Cost Sharing Network Routing Game

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1 The Game

Consider a graph $G(V,E)$ and n players, where the i^{th} player is interested in moving from his source node $s_i \in V$ to a common sink $t \in V$. Each player strategically chooses a path, p_i, from his source to the sink in order to minimize his own cost C_i. The cost of a path p_i depends on all the rest p_j and decreases with the number of players *sharing* the path [1].

To fix ideas, we can think of the graph as a public transportation network, where each edge represents a bus route connecting two stations. The n players can be thought of as employees that want to commute from their home stations s_i to a company located at station t. Assume that the operation of each route requires a fixed amount of money, say one unit of cost, which people using that route can share. Now, the natural objective of each employee is to find a path that minimizes his commuting expenses.

More specifically, the cost function of the i^{th} player can be written as

$$C_i(\mathtt{P}) = \sum_{e \in p_i} \frac{1}{n_p(e)} \tag{1}$$

where $\mathtt{P} = [p_1, \dots, p_i, \dots, p_n]^T$ is the strategy vector and $n_p(e) = \big|\{p_j \in \mathtt{P} | e \in p_j\}\big|$ is the number of paths in $\mathtt{P}$ sharing an edge e.

Under **best-response dynamics**, the i^{th} player "wakes up" arbitrarily, observes the current strategic configuration $\mathtt{P} = [p_1, \dots, p_i, \dots, p_n]^T$, and chooses the ${p_i}'$ that minimizes (and strictly improves) his own cost $C_i(\mathtt{P}')$, $\mathtt{P}' = [p_1, \dots, p_i', \dots, p_n]^T$. We say that the game reaches an *equilibrium* if no player has incentive to deviate from his current strategy, i.e., change his path. More formally, if $\mathtt{P}^{[k]}$ denotes the strategic configuration at time k, then there is a time $1 < t_{eq} < \infty$ such that if we wake any player i at time t_{eq}, then we have $\mathtt{P}^{[t_{eq}+1]} = \mathtt{P}^{[t_{eq}]}$. In fact, $\mathtt{P}^{[k]} = \mathtt{P}^{[t_{eq}]}$, $\forall k \geq t_{eq}$.

A cost sharing network routing (CSNR) instance can be described by the following tuple:

$$CSNR = \big(\mathcal{N}, G(V,E), \{\mathcal{P}_i\}_{i\in\mathcal{N}}, \{C_i : \times_{j\in\mathcal{N}}\mathcal{P}_j \to \mathbb{R}\}_{i\in\mathcal{N}}\big)$$

where $\mathcal{N} = \{1, 2, \dots, n\}$ is the index set of the players, $G(V,E)$ is the network graph, $\mathcal{P}_i$ is the set of all simple paths from $s_i \in V$ to $t \in V$ in $G(V,E)$ (all the i^{th} player's strategies), and C_i is the i^{th} player's cost function as defined in (1).

[1] Given a graph $G(V,E)$, a path p from $s \in V$ to $t \in V$ can be formally defined as the set of adjacent edges: $p = \{(s,v_1),(v_1,v_2),\dots,(v_{k-1},v_k),(v_k,t) | v_i \in V\}$

2 Existence of Equilibrium

2.1 Best Response Graphs

The best response dynamics of a game can be modeled as a directed graph $F(P, R)$. The vertices, $v \in P$, in the graph are the configurations of the strategic choices and the edges, $e \in R$, correspond to player's best responses. In our game, the vertices are $\mathtt{P} = [p_1, \ldots, p_i, \ldots, p_n]^T$, $p_j \in \mathcal{P}_j$ and there is an edge from $\mathtt{P}$ to $\mathtt{P}' = [p_1, \ldots, p_i', \ldots, p_n]^T$ if $p_i{}'$ is the minimizer of $C_i(\mathtt{P}')$, with $C_i(\mathtt{P}) - C_i(\mathtt{P}') > 0$. Note that in CSNR game the set of all possible configurations $\mathtt{P}$ is finite and the graph is finite.

