*}, Income^{(m)*} + \delta)$ can be achieved by a convex combination of at most two of the $\boldsymbol{\omega}_i^{(m)}$ values. Hence, for each $m \in \mathcal{M}$, we can find a convex combination of two elements of $\Omega^{(m)}$, defining a stationary randomized pricing policy with two business-price choices $(Z_1^{(m)}, p_1^{(m)})$, $(Z_2^{(m)}, p_2^{(m)})$ and two probabilities $q_1^{(m)}$, $q_2^{(m)}$. This new policy yields exactly the same average arrival rate λ^*, and has an average income that is greater than or equal to $Income^*$. □

Most work in network pricing has focused on achieving optimality over the class of single-price solutions, and indeed in some cases it can be shown that optimality can be achieved over this class (so that two prices are not needed). However, such optimality requires special properties of the demand function. Theorem 3 shows that for *any* demand function $F(m, p)$, the AP can optimize its average profit by using only two prices for every demand state $m \in \mathcal{M}$. In fact, the following example shows that the number two is *tight*, in that a single fixed price does not always suffice to achieve optimality.

C. Example Demonstrating Necessity of Two Prices

For simplicity, we consider a static situation where the transmission rate is equal to $\mu = 2.28$ with zero cost for all t (so that $\Phi(cost(t), S(t)) = 2.28$ for all $S(t)$ and all $cost(t)$, including $cost(t) = 0$). The demand state $M(t)$ is also assumed to be fixed for all time, so that $F(m, p)$ can be simply written as $F(p)$. Let $\mathcal{P}$ represent the interval $0 \leq p \leq p_{max}$, with $p_{max} = 10$. We consider the following $F(p)$ function:

$$F(p) = \begin{cases} 4 & 0 \leq p \leq 1 \\ -6p + 10 & 1 < p \leq \frac{3}{2} \\ -\frac{2}{17}p + \frac{20}{17} & \frac{3}{2} < p \leq 10 \end{cases} \quad (19)$$

[6]Because the new average income is greater than or equal to $Income^*$, the new average profit is greater than or equal to $Profit^*_{av}$ when this new pricing policy is used together with the old $cost^*(t)$ scheduling policy.

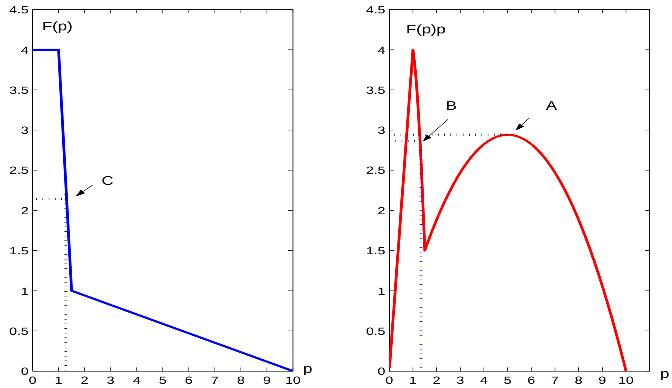

Fig. 2. $F(p)$ and $F(p)p$, A=$(5, \frac{50}{17})$,B=$(1.2867, 2.9285)$,C=$(1.2867, 2.28)$.

The $F(p)$ and $pF(p)$ functions corresponding to (19) are plotted in Fig. 2. Now consider the situation when the AP only uses one price. First we consider the case when $Z(t) = 1$ for all time. Since $\mu = 2.28$, in order to stabilize the queue, the AP has to choose a price p such that $\lambda = F(p) < 2.28$. Thus we obtain that p has to be greater than 1.2867 (points B and C in Fig. 2 show $F(p)$ and $F(p)p$ for $p = 1.2867$).[7] It is easy to show that in this case the best single-price is $p = 5$ (point A in Fig. 2), which yields an average profit $Profit_{single}$ of $Profit_{single} = 50/17 \approx 2.9412$. However, we see that in this case the average arrival rate $F(p)$ is only $10/17 \approx 0.5882$, which is far smaller than μ. Now consider an alternative scheme that uses two prices $p_1 = 31/30$ and $p_2 = 5$, each with probability 0.5. Then the resulting profit is:

$$\begin{aligned} Profit_{Two} &= 0.5F(p_1)p_1 + 0.5F(p_2)p_2 \\ &= 0.5(3.8 \cdot \frac{31}{30} + \frac{50}{17}) \approx 3.4339 \\ &> Profit_{single} \end{aligned} \quad (20)$$

Further, the resulting arrival rate is only:

$$\lambda_{Two} = 0.5F(p_1) + 0.5F(p_2) = 0.5(3.8 + \frac{10}{17}) \approx 2.1941$$

which is strictly less than $\mu = 2.28$. Therefore the queue is stable under this scheme [19].

