CS675: Convex and Combinatorial Optimization
Fall 2019
Consequences of the Ellipsoid Algorithm

Instructor: Shaddin Dughmi
1. Recapping the Ellipsoid Method
2. Complexity of Convex Optimization
3. Complexity of Linear Programming
4. Equivalence of Separation and Optimization
Recall: Feasibility Problem

The ellipsoid method solves the following problem.

**Convex Feasibility Problem**

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing $K$
- A rational number $R > 0$ satisfying $\text{vol}(E) \leq R$
- A rational number $r > 0$ such that if $K$ is nonempty, then $\text{vol}(K) \geq r$

Find a point $x \in K$ or declare that $K$ is empty.

Equivalent variant: drop the requirement on volume $\text{vol}(K)$, and either find a point $x \in K$ or an ellipsoid $E \supseteq K$ with $\text{vol}(E) < r$. 
All the ellipsoid method needed was the following subroutine

**Separation oracle**

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separating $x$ from $K$.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^T x \geq h^T y$ for all $y \in K$.
- Equivalently, $K$ is contained in the open halfspace

$$H(h, x) = \{ y : h^T y < h^T x \}$$

with $x$ at its boundary.
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Examples:
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**Examples:**

- Explicitly written polytope $Ay \leq b$: take $h = a_i$ to the row of $A$ corresponding to a constraint violated by $x$. 

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- Convex set given by a family of convex inequalities $f_i(y) \leq 0$: Let $h = \nabla f_i(x)$ for some violated constraint.
- The positive semi-definite cone $S_n^+$: Let $H$ be $-vv^T$ for an eigenvector $v$ with a negative eigenvalue.
Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$

2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output $c$.
   - Otherwise, we get a separating hyperplane $h$ such that $K$ is contained in the half-ellipsoid $E \cap \{ y : h^Ty \leq h^Tc \}$

3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.

4. If $\text{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return “empty”.

Recapping the Ellipsoid Method
Ellipsoid Method

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Recapping the Ellipsoid Method
Properties

Using $T$ to denote the runtime of the separation oracle

Theorem

The ellipsoid algorithm terminates in time polynomial $n$, $\ln \frac{R}{r}$, and $T$, and either outputs $x \in K$ or correctly declares that $K$ is empty.

We proved most of this (modulo the ellipsoid updating Lemma which we cited and briefly discussed).
Properties

Using $T$ to denote the runtime of the separation oracle

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Note

For runtime polynomial in input size we need

- $T$ polynomial in input size
- $\frac{R}{r}$ exponential in input size
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Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \mathcal{X}
\end{align*}
\]

Where \( \mathcal{X} \subseteq \mathbb{R}^n \) is convex and closed, and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex.
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- Recall: A problem \( \Pi \) is a family of instances \( I = (f, \mathcal{X}) \)
- When represented explicitly, often given in standard form

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \quad \text{for} \ i \in C_1. \\
& \quad a_i^T x = b_i, \quad \text{for} \ i \in C_2.
\end{align*}
\]

- The functions \( f, \{g_i\}_i \) are given in some parametric form allowing evaluation of each function and its derivatives.
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- We will abstract away details of how instances of a problem are represented, but denote the length of the description by \( \langle I \rangle \).
- Require polynomial time (in \( \langle I \rangle \) and \( n \)) implementation of separation oracle, and other subroutines.
There are many subtly different “solvability statements”. This one is the most useful, yet simple to describe, IMO.

**Requirements**

We say an algorithm **weakly solves** a convex optimization problem in **polynomial time** if it:

- Takes an approximation parameter $\epsilon > 0$
- Terminates in time $\text{poly}(\langle I \rangle, n, \log(\frac{1}{\epsilon}))$
- Returns an $\epsilon$-optimal $x \in \mathcal{X}$:

\[
f(x) \leq \min_{y \in \mathcal{X}} f(y) + \epsilon \left[ \max_{y \in \mathcal{X}} f(y) - \min_{y \in \mathcal{X}} f(y) \right]
\]
Consider a family $\Pi$ of convex optimization problems $I = (f, \mathcal{X})$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):

- A **separation oracle** for the feasible set $\mathcal{X} \subseteq \mathbb{R}^n$
- A **first order oracle** for $f$: evaluates $f(x)$ and $\nabla f(x)$.
- An algorithm which **computes a starting ellipsoid** $E \supseteq \mathcal{X}$ with $\frac{\text{vol}(E)}{\text{vol}(\mathcal{X})} = O(\exp(\langle I \rangle, n))$.