2.2 Potential Games

Consider a function Φ that maps configurations in a game to a real number. In our example Φ maps $\mathtt{P} = [p_1, \ldots, p_i, \ldots, p_n]^T$ to the reals. Now consider an edge in the best response graph from $\mathtt{P}$ to $\mathtt{P}' = [p_1, \ldots, p_i', \ldots, p_n]^T$. Call this change $A \to B_i$. We now define two functions.

$$\Delta\Phi(A, B_i) := \Phi(B_i) - \Phi(A) \tag{2}$$

$$\Delta C_i(A, B_i) := C_i(B_i) - C_i(A) \tag{3}$$

Recall that C_i is the cost function of the i^{th} player. Note that the function $\Delta C_i(A, B_i)$ is defined for that i that made the best response in $A \to B_i$. A game is a *potential game* if there exists a function Φ such that, for all A and for all i, $\Delta\Phi(A, B_i)$ and $\Delta C_i(A, B_i)$ have the same sign.

2.3 Existence of Equilibrium for Potential Games

We now argue that potential games have an equilibrium that can be achieved through best response dynamics. Consider the best response graph of a potential game. Observe that for every edge (A, B_i) in the graph, $\Delta C_i(A, B_i)$ is strictly negative. Which implies that $\Phi(B_i) < \Phi(A)$. That is for all edges (A, B) in the graph, $\Phi(B) < \Phi(A)$.

Assume there exists a cycle in the graph. Then clearly there exists an edge (A, B) in the cycle such that $\Phi(B) \geq \Phi(A)$. This is a contradiction. Thus, our graph is a **finite directed acyclic graph**. Such graphs always have a sink node. Clearly, such a sink node represents an equilibrium.

2.4 CSNR game is a Potential Game

Here we will prove that CSNR game is a potential game and thus has an equilibrium that can be achieved through best response dynamics.

Proof. Define a function $\Phi : \times_{j \in \mathcal{N}} \mathcal{P}_j \to \mathbb{R}$ as follows:

$$\Phi(\mathtt{P}) = \sum_{e \in E} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)} \right) \tag{4}$$

where $\mathtt{P} = [p_1, \ldots, p_i, \ldots, p_n]^T$, $p_j \in \mathcal{P}_j$, and $n_p(e) = \big|\{p_j \in \mathtt{P} | e \in p_j\}\big|$. We will show that Φ is a potential function for the CSNR game.

Let $\mathtt{P}' = [p_1, \ldots, p_i', \ldots, p_n]^T$ be the strategic configuration after the i^{th} player's best response, with $\Delta C_i(\mathtt{P}, \mathtt{P}') < 0$. We want to show that $\Delta\Phi(\mathtt{P}, \mathtt{P}') < 0$.

Let $E^+ = \{e \in p_i' \setminus p_i\}$ and $E^- = \{e \in p_i \setminus p_i'\}$ be the set of edges included only in p_i' and p_i respectively. We can partition the edges as the union of four disjoint sets:

$$E = E^+ \cup E^- \cup \{e \in p_i' \cap p_i\} \cup \{e \notin p_i' \cup p_i\}$$

It is evident that $n_{p'}(e) \neq n_p(e)$ only for edges $e \in E^+ \cup E^-$, since the number, $n_{p'}(e)$, of players using an edge, changes only for the edges where the paths p_i and p_i' differ. More specifically,

$$n_{p'}(e) = \begin{cases} n_p(e) + 1, , & \text{if } e \in E^+ \\ n_p(e) - 1, & \text{if } e \in E^- \end{cases}$$

We can write:

$$\begin{aligned}
\Delta\Phi(\mathtt{P}, \mathtt{P}') &= \Phi(\mathtt{P}') - \Phi(\mathtt{P}) = \sum_{e \in E}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_{p'}(e)}\right) - \sum_{e \in E}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)}\right) \\
&= \sum_{e \in E^+}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_{p'}(e)}\right) + \sum_{e \in E^-}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_{p'}(e)}\right) \\
&\quad - \left[\sum_{e \in E^+}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)}\right) + \sum_{e \in E^-}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)}\right)\right] \\
&= \sum_{e \in E^+}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)+1}\right) + \sum_{e \in E^-}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)-1}\right) \\
&\quad - \sum_{e \in E^+}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)}\right) - \sum_{e \in E^-}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n_p(e)}\right) \\
&= \sum_{e \in E^+} \frac{1}{n_p(e)+1} - \sum_{e \in E^-} \frac{1}{n_p(e)} \\
&= \sum_{e \in E^+} \frac{1}{n_p(e)+1} + \sum_{e \in p_i' \cap p_i} \frac{1}{n_p(e)} - \sum_{e \in E^-} \frac{1}{n_p(e)} - \sum_{e \in p_i' \cap p_i} \frac{1}{n_p(e)} \\
&= \sum_{e \in p_i'} \frac{1}{n_{p'}(e)} - \sum_{e \in p_i} \frac{1}{n_p(e)} = C_i(\mathtt{P}') - C_i(\mathtt{P}) = \Delta C_i(\mathtt{P}, \mathtt{P}') < 0
\end{aligned}$$

Hence, $\Phi(\mathtt{P})$ is a potential function for the CSNR game.