Now consider the case when the AP uses a varying $Z(t)$ and a single fixed price. From Theorem 1 we see that this is equivalent to using a probability $0 < \phi < 1$ to decide whether or not to allow new data for all time.[8] In order to stabilize the queue, the AP has to choose a price p such that $F(p)\phi < \mu$. Thus the average profit in this case would be $F(p)p\phi < p\mu$. If $p \leq 1.5$, then $F(p)p\phi < 1.5 \cdot 2.28 = 3.42$ (note that this is indeed just an upper bound); else if $1.5 < p \leq 10$, $F(p)p\phi < F(5) \cdot 5 = 50/17$. Both are less than $Profit_{Two}$ obtained above.

One may think that the two optimum prices chosen by the AP are the two prices that generate the two local maximums in the function $F(p)p$, i.e. $p = 1$ and $p = 5$. However, it is easy to show that if one only uses prices $p = 1$ and $p = 5$, the

[7]Throughout the paper, numbers of this type are numerical results and are accurate enough for our arguments.

[8] The case when ϕ=0 is trivial and thus is excluded.

maximum average profit is 3.4662. On the other hand, one can achieve profit 3.4676 with prices $p = 1$ and $p = 5.15$. Thus we see that even in this simple example, knowing the exact $F(p)$ function does not guarantee a simple way of finding the optimal pricing strategy.

IV. Achieving the Maximum Profit

Even though Theorem 2 and 3 show the possibility of achieving the optimum average profit by using only two prices for each demand state, in practice, we still need to solve the problem in Theorem 1. This often involves a very large number of variables and would require the exact demand state and channel state distributions, which are usually hard to obtain. To overcome these difficulties, here we develop the dynamic Pricing and Transmission Scheduling Algorithm (PTSA). The algorithm offers a control parameter $V > 0$ that determines the tradeoff between the queue backlog and the proximity to the optimal average profit.

Admission Control: Every slot t, the AP observes the current backlog $U(t)$ and the user demand $M(t)$ and chooses the price $p(t)$ to be the solution of the following problem:

$$\begin{aligned} Max: \quad & VF(M(t),p)p - 2U(t)F(M(t),p) \\ s.t. \quad & p \in \mathcal{P} \end{aligned} \tag{21}$$

If for all $p \in \mathcal{P}$ the resulting maximum is less than or equal to zero, the AP sends the "CLOSED" signal ($Z(t) = 0$) and does not accept new data. Else, it sets $Z(t) = 1$ and announces the chosen $p(t)$.

Cost/Transmission: Every slot t, the AP observes the current channel state $S(t)$ and backlog $U(t)$ and chooses $cost(t)$ to be the solution of the following problem:

$$\begin{aligned} Max: \quad & 2U(t)\Phi(cost, S(t)) - Vcost \\ s.t. \quad & cost \in \mathcal{C} \end{aligned} \tag{22}$$

The AP then sends out packets according to $\mu(t) = \Phi\left(cost(t), S(t)\right)$.

The control policy is thus decoupled into separate algorithms for pricing and transmission scheduling. Note from (21) that a larger $U(t)$ increases the negative term $-2U(t)F(M(t),p)$ in the optimization metric, and hence tends to lead to a higher price $p(t)$. Intuitively, such a slow down of the packet arrival helps alleviate the congestion in the AP. Note that the metric in (21) can be written as $F(M(t),p)\big(Vp - 2U(t)\big)$. This is positive only if p is larger than $2U(t)/V$. Thus, we have the following simple fact:

***Lemma** 1:* Under the PTSA algorithm, if $2U(t)/V > p_{max}$, then $Z(t) = 0$. □

A. Performance Analysis

In this section we evaluate the performance of PTSA. The following theorem summarizes the performance results:

***Theorem** 4:* PTSA stabilizes the AP and achieves the following bounds (assuming $U(0) = 0$):

$$U(t) \leq U_{max} \triangleq Vp_{max}/2 + R_{max} \qquad \forall\, t \tag{23}$$

$$Profit_{av} \geq Profit_{av}^{opt} - \frac{\tilde{B}}{V} \tag{24}$$

where:

$$Profit_{av} \triangleq \liminf_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\left\{Z(\tau)P(\tau)R(\tau) - cost(\tau)\right\}$$

and where $Profit_{av}^{opt}$ is the optimal profit characterized by (6) in Theorem 1, and $\tilde{B}$ is defined in equation (41) of the proof, and $\tilde{B} = O(\log(V))$.

The V parameter can be increased to push the profit arbitrarily close to the optimum value, while the worst case backlog bound grows linearly with V. In fact, we can see from (21) and (22) that these results are quite intuitive: when using a larger V, the AP is more inclined to admit packets (setting $p(t)$ to a smaller value and only requiring $p(t) \geq 2U(t)/V$). Also, a larger V implies that the AP is more careful in choosing the transmission opportunities (indeed, $\Phi\left(cost(t), S(t)\right)$ must be more cost effective, i.e. larger than $Vcost(t)/2U(t)$). Therefore a larger V would yield a better profit, at the cost of larger backlog.