Then there is a polynomial time algorithm which weakly solves $\Pi$. 

Let's now prove this, by reducing to the ellipsoid method.
Solvability of Convex Optimization

Theorem (Polynomial Solvability of CP)

Consider a family $\Pi$ of convex optimization problems $I = (f, \mathcal{X})$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):

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Proof (Simplified)

Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$. 
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Our task reduces to the following convex feasibility problem:

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\text{find } x \\
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f(x) \leq \epsilon
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We can feed this into the Ellipsoid method!

Needed Ingredients

1. Separation oracle for new feasible set $K$:

2. Ellipsoid $E$ containing $K$:

3. Guarantee that $\frac{\text{vol}(E)}{\text{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$:
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Needed Ingredients

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Needed Ingredients

1. Separation oracle for new feasible set $K$: Use the separation oracle for $\mathcal{X}$ and first order oracle for $f$

2. Ellipsoid $E$ containing $K$: Use the ellipsoid containing $\mathcal{X}$

3. Guarantee that $\frac{\text{vol}(E)}{\text{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$: Not obvious, but true!
Proof (Simplified)

\[ K = \{ x \in \mathcal{X} : f(x) \leq \epsilon \} \]

**Lemma**

\[ \text{vol}(K) \geq \epsilon^n \text{vol}(\mathcal{X}). \]

This shows that \( \text{vol}(K) \) is only exponentially smaller (in \( n \) and \( \log \frac{1}{\epsilon} \)) than \( \text{vol}(\mathcal{X}) \), and therefore also \( \text{vol}(E) \), so it suffices.
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- Assume wlog \( 0 \in \mathcal{X} \) and \( f(0) = \min_{x \in \mathcal{X}} f(x) = 0 \).
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- \( \text{vol}(\epsilon \mathcal{X}) = \epsilon^n \text{vol}(\mathcal{X}). \)
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- We show that \( \epsilon \mathcal{X} \subseteq K \) by showing \( f(y) \leq \epsilon \) for all \( y \in \epsilon \mathcal{X} \).
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- Assume wlog \( 0 \in X \) and \( f(0) = \min_{x \in X} f(x) = 0 \).
- Consider scaling \( X \) by \( \epsilon \) to get \( \epsilon X \).
- \( \text{vol}(\epsilon X) = \epsilon^n \text{vol}(X) \).
- We show that \( \epsilon X \subseteq K \) by showing \( f(y) \leq \epsilon \) for all \( y \in \epsilon X \).
- Let \( y = \epsilon x \) for \( x \in X \), and invoke Jensen's inequality

\[
 f(y) = f(\epsilon x + (1 - \epsilon)0) \leq \epsilon f(x) + (1 - \epsilon)f(0) \leq \epsilon
\]
Proof (General)

- Denote $L = \min_{y \in \mathcal{X}} f(y)$ and $H = \max_{y \in \mathcal{X}} f(y)$.
- If we knew the target $T = L + \epsilon [H - L]$, we can reduce to solving the feasibility problem over $K = \{x \in \mathcal{X} : f(x) \leq T\}$.
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If we knew the target $T = L + \epsilon [H - L]$, we can reduce to solving the feasibility problem over $K = \{x \in X : f(x) \leq T\}$.
If we knew it lied in a sufficiently narrow range, we could binary search for $T$. 
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If we knew the target $T = L + \epsilon [H - L]$, we can reduce to solving the feasibility problem over $K = \{ x \in X : f(x) \leq T \}$.

If we knew it lied in a sufficiently narrow range, we could binary search for $T$

We don’t need to know anything about $T$!