$\square$

3 Convergence to Equilibrium

In the previous sections we proved that an equilibrium point *exists* for the CSNR game. A natural question, now, is *how fast* the players can reach it under best response dynamics.

3.1 How long can a sequence of best responses be?

Consider the best response graph $F(P,R)$ for the CSNR game, as described in section 2.1, where vertices correspond to strategic configurations $\mathtt{P}$, and edges represent best responses. We have seen that $F(P,R)$ is a finite directed acyclic graph and thus any path from a vertex $\mathtt{P}$ to a sink $\mathtt{P}^*$ is of finite length. But how does this path length *scale* with respect to the number of players and the size of the network?

Let $r(s)$ be a the path in $F(P,R)$, from a vertex $s \in P$ to its corresponding sink $t_s \in P$. We are interested in bounding the number of edges in $r(s)$ for all $s \in P$. Since $r(s)$ is a best-response sequence of a potential game, there is a strictly negative number $\Delta\Phi(v_k, v_l)$ associated with every edge $(v_k, v_l) \in r(s)$, that corresponds to the reduction in the potential function after some player's response from v_k to v_l.

For a path $r(s)$ in $F(P,R)$ we have:

$$\Phi(t_s) \;=\; \Phi(s) \;-\sum_{(v_k,v_l)\in r(s)} |\Delta\Phi(v_k,v_l)| \tag{5}$$

where $\Phi(t_s)$ is a local minimum of the potential function that is reached by starting from $\Phi(s)$ and following the best response path $r(s)$. See Fig.1 for an illustration of Eq.5.

To upper bound the number of edges in $r(s)$ we can write:

$$\begin{aligned}\Phi(t_s) \;&\leq\; \Phi(s) \;-\sum_{(v_k,v_l)\in r(s)} \min_{(v_k,v_l)\in r(s)} \big\{|\Delta\Phi(v_k,v_l)|\big\} \\ &=\; \Phi(s) \;- A(s) \min_{(v_k,v_l)\in r(s)} \big\{|\Delta\Phi(v_k,v_l)|\big\}\end{aligned}$$

or, equivalently:

$$A(s) \leq \frac{\Phi(s) - \Phi(t_s)}{\min_{(v_k,v_l)\in r(s)}\big\{|\Delta\Phi(v_k,v_l)|\big\}}$$

where $A(s) = \big|\{(v_k,v_l) \in r(s)\}\big|$, is the number of edges in $r(s)$.

Since $0 \leq \Phi(a) \leq |\mathrm{E}|(\ln(n)+1)$, $\forall a \in P$, (this can be shown directly from in Eq.4, by setting $n_p(e) = n$, $\forall e \in E$ and upper bounding the harmonic number H_n by $\ln(n)+1$), we can further upper bound the number of edges *for all paths* $r(s)$ *in* $F(P,R)$ by writing:

$$A(s) \leq \frac{|\mathrm{E}|(\ln(n)+1)}{\min_{(v_k,v_l)\in R}\big\{|\Delta\Phi(v_k,v_l)|\big\}}\ ,\ \ \forall s \in P \tag{6}$$

Hence, the length of all best-response sequences is *upper bounded* by a quantity inversely proportional to the minimum possible change $|\Delta\Phi(\mathtt{P},\mathtt{P}')|$ in the potential function.