We first prove (23) in Theorem 4:

Proof: ((23) in Theorem 4) We prove this by induction. It is easy to see that (23) is satisfied at time 0. Now assume $U(t) \leq Vp_{max}/2 + R_{max}$ for some integer slot $t \geq 0$. We will prove that $U(t+1) \leq Vp_{max}/2 + R_{max}$. We have the following two cases:

(a) $U(t) \leq Vp_{max}/2$: In this case, $U(t+1) \leq Vp_{max}/2 + R_{max}$ by the definition of R_{max}.

(b) $U(t) > Vp_{max}/2$: In this case, $2U(t)/V > p_{max}$. Hence, by Lemma 1 the AP will decide not to admit any new data. Therefore $U(t+1) \leq U(t) \leq Vp_{max}/2 + R_{max}$. □

In the following we prove (24) in Theorem 4 via a Lyapunov analysis, using the framework of [19]. First define the Lyaponov function $L(U(t))$ to be:

$$L(U(t)) \triangleq U^2(t) \tag{25}$$

Define the one-step *unconditional Lyapunov drift* as $\Delta(t) \triangleq \mathbb{E}\{L(U(t+1)) - L(U(t))\}$. Squaring both sides of (4) and rearranging the terms, we see that the drift satisfies:

$$\Delta(t) \leq B - \mathbb{E}\{2U(t)\big[\Phi(cost(t), S(t)) - Z(t)R(t)\big]\} \tag{26}$$

where $B = R_{max}^2 + \mu_{max}^2$. For a given number $V > 0$, we subtract from both sides the instantaneous profit (scaled by V) and rearrange terms to get:

$$\begin{aligned} \Delta(t) - V\mathbb{E}\big\{Z(t)p(t)R(t) - cost(t)\big\} & \\ \leq B - \mathbb{E}\big\{2U(t)\Phi(cost(t), S(t)) - Vcost(t)\big\} & \\ -\mathbb{E}\big\{Z(t)\big[Vp(t)R(t) - 2U(t)R(t)\big]\big\} & \end{aligned} \tag{27}$$

Now we see that the PTSA algorithm is designed to *minimize the right hand side of the drift expression (27) over all alternative control decisions that could be chosen on slot t.* Thus, we have that the drift of PTSA satisfies:

$$\begin{aligned} \Delta^P(t) - V\mathbb{E}\big\{Z^P(t)p^P(t)R^P(t) - cost^P(t)\big\} & \\ \leq B - \mathbb{E}\big\{2U^P(t)\Phi(cost^*(t), S(t)) - Vcost^*(t)\big\} & \\ -\mathbb{E}\big\{Z^*(t)\big[Vp^*(t)R^*(t) - 2U^P(t)R^*(t)\big]\big\} & \end{aligned} \tag{28}$$

where the decisions $Z^*(t)$, $p^*(t)$, and $cost^*(t)$ (and the resulting random arrival $R^*(t)$) correspond to any other feasible control action that can be implemented on slot t (subject to the same constraints $p^*(t) \in \mathcal{P}$ and $cost^*(t) \in \mathcal{C}$). Note that we have used notation $\Delta^P(t)$, $Z^P(t)$, $p^P(t)$, $R^P(t)$, and $cost^P(t)$ on the left hand side of the above inequality to emphasize that this left hand side corresponds to the variables associated with the PTSA policy. Note also that, because the PTSA policy has been implemented up to slot t, the queue backlog on the right hand side at time t is the backlog associated with the PTSA algorithm and hence is also denoted $U^P(t)$. We emphasize that the right hand side of the drift inequality (28) has been modified *only* in those control variables that can be chosen on slot t. Note further that $R^*(t)$ is a random variable that is conditionally independent of the past given the $p^*(t)$ price and the current value of $M(t)$.

Now consider the alternative control policy $STAT^*$ described in Corollary 1, which chooses decisions $Z^*(t), p^*(t)$ and $cost^*(t)$ on slot t as a pure function of the observed $M(t)$ and $S(t)$ states and yields:

$$\begin{aligned} Profit_{av}^{opt} &= \mathbb{E}_m\{Z^*(t)R^*(t)p^*(t)\} - \mathbb{E}_s\{cost^*(t)\} \quad (29)\\ \lambda_{av}^* &\triangleq \mathbb{E}_m\{Z^*(t)R^*(t)\}\\ &\leq \mu_{av}^* \triangleq \mathbb{E}_s\{\mu^*(t)\} \quad (30)\end{aligned}$$

where $Profit_{av}^{opt}$ is the optimal average profit defined in Theorem 1, $\mu^*(t) = \Phi(cost^*(t), S(t))$, and $R^*(t)$ is the random arrival for a given $p^*(t)$ and $M(t)$. Recall that $\mathbb{E}_m\{\}$ and $\mathbb{E}_s\{\}$ denote expectations over the steady state distributions for $M(t)$ and $S(t)$, respectively. Of course, the expectations in (29) and (30) cannot be directly used in the right hand side of (28) because the $M(t)$ and $S(t)$ distributions at time t may not be the same as their steady state distributions. However, regardless of the initial condition of $M(0)$ and $S(0)$ we have:

$$\lim_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\left\{Z^*(\tau)p^*(\tau)R^*(\tau) - cost^*(\tau)\right\} = Profit_{av}^{opt} \quad (31)$$