Key Observation

We don’t really need to know $T$, $H$, or $L$ to simulate the same execution of the ellipsoid method on $K$!!
Proof (General)

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\begin{align*}
\text{find} & \quad x \\
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& \quad f(x) \leq T = L + \epsilon[H - L]
\end{align*}
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- Simulate the execution of the ellipsoid method on $K$
- Polynomial number of iterations, terminating with point in $K$
Proof (General)

find $x$
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- Simulate the execution of the ellipsoid method on $K$
- Polynomial number of iterations, terminating with point in $K$
- Require separation oracle for $K$ to use $\nabla f$ only as a last resort
  - This is allowed.
  - Tries to get feasibility whenever possible.
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- Action of algorithm in each iteration other than the last can be described independently of \( T \)
  - If ellipsoid center \( c \notin \mathcal{X} \), use separating hyperplane with \( \mathcal{X} \).
  - Else use \( \nabla f(c) \)
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- Action of algorithm in each iteration other than the last can be described independently of $T$
  - If ellipsoid center $c \notin \mathcal{X}$, use separating hyperplane with $\mathcal{X}$.
  - Else use $\nabla f(c)$
- Run this simulation until enough iterations have passed, and take the best feasible point encountered. This must be in $K$. 

Complexity of Convex Optimization
Outline

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2. Complexity of Convex Optimization
3. Complexity of Linear Programming
4. Equivalence of Separation and Optimization
Recall: Linear Programming Problem

A problem of maximizing a linear function over a polyhedron.

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\begin{align*}
\text{maximize} & \quad c^\top x \\
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- When stated in standard form, optimal solution occurs at a vertex.
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- We will consider both explicitly and implicitly described LPs
  - Explicit: given by \(A, b\) and \(c\)
  - Implicit: Given by \(c\) and a separation oracle for \(Ax \leq b\).
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- In both cases, we require all numbers to be rational
- In the explicit case, we require polynomial time in \(\langle A \rangle, \langle b \rangle, \text{ and } \langle c \rangle\), the number of bits used to represent the parameters of the LP.
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  - Explicit: given by \( A, b \) and \( c \)
  - Implicit: Given by \( c \) and a separation oracle for \( Ax \leq b \).
- In both cases, we require all numbers to be rational
- In the explicit case, we require polynomial time in \(<A>, <b>, \text{and } <c>,\) the number of bits used to represent the parameters of the LP.
- In the implicit case, we require polynomial time in the bit complexity of individual entries of \( A, b, c \).
Theorem (Polynomial Solvability of Explicit LP)

There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.

Proof Sketch (Informal)

Using result for weakly solving convex programs, we need 4 things:

- A separation oracle for $Ax \preceq b$: trivial when explicitly represented
- A first order oracle for $c^T x$: also trivial
- A bounding ellipsoid of volume at most an exponential times the volume of the feasible polyhedron: tricky
- A way of “rounding” an $\epsilon$-optimal solution to an optimal vertex solution: tricky
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Solution to both issues involves tedious accounting of numerical issues
Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

**Lemma**

Let $v$ be vertex of the polyhedron $Ax \leq b$. It is the case that $v$ has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.

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- Bounding ellipsoid: all vertices contained in the box $-2^M \leq x \leq 2^M$, which in turn is contained in an ellipsoid of volume exponential in $M$ and $n$. 
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To guarantee volume lower bound, need to instead solve a “relaxed problem”. Specifically, relaxing to $Ax \leq b + \epsilon$, for sufficiently small $\epsilon$ with $\langle \epsilon \rangle = poly(M)$. Gives volume exponentially small in $M$, but no smaller. Still close enough to original polyhedron so solution to relaxed problem can be “rounded” to solution of the original problem.
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Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- Rounding to a vertex: If a point $y$ is $\epsilon$-optimal for the $\epsilon$-relaxed problem, for sufficiently small $\epsilon$ chosen carefully to polynomial in description of input, then rounding to the nearest $x$ with $M$ bits recovers the vertex.
Theorem (Polynomial Solvability of Implicit LP)

Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):

- A separation oracle for the polyhedron $Ax \preceq b$
- Explicit access to $c$

Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\text{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).

Informal Proof Sketch (Primal)

Separation oracle and first order oracle are given. Rounding to a vertex exactly as in the explicit case. Every vertex $v$ still has polynomial bit complexity $M$. Bounding ellipsoid: Still true that we get a bounding ellipsoid of volume exponential in $\langle I \rangle$ and $n$. However, no lower bound on the volume of $Ax \preceq b$, and can't relax to $Ax \preceq b + \epsilon$ as in the explicit case. (In fact, volume may be zero!) Turns out this is still OK, but takes a lot of work (see references).
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For the dual, we need equivalence of separation and optimization. Also, we necessarily get a solution to a normalized version of the dual.

