Outline

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Mathematical Optimization

The task of selecting the “best” configuration of a set of variables from a “feasible” set of configurations.

\[
\begin{align*}
\text{minimize (or maximize) } & \quad f(x) \\
\text{subject to } & \quad x \in \mathcal{X}
\end{align*}
\]

- Terminology: decision variable(s), objective function, feasible set, optimal solution, optimal value
- Two main classes: continuous and combinatorial
Optimization problems where feasible set $\mathcal{X}$ is a connected subset of Euclidean space, and $f$ is a continuous function.

Instances typically formulated as follows.

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq b_i, \quad \text{for } i \in \mathcal{C}.
\end{align*}
\]

- Objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.
- Constraint functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$. The inequality $g_i(x) \leq b_i$ is the $i$’th constraint.
- In general, intractable to solve efficiently (NP hard)
Convex Optimization Problem

A continuous optimization problem where $f$ is a convex function on $\mathcal{X}$, and $\mathcal{X}$ is a convex set.

- **Convex function**: $f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y)$ for all $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$
- **Convex set**: $\alpha x + (1 - \alpha) y \in \mathcal{X}$, for all $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$
- Convexity of $\mathcal{X}$ implied by convexity of $g_i$'s
- For maximization problems, $f$ should be **concave**
- Typically solvable efficiently (i.e. in polynomial time)
- Encodes optimization problems from a variety of application areas
Convex Optimization Example: Least Squares Regression

Given a set of measurements \((a_1, b_1), \ldots, (a_m, b_m)\), where \(a_i \in \mathbb{R}^n\) is the \(i\)'th input and \(b_i \in \mathbb{R}\) is the \(i\)'th output, find the linear function \(f : \mathbb{R}^n \to \mathbb{R}\) best explaining the relationship between inputs and outputs.

\begin{itemize}
  \item \(f(a) = x^\top a\) for some \(x \in \mathbb{R}^n\)
  \item Least squares: minimize mean-square error.
\end{itemize}

\[
\text{minimize} \quad \|Ax - b\|_2^2
\]
Given a directed network \( G = (V, E) \) with cost \( c_e \in \mathbb{R}^+ \) per unit of traffic on edge \( e \), and capacity \( d_e \), find the minimum cost routing of \( r \) divisible units of traffic from \( s \) to \( t \).
Given a directed network $G = (V, E)$ with cost $c_e \in \mathbb{R}_+$ per unit of traffic on edge $e$, and capacity $d_e$, find the minimum cost routing of $r$ divisible units of traffic from $s$ to $t$.

minimize  \[ \sum_{e \in E} c_e x_e \]
subject to  \[ \sum_{e \leftarrow v} x_e = \sum_{e \rightarrow v} x_e, \quad \text{for } v \in V \setminus \{s, t\}. \]
\[ \sum_{e \leftarrow s} x_e = r \]
\[ x_e \leq d_e, \quad \text{for } e \in E. \]
\[ x_e \geq 0, \quad \text{for } e \in E. \]
Given a directed network \( G = (V, E) \) with cost \( c_e \in \mathbb{R}_+ \) per unit of traffic on edge \( e \), and capacity \( d_e \), find the minimum cost routing of \( r \) divisible units of traffic from \( s \) to \( t \).

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\end{align*}
\]

\[
\sum_{e \leftarrow s} x_e = r
\]

\[
x_e \leq d_e, \quad \text{for } e \in E.
\]

\[
x_e \geq 0, \quad \text{for } e \in E.
\]

Generalizes to traffic-dependent costs. For example

\[
c_e(x_e) = a_e x_e^2 + b_e x_e + c_e.
\]

Course Overview
Combinatorial Optimization

Combinatorial Optimization Problem

An optimization problem where the feasible set $\mathcal{X}$ is finite.

- e.g. $\mathcal{X}$ is the set of paths in a network, assignments of tasks to workers, etc...
- Again, NP-hard in general, but many are efficiently solvable (either exactly or approximately)
Combinatorial Optimization Example: Shortest Path

Given a directed network $G = (V, E)$ with cost $c_e \in \mathbb{R}_+$ on edge $e$, find the minimum cost path from $s$ to $t$. 

![Graph Diagram]

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Combinatorial Optimization Example: Traveling Salesman Problem

Given a set of cities $V$, with $d(u, v)$ denoting the distance between cities $u$ and $v$, find the minimum length tour that visits all cities.
Continuous vs Combinatorial Optimization

- Some optimization problems are best formulated as one or the other.
- Many problems, particularly in computer science and operations research, can be formulated as both.
- This dual perspective can lead to structural insights and better algorithms.
The shortest path problem can be encoded as a minimum cost flow problem, using distances as the edge costs, unit capacities, and desired flow rate 1

The optimum solution of the (linear) convex program above will assign flow only on a single path — namely the shortest path.
Course Goals

- Recognize and model convex optimization problems, and develop a general understanding of the relevant algorithms.
- Formulate combinatorial optimization problems as convex programs
- Use both the discrete and continuous perspectives to design algorithms and gain structural insights for optimization problems
Who Should Take this Class

Anyone planning to do research in the design and analysis of algorithms
- Convex and combinatorial optimization have become an indispensable part of every algorithmist’s toolkit

Students interested in theoretical machine learning and AI
- Convex optimization underlies much of machine learning
- Submodularity has recently emerged as an important abstraction for feature selection, active learning, planning, and other applications

Anyone else who solves or reasons about optimization problems: electrical engineers, control theorists, operations researchers, economists . . .
- If there are applications in your field you would like to hear more about, let me know.
Who Should Not Take this Class

- You don’t satisfy the prerequisites “in practice”
- You are looking for a “cookbook” of optimization algorithms, and/or want to learn how to use CPLEX, CVX, etc
  - This is a THEORY class
  - We will bias our attention towards simple yet theoretically insightful algorithms and questions
  - We will not write code
Weeks 1-5: Convex optimization basics and duality theory
Weeks 6-7: Combinatorial problems posed as linear and convex programs
Weeks 8-9: Algorithms for convex optimization
Weeks 10-11: Matroid theory and optimization
Weeks 12-13: Submodular Function optimization
Week 14: Semidefinite programming and constraint satisfaction problems
Week 15: Additional topics
Outline

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2. Administrivia
Basic Information

- Lecture time: Mondays and Wednesdays 4:00pm - 5:50pm
- Lecture place: VHE 214
- Instructor: Shaddin Dughmi
  - Email: shaddin@usc.edu
  - Office: SAL 234
  - Office Hours: TBD
- TA: TBA
  - Email: TBA
  - Office Hours: TBA
- Course Homepage: http://www-bcf.usc.edu/shaddin/cs675fa19/index.html
- References: Convex Optimization by Boyd and Vandenberghe, and Combinatorial Optimization by Korte and Vygen. (Available online through USC libraries. Will place on reserve)
- Additional References: Schrijver, Luenberger and Ye (available online through USC libraries)
Prerequisites

- Mathematical maturity: Be good at proofs, at the graduate level.
- Linear algebra at advanced undergrad / beginning grad level
- Exposure to algorithms or optimization at advanced undergrad / beginning grad level
  - CS570 or equivalent, or
  - CS270 and you did really well
This is an advanced elective class, so grade is not the point.
  I assume you want to learn this stuff.

4-6 homeworks, 75% of grade.
  Proof based.
  Challenging.
  Discussion allowed, even encouraged, but must write up solutions independently.

Research project worth 25% of grade. Project suggestions will be posted on website.

5 late days allowed total (use in integer amounts)