Let $f^P(t)$ represent a short-hand notation for the left-hand side of (28), and define $g^*(t)$ as the right hand side of (28), so that:

$$\begin{aligned} g^*(t) \triangleq B &- \mathbb{E}\{2U^P(t)[\mu^*(t) - Z^*(t)R^*(t)]\}\\ &- V\mathbb{E}\{Z^*(t)p^*(t)R^*(t) - cost^*(t)\} \quad (32)\end{aligned}$$

where we have rearranged terms and have used $\mu^*(t)$ to represent $\Phi(cost^*(t), S(t))$. Thus, the inequality (28) is equivalent to $f^P(t) \leq g^*(t)$. To compute a simple upper bound on $g^*(t)$, note that for any integer $d \geq 0$, we have:

$$\begin{aligned} U^P(t) &\leq U^P(t-d) + dR_{max}\\ U^P(t) &\geq U^P(t-d) - d\mu_{max}\end{aligned}$$

These inequalities hold since the backlog at time t is no smaller than the backlog at time $t-d$ minus the maximum departures during the interval from $t-d$ to t, and is no larger than the backlog at time $t-d$ plus the largest possible arrivals during this interval. Plugging these two inequalities directly into the definition of $g^*(t)$ in (32) yields:

$$\begin{aligned} g^*(t) \leq\ & B + 2d(\mu_{max}^2 + R_{max}^2)\\ &- \mathbb{E}\left\{2U^P(t-d)[\mu^*(t) - Z^*(t)R^*(t)]\right\}\\ &- V\mathbb{E}\left\{Z^*(t)p^*(t)R^*(t) - cost^*(t)\right\} \quad (33)\end{aligned}$$

Also note that (by the law of iterated expectations):

$$\begin{aligned} &\mathbb{E}\{U^P(t-d)\left[\mu^*(t) - Z^*(t)R^*(t)\right]\}\\ &= \mathbb{E}\left\{U^P(t-d)\mathbb{E}\{\left[\mu^*(t) - Z^*(t)R^*(t)\right] \mid \chi(t-d)\}\right\} \quad (34)\end{aligned}$$

where $\chi(t) \triangleq [M(t), S(t), U(t)]$ is the joint demand state, channel state, and queue state of the system. Since $M(t)$ and $S(t)$ are Markovian and both have well defined steady state distributions, and the $STAT^*$ policy makes $p^*(t)$ and $cost^*(t)$ decisions as a stationary and random function of the observed $M(t)$ and $S(t)$ states (and independent of queue backlog), we see that the resulting processes $\mu^*(t)$ and $Z^*(t)R^*(t)$ are Markovian and have well defined steady state averages. Further, they converge exponentially fast to their steady state values [23] (one such example is provided in Appendix B). Of course, we know the steady state averages are given by μ_{av}^* and λ_{av}^*, respectively. Therefore there exist positive constants θ_1, θ_2, γ_1, and γ_2 with $0 < \gamma_1, \gamma_2 < 1$, such that:

$$\begin{aligned} \mathbb{E}\{\mu^*(t) \mid \chi(t-d)\} &\geq \mu_{av}^* - \theta_1\gamma_1^d \quad (35)\\ \mathbb{E}\{Z^*(t)R^*(t) \mid \chi(t-d)\} &\leq \lambda_{av}^* + \theta_2\gamma_2^d \quad (36)\end{aligned}$$

Plugging (35) and (36) into (34) yields:

$$\begin{aligned} &\mathbb{E}\{U^P(t-d)\left[\mu^*(t) - Z^*(t)R^*(t)\right]\}\\ &\geq -\mathbb{E}\left\{U^P(t-d)\left[\theta_1\gamma_1^d + \theta_2\gamma_2^d\right]\right\} \quad (37)\end{aligned}$$

where we have used the fact that $\lambda_{av}^* \leq \mu_{av}^*$ (from (30)). Plugging (37) directly into (33) yields:

$$\begin{aligned} g^*(t) \leq\ & B_1 + 2\mathbb{E}\{U^P(t-d)(\theta_1\gamma_1^d + \theta_2\gamma_2^d)\}\\ &- V\mathbb{E}\left\{Z^*(t)p^*(t)R^*(t) - cost^*(t)\right\} \quad (38)\end{aligned}$$

where $B_1 \triangleq B + 2d(\mu_{max}^2 + R_{max}^2)$. However, the queue backlog under PTSA is always bounded by U_{max} (by (23) in Theorem 4). We now choose d large enough so that $\theta_i\gamma_i^d \leq 1/(2U_{max})$ for $i \in \{1, 2\}$. Specifically, by choosing:

$$d \triangleq \left\lceil \max_{i=1,2}\left\{\frac{\log\left(2\theta_i U_{max}\right)}{\log\left(1/\gamma_i\right)}\right\}\right\rceil \quad (39)$$

we have $2U_{max}[\theta_1\gamma_1^d + \theta_2\gamma_2^d] \leq 2$. Inequality (38) becomes:

$$g^*(t) \leq B_1 + 2 - V\mathbb{E}\left\{Z^*(t)p^*(t)R^*(t) - cost^*(t)\right\} \quad (40)$$