(HW)
Outline

1. Recapping the Ellipsoid Method
2. Complexity of Convex Optimization
3. Complexity of Linear Programming
4. Equivalence of Separation and Optimization
One interpretation of the previous theorem is that optimization of linear functions over a polytope of polynomial bit complexity reduces to implementing a separation oracle.

As it turns out, the two tasks are polynomial-time equivalent.
Separation and Optimization

- One interpretation of the previous theorem is that optimization of linear functions over a polytope of polynomial bit complexity reduces to implementing a separation oracle.
- As it turns out, the two tasks are polynomial-time equivalent.

Let's formalize the two questions, parametrized by a polytope $P$.

**Linear Optimization Problem**
- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $\arg\max_{x \in P} c^T x$.

**Separation Problem**
- Input: $y \in \mathbb{R}^n$.
- Output: Decide that $y \in P$, or else find $h \in \mathbb{R}^n$ s.t. $h^T x < h^T y$ for all $x \in P$. 

Equivalence of Separation and Optimization
Recall: Minimum Cost Spanning Tree

Given a connected undirected graph $G = (V, E)$, and costs $c_e$ on edges $e$, find a minimum cost spanning tree of $G$. 

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Spanning Tree Polytope

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\sum_{e \subseteq X} x_e \leq |X| - 1, \quad \text{for } X \subset V.
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\sum_{e \in E} x_e = n - 1
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- Optimization: Find the minimum/maximum weight spanning tree
- Separation: Find \( X \subset V \) with \( \sum_{e \subseteq X} x_e > |X| - 1 \), if one exists
  - i.e. When edge weights are \( x \), find a “dense” subgraph

Equivalence of Separation and Optimization
Theorem (Equivalence of Separation and Optimization for Polytopes)

Consider a family $\mathcal{P}$ of polytopes $P = \{x : Ax \leq b\}$ described implicitly using $\langle P \rangle$ bits, and satisfying $\langle a_{ij} \rangle, \langle b_i \rangle \leq \text{poly}(\langle P \rangle, n)$. Then the separation problem is solvable in $\text{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the linear optimization problem is solvable in $\text{poly}(\langle P \rangle, n, \langle c \rangle)$ time.

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- We already saw the proof of the forward direction, via Ellipsoid method
  - Separation $\Rightarrow$ optimization
- For the other direction, we need polars
Recall: Polar Duality of Convex Sets

One way of representing the all halfspaces containing a convex set.

**Polar**

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The **polar** of $S$ is defined as follows:

$$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$$

**Note**

- Every halfspace $a^T x \leq b$ with $b \neq 0$ can be written as a “normalized” inequality $y^T x \leq 1$, by dividing by $b$.
- $S^\circ$ can be thought of as the normalized representations of halfspaces containing $S$. 

Equivalence of Separation and Optimization
Properties of the Polar

1. If $S$ is bounded and $0 \in \text{interior}(S)$, then the same holds for $S^\circ$.
2. $S^{oo} = S$

$S = \{x : y \cdot x \leq 1 \text{ for all } y \in S^\circ\}$

$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$
Given a polytope $P$ represented as $Ax \leq \mathbf{1}$, the polar $P^\circ$ is the convex hull of the rows of $A$.

- Facets of $P$ correspond to vertices of $P^\circ$.
- Dually, vertices of $P$ correspond to facets of $P^\circ$. 