Now define $\tilde{B}$ as follows:

$$\tilde{B} \triangleq B_1 + 2 = (2d+1)(R_{max}^2 + \mu_{max}^2) + 2 \quad (41)$$

where d is defined in (39). Because $U_{max} = Vp_{max}/2 + R_{max}$ (by (23) in Theorem 4), the value of d is $O(\log(V))$, and hence $\tilde{B} = O(\log(V))$. Recalling that $f^P(t) \leq g^*(t)$, where $f^P(t)$ is the left hand side of (28), we have:

$$\begin{aligned} &\Delta^P(t) - V\mathbb{E}\{Z^P(t)p^P(t)R^P(t) - cost^P(t)\}\\ &\quad \leq \tilde{B} - V\mathbb{E}\left\{Z^*(t)p^*(t)R^*(t) - cost^*(t)\right\}\end{aligned}$$

The above inequality holds for all t. Summing both sides over $\tau \in \{0, 1, \ldots, t-1\}$ and using $\Delta^P(t) = \mathbb{E}\left\{L(U^P(t+1)) - L(U^P(t))\right\}$, we get:

$$\begin{aligned} &\mathbb{E}\left\{L(U^P(t))\right\} - \mathbb{E}\left\{L(U^P(0))\right\} \\ &-V\sum_{\tau=0}^{t-1}\mathbb{E}\left\{Z^P(\tau)p^P(\tau)R^P(\tau) - cost^P(\tau)\right\} \\ &\leq \tilde{B}t - V\sum_{\tau=0}^{t-1}\mathbb{E}\left\{Z^*(\tau)p^*(\tau)R^*(\tau) - cost^*(\tau)\right\} \end{aligned}$$

Dividing by Vt, using the fact that $L(U^P(t)) \geq 0$, $L(U(0)) = 0$, and taking limits yields:

$$\liminf_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}\left\{Z^P(\tau)p^P(\tau)R^P(\tau) - cost^P(\tau)\right\} \geq Profit_{av}^{opt} - \tilde{B}/V \quad (42)$$

where we have used (31). The left hand side of the above inequality is the liminf time average profit of the PTSA algorithm. This completes the proof of Theorem 4.[9] □

B. Demand Blind Pricing

In the special case when the demand function $F(m, p)$ takes the form of $F(m, p) = m\hat{F}(p)$, PTSA can in fact choose the current price *without* looking at the current demand state $M(t)$. To see this, note in this case that (21) can be written:

$$\begin{aligned} Max: \quad & M(t)\left[V\hat{F}(p)p - 2U(t)\hat{F}(p)\right] \\ s.t. \quad & p \in \mathcal{P} \end{aligned} \quad (43)$$

Thus we see that the price set by the AP under PTSA is *independent* of $M(t)$. So in this case, PTSA can make decisions just by looking at the queue backlog value $U(t)$.

V. Simulation

In this section, we provide simulation results for the PTSA algorithm. We simulate the same example that we use in Section III. That is, the system has transmission rate $\mu = 2.28$ for all time (with zero cost), and has a static demand function $F(p)$ given by (19). We compare two cases of arrival processes. In the first case, the arrival $R(t)$ is deterministic and is exactly equal to $F(p(t))$. In the other case, we assume that $R(t)$ is a Bernoulli random variable, and satisfies:[10]

$$R(t) = \begin{cases} 2F(p(t)) & \text{w.p. } 0.5 \\ 0 & \text{w.p. } 0.5 \end{cases} \quad (44)$$

At each time slot, the AP chooses a price according to (21) and admits all incoming packets. The simulation is conducted with control parameters $V \in \{1, 2, 5, 10, 100, 200\}$ and we run each simulation over $5,000,000$ timeslots. Figure 3 and 4 show the backlog and the average profit performances.

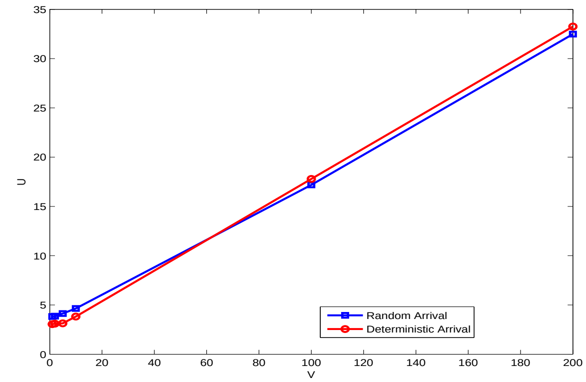

Fig. 3. Average Backlog v.s. V

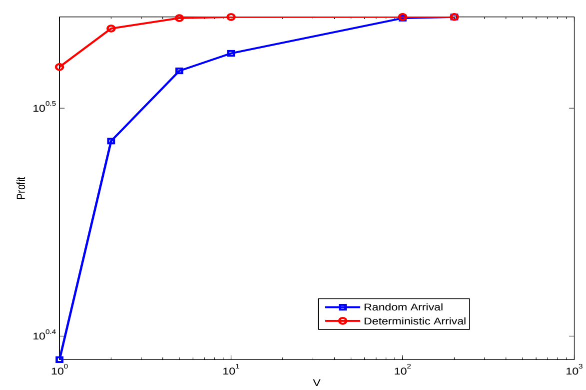

Fig. 4. Average Profit v.s. V

We see from Fig. 3 that the average backlog grows linearly in V and the average backlog is smaller than the worst case bound $Vp_{max}/2 + R_{max}$. We also see that the results for both arrival processes are very close to each other. Fig. 4 shows the achieved average profit versus the parameter V. We see that the average profit converges quickly as V grows. The average profits are almost indistinguishable from the optimal value when $V \geq 100$.

We also observe an interesting fact that in both cases, the prices chosen by the PTSA algorithm exhibit a "two-value" property, i.e. the prices switch between values that are either 1 or close to 5.15. This fact is shown in Fig. 5.

In the deterministic arrival case, we observe that the price jumps from one to the other every time slot with some occasional "phase inversions," i.e., the price occasionally stays at the same level for one more slot and then starts jumping again. While in the random arrival case, the price sometimes stays at a value for a few slots before jumping to the other value, and when the price near 5.15 does not jump to the other side, it gradually decreases until it makes the jump.

Intuitively, this happens in the random arrival case because of the following: There is a probability of 1/2 that the AP gets no new data even if $p < 10$. If at one slot the AP sets a price $p = 1$ but does not get any new data, then the AP only serves some data at that slot. Since $p = 1$, it follows that the AP plans to admit new data. The plan will be preserved at the next slot, since the AP just sends out some packets and further reduces its queue. Thus the price would still be 1 in

[9] In the case when $F(m, p)$ is not necessarily continuous, the same proof holds with $Profit_{av}^*$ replacing $Profit_{av}^{opt}$, where $Profit_{av}^*$ represents the profit of any particular stationary randomized algorithm. The bound (42) can then be optimized by taking a supremum over all such $Profit_{av}^*$.

[10] For simplicity here, we assume $R(t)$ can take fractional values. Alternatively, we could restrict packet sizes to integral units and make the probabilities be such that $\mathbb{E}\{R(t) \mid p(t)\} = F(p(t))$.

the next slot. If instead the AP sets p near 5.15, it in fact needs to reduce the number of new packets. Thus if the AP does not get any incoming packet at this slot, it can use that slot to serve some packets. Therefore in the next slot, it can lower the price and allow more new data.

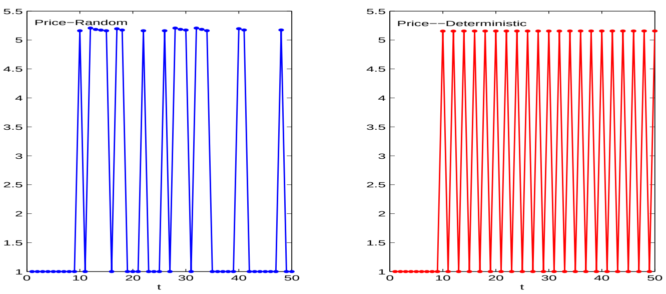

Fig. 5. The chosen prices in the first 50 slots in both cases with V=100

VI. CONCLUSION

In this paper, we first characterized the optimum average profit for the AP, and proved the "Optimality of Two Prices" theorem. We then developed a joint pricing and transmission scheduling algorithm, PTSA, which can achieve any average profit that is arbitrarily close to the optimum while ensuring queue backlog is bounded. PTSA uses the admission price and the business decision as tools to regulate the incoming traffic. It also provides a parameter V to tradeoff the worst case backlog with the profit loss. The analysis uses a Lyapunov drift technique which jointly takes into account the stability issue and the performance optimization issue.

APPENDIX A – PROOF OF THEOREM 1

Proof: (Part I) We prove the first part by using a similar analysis as in [18]: *Consider any rule for choosing the business decision $Z(t) \in \{0,1\}$ and price $p(t) \in \mathcal{P}$, and any rule for choosing $cost(t) \in \mathcal{C}$. If the policy stabilizes the AP, then:*

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\left\{Z(\tau)p(\tau)R(\tau) - cost(\tau)\right\} \leq Profit_{av}^{opt} \quad (45)$$

where $Profit_{av}^{opt}$ is defined by the optimization in Theorem 1.