Polytopes
Proof Outline: Optimization $\Rightarrow$ Separation

Separation on $P$

Ellipsoid

Optimization on $P$
Proof Outline: Optimization $\Rightarrow$ Separation

- Separation on $P$
- Optimization on $P$
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- Optimization on $P^o$
- Separation on $P^o$
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Equivalence of Separation and Optimization
\[ S = \{ x : y \cdot x \leq 1 \text{ for all } y \in S^\circ \} \quad \text{and} \quad S^\circ = \{ y : x \cdot y \leq 1 \text{ for all } x \in S \} \]
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**Lemma**

Separation over \( S \) reduces in constant time to optimization over \( S^\circ \), and vice versa since \( S^{\circ\circ} = S \).
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- We are given vector \( x \), and must check whether \( x \in S \), and if not output separating hyperplane.
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- equivalently, iff \( \max_{y \in S^\circ} y \cdot x \leq 1 \).
- If we find \( y \in S^\circ \) s.t. \( y \cdot x > 1 \), then \( y \) is the separating hyperplane
  - \( y^T z \leq 1 < y^T x \) for every \( z \in S \).
Equivalence of Separation and Optimization
Optimization $\iff$ Separation

**Technical Note 1**

Need to “center” polytopes about origin. Can do that by running ellipsoid method to find a strictly feasible point in $P$. 

Equivalence of Separation and Optimization
For up arrow (applying ellipsoid to $P^\circ$), need polynomial bit complexity of facets of $P^\circ$. Follows from polynomial bit complexity of vertices of $P$. 

Technical Note 2
Essentially everything we proved about equivalence of separation and optimization for polytopes extends (approximately) to arbitrary convex sets, so long as you can circumscribe the convex set.
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Given closed convex $P \subseteq \mathbb{R}^n$, and radius $R$ s.t. $P \subseteq B(0, R)$:

### Weak Optimization Problem
- **Input:** Linear objective $c \in \mathbb{R}^n$.
- **Output:** $x \in P + \epsilon$, and $c^\top x \geq \max_{x' \in P} c^\top x' - \epsilon$

### Weak Separation Problem
- **Input:** $y \in \mathbb{R}^n$
- **Output:** Decide that $y \in P - \epsilon$, or else find $h \in \mathbb{R}^n$ with $\|h\| = 1$ s.t. $h^\top x < h^\top y + \epsilon$ for all $x \in P$. 
Theorem (Equivalence of Separation and Optimization for Convex Sets)

Consider a family $\mathcal{P}$ of convex sets described implicitly using $\langle P \rangle$ bits, and suppose that for each $P \in \mathcal{P}$ we are also given rational $R$ s.t. $P \subseteq B(0, R)$. The weak separation problem is solvable in $\text{poly}(\langle P \rangle, \langle R \rangle, n, \langle y \rangle, \log(1/\epsilon))$ time for $P \in \mathcal{P}$ if and only if the weak optimization problem is solvable in $\text{poly}(\langle P \rangle, \langle R \rangle, n, \langle c \rangle, \log(1/\epsilon))$ time.
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- Weak separation suffices for ellipsoid, which is only approximately optimal anyways.
- By polarity, weak optimization is equivalent to weak separation.
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- For proof / details, see the GLS book.
Implication: Operations preserving solvability

- Assume you can efficiently optimize over two convex sets $P$ and $Q$

**Question**

What about $P \cap Q$ and $P \cup Q$?
Implication: Operations preserving solvability

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**Question**

What about $P \cap Q$ and $P \cup Q$?

**$P \cap Q$**

- Yes! Follows from equivalence of separation and optimization.
- Specifically, can separate over $P$ and $Q$ individually, therefore can separate over $P \cap Q$, and then can optimize over $P \cap Q$.
- Applications: colorful spanning tree, cardinality-constrained matching, ...
Implication: Operations preserving solvability

Assume you can efficiently optimize over two convex sets $P$ and $Q$.

**Question**

What about $P \cap Q$ and $P \cup Q$?

**$P \cup Q$**

- Yes! Simply optimize over each separately, and take the better of the two outcomes.
- Equivalent to optimizing over the convex hull of $P \cup Q$.
- Implication of Separation/optimization equivalence: there is a separation oracle for $\text{convexhull}(P \cup Q)$.
Implication: Constructive Caratheodory

Problem

Given a point \( x \in \mathcal{P} \), where \( \mathcal{P} \subseteq \mathbb{R}^n \) is a solvable polytope, write \( x \) as a convex combination of \( n + 1 \) vertices of \( \mathcal{P} \), and do so in polynomial time.

- Existence: Caratheodory’s theorem.
- E.g. Birkhoff Von-Neumann, fractional spanning trees, fractional matchings, . . .
- Follows from equivalence of separation and optimization. See HW.