To show this, let $Alg1$ be a pricing and scheduling algorithm that stabilizes the queue. Let $\{(Z(0),p(0),cost(0)),(Z(1),p(1),cost(1)),\ldots\}$ be the sequence of control decisions used by $Alg1$ over time. Then there is a sub-sequence of times $\{H_i\}$ such that $H_i \to \infty$ and such that the limiting time average expected profit over times H_i is equal to the $\limsup$ average profit under $Alg1$ (defined by the left hand side of (45)). Now define the conditional average of income and packet rate over H slots:

$$\big(Income_{av}^m(H);\lambda_{av}^m(H)\big)\triangleq \frac{1}{H}\sum_{\tau=0}^{H-1}\mathbb{E}\left\{\big(Z(\tau)R(\tau)p(\tau);Z(\tau)R(\tau)\big)\mid M(\tau)=m\right\}$$

The above can be rewritten as:

$$\big(Income_{av}^m(H);\lambda_{av}^m(H)\big)\triangleq\frac{1}{H}\sum_{\tau=0}^{H-1}\mathbb{E}\left\{\big(Z(\tau)F(m,p(\tau))p(\tau)\right.$$
$$\left.;Z(\tau)F(m,p(\tau))\big)\mid M(\tau)=m\right\}$$

***Lemma** 2:* For every H, there exist probabilities $\phi^{(m)}(H)$, $\alpha_k^{(m)}(H)$ and price values $p_k^{(m)}(H) \in \mathcal{P}$ such that:

$$Income_{av}^m(H) = \phi^{(m)}(H)\sum_{k=1}^{3}\alpha_k^{(m)}(H)F(m,p_k^{(m)}(H))p_k^{(m)}(H) \quad (46)$$

$$\lambda_{av}^m(H) = \phi^{(m)}(H)\sum_{k=1}^{3}\alpha_k^{(m)}(H)F(m,p_k^{(m)}(H)) \quad (47)$$

Proof: Define $\Psi^m(Z,p) \triangleq \big(ZF(m,p)p;ZF(m,p)\big)$ as a function mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$. Then $(Income_{av}^m(H),\lambda_{av}^m(H))$ is equal to:

$$\frac{1}{H}\sum_{\tau=0}^{H-1}\mathbb{E}\left\{\Psi^m(Z(\tau),p(\tau))\mid M(\tau)=m\right\}$$

The above is an expectation over points in the image of $\Psi^m(Z,p)$, and hence is in the convex hull of the image. Hence, it can be expressed as a convex combination of at most three elements of the image (by Caratheodory's theorem). Indeed, note that the image of $\Psi^m(Z,p)$ consists of two sets: $\{(0,0)\}$ and $\{(pF(m,p),F(m,p)) \mid p \in \mathcal{P}\}$, corresponding to Z=0 and Z=1, respectively. Thus the convex combination can be expressed as $\rho_0(H)(0,0) + \sum_{k=1}^{3}\rho_k(H)(p_k^{(m)}(H)F(m,p_k^{(m)}(H)),F(m,p_k^{(m)}(H)))$, so that $\sum_{k=0}^{3}\rho_k(H)=1$ and $p_k^{(m)}(H)\in\mathcal{P}$. Define $\phi^{(m)}(H) = 1-\rho_0(H)$. If $\rho_0(H)\neq 1$, define $\alpha_k^{(m)}(H) = \frac{\rho_k(H)}{1-\rho_0(H)}$ for all $k\geq 1$; else if $\rho_0(H)=1$ define $\alpha_k^{(m)}(H)=0$ for all $k\geq 1$, we see that the lemma follows. □

Now define:

$$(Income_{av}(H),\lambda_{av}(H))\triangleq \sum_{m\in\mathcal{M}}\pi_m(Income_{av}^m(H),\lambda_{av}^m(H))$$

Using continuity of $F(p,m)$ and compactness of $\mathcal{P}$ and following the same line of analysis as in [18], we see that we can find a sub-subsequence $\{\tilde{H}_i\}$ of the subsequence of times $\{H_i\}$ such that $\tilde{H}_i \to \infty$ as $i\to\infty$, and there exist probabilities $\phi^{(m)}$, $\alpha_k^{(m)}$ and price values $p_k^{(m)}\in\mathcal{P}$ such that:

$$p_k^{(m)}(\tilde{H}_i)\to p_k^{(m)}, \alpha_k^{(m)}(\tilde{H}_i)\to\alpha_k^{(m)}, \phi^{(m)}(\tilde{H}_i)\to\phi^{(m)}$$
$$\lambda_{av}^m(\tilde{H}_i)\to\lambda_{av}^m\ ,\ \ F(m,p_k^{(m)}(\tilde{H}_i))\to F(m,p_k^{(m)}) \quad (48)$$

It is easy to see that the $p_k^{(m)}$ values satisfy (13), the $\phi^{(m)}$ probabilities satisfy (11), and the $\alpha_k^{(m)}$ values satisfy (15). Further, because $\{\pi_m\}$ are the steady state values for $M(t)$, the corresponding λ_{av} and $Income_{av}$ values satisfy (9) and (7). Similarly, one can take limiting time average expected values over the same sequence of times $\tilde{H}_i$ (possibly passing to a convergent subsequence if necessary), to define limiting probabilities $\beta_k^{(s)}$ and $cost_k^{(s)}$ and a limiting time average service rate μ_{av} that satisfy constraints (16), (14), (10), (8). Finally, as in [18], because $Alg1$ is stable we can infer that $\lambda_{av}\leq\mu_{av}$ for these particular limiting values. Thus, we now have a stationary randomized policy that satisfies all constraints (7)-(16) of the optimization problem of Theorem 1, and yields an average profit that is identical to the $\limsup$ average profit of $Alg1$. However, $Profit_{av}^{opt}$ is defined as the *maximum* time average profit over any such stationary randomized policy that satisfies the constraints (7)-(16), and hence (45) holds. The proof of Part I is thus completed. □

Proof: (Part II) We want to show that: *if* $Profit_{av}^{opt} > 0$, *then for any arbitrary small* $\epsilon > 0$, *the profit* $Profit_{av}^{opt} - \epsilon$ *can be achieved with an algorithm, which yields an average arrival rate* λ_{av} *and an average transmission rate* μ_{av} *that satisfy* $\lambda_{av} < \mu_{av}$.

First let Alg^* be the stationary randomized algorithm that solves (6) with $Profit_{av}^{opt} > 0$. Now let λ_{av}^* and μ_{av}^* denote the average arrival rate and the average transmission rate that Alg^* yields, let $\{\phi^{m*}\}$ denote the probability values used by Alg^*, and let $Income_{av}^*$ and $Cost_{av}^*$ denote the average income and average of Alg^*. We see from (12) that $\mu_{av}^* \geq \lambda_{av}^*$. Since $Profit_{av}^{opt} > 0$, we see that $\lambda_{av}^* > 0$. Consider the following algorithm $Alg2$: $Alg2$ is a stationary randomized algorithm that is exactly the same as Alg^*, except that it uses probability values $\{\phi^{m'}\} = \{\rho\phi^{m*}\}$, with some constant $0 < \rho < 1$.

It is easy to see that $Alg2$ will yield the same average transmission rate μ_{av}^* and average cost $Cost_{av}^*$ since the business factors do not affect the transmission under both Alg^* and $Alg2$. But we can see that under $Alg2$, the average income and average input rate become:

$$Income_{av}^{Alg2} = \rho Income_{av}^* \quad , \quad \lambda_{av}^{Alg2} = \rho\lambda_{av}^*$$

Now it is easy to see that $\lambda_{av}^{Alg2} < \mu_{av}^* = \mu_{av}^{Alg2}$ and that:

$$\begin{aligned} ProfitLoss &= Income_{av}^* - Income_{av}^{Alg2} \\ &= (1-\rho)Income_{av}^* \end{aligned}$$

To achieve an average profit no smaller than $Profit_{av}^{opt} - \epsilon$ for some $\epsilon \geq 0$, we only need:

$$(1-\rho)Income_{av}^* \leq \epsilon \tag{49}$$

Since $Income_{av}^* \leq R_{max}p_{max}$, we see that (49) can easily be satisfied by choosing $\rho \geq 1 - \frac{\epsilon}{R_{max}p_{max}}$.

Thus we see that $Alg2$ stabilizes the AP ($\lambda_{av}^{Alg2} < \mu_{av}^{Alg2}$) and achieves an average profit $Profit_{av}^{Alg2} > Profit_{av}^{opt} - \epsilon$. Also, by taking $\epsilon \to \infty$, we see that $Profit_{av}^{Alg2} \to Profit_{av}^{opt}$. This proves Part II. □

Appendix B – An Example of Exponential Convergence for Markov Processes

Here we provide a simple example about the exponential convergence of a Markov process. We consider the two state Markov chain in Fig. 6. Let the initial distribution be given by $[P_{ON}(0), P_{OFF}(0)]$. It is easy to show that the probabilities of being in state ON and OFF at time t are given by:

$$P_{ON}(t) = \frac{\delta}{\delta+\epsilon} + \frac{P_{ON}(0)\epsilon - P_{OFF}(0)\delta}{\delta+\epsilon}(1-\delta-\epsilon)^t$$
$$P_{OFF}(t) = \frac{\epsilon}{\delta+\epsilon} - \frac{P_{ON}(0)\epsilon - P_{OFF}(0)\delta}{\delta+\epsilon}(1-\delta-\epsilon)^t$$

We thus see that $P_{ON}(t)$ and $P_{OFF}(t)$ converge exponentially fast to their steady state distributions $\frac{\delta}{\delta+\epsilon}$ and $\frac{\epsilon}{\delta+\epsilon}$.

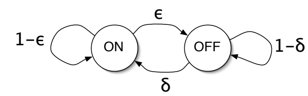

Fig. 6. A two state Markov Chain with transition probabilities ϵ and δ.

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