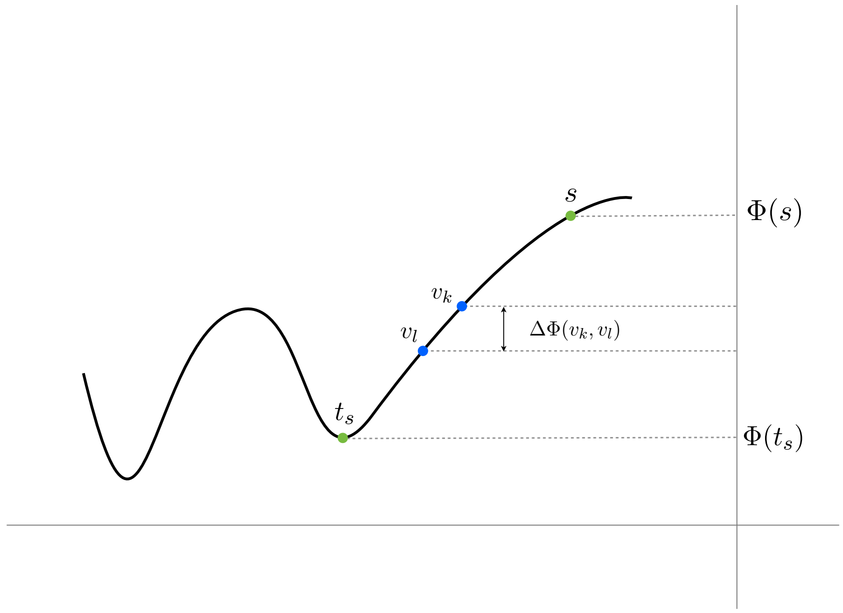

Figure 1: A best-response path to equilibrium and strictly negative reductions in the potential function

3.2 How small can $|\Delta\Phi(\mathtt{P}, \mathtt{P}')|$ be?

Let $\mathtt{P} = [p_1, \ldots, p_i, \ldots, p_n]^T$ be an arbitrary strategic configuration and let $\mathtt{P}' = [p_1, \ldots, p_i', \ldots, p_n]^T$ be the strategic configuration after the i^{th} player's best response. Then,

$$\begin{aligned} \Delta\Phi(\mathtt{P}, \mathtt{P}') &= \Delta C_i(\mathtt{P}, \mathtt{P}') \\ &= C_i(\mathtt{P}') - C_i(\mathtt{P}) \\ &= \sum_{e \in p_i'} \frac{1}{n_{p'}(e)} - \sum_{e \in p_i} \frac{1}{n_p(e)} \end{aligned} \tag{7}$$

Lemma 1. $d(K, n) \triangleq \left| \sum_{i=1}^{K_1} \frac{1}{a_i} - \sum_{j=1}^{K_2} \frac{1}{b_j} \right| \geq n^{-2K}$, *for all integers* $a_i, b_j \in \{1, \ldots, n\}$ *and* $K_1, K_2 \in \{1, \ldots, K\}$: $d(K, n) \neq 0$

Proof. (by Elsholtz) First notice that a sum of K_1 unit fractions, each of denominator $a_i \leq n$, can be rewritten as a single fraction with a denominator bounded by the product of the a_i, i.e, by n^{K_1}. Thus, $\sum_{i=1}^{K_1} \frac{1}{a_i} = \frac{x_a}{m_a}$, with $m_a \leq n^{K_1}$ and $\sum_{i=1}^{K_2} \frac{1}{b_j} = \frac{x_b}{m_b}$, with $m_b \leq n^{K_2}$, $\forall a_i, b_j \in \{1, \ldots, n\}$, and their absolute difference can be written as $d(K, n) = \left| \frac{x_a}{m_a} - \frac{x_b}{m_b} \right| = \frac{|m_b x_a - m_a x_b|}{m_a m_b}$. Since $d(K, n)$ is constrained to be nonzero, the smallest possible value for $|m_b x_a - m_a x_b|$ is one (must be integer). Hence, $d(K, n) \geq \frac{1}{m_a m_b} \geq \frac{1}{n^{(K_1+K_2)}} \geq n^{-2K}$, $\forall K_1, K_2 \in \{1, \ldots, K\}$. $\square$

Now, we can use Lemma 1 to lower bound the expression in Eq.7 and get :

$$\left|\Delta\Phi(\mathtt{P},\mathtt{P}')\right| \geq n^{-2|\mathrm{E}|} \tag{8}$$

Hence, a player's response reduces the potential function $\Phi(\mathtt{P})$ by at least $1/n^{2|\mathrm{E}|}$.

3.3 The speed of convergence

Putting Eq.6 and Eq.8 together, we get:

$$\begin{aligned} A(s) &\leq \frac{|\mathrm{E}|(\ln(n)+1)}{\min_{(v_k,v_l)\in R}\{|\Delta\Phi(v_k,v_l)|\}} \\ &\leq |\mathrm{E}|(\ln(n)+1)/n^{-2|E|}\,,\ \forall s \in P \end{aligned} \tag{9}$$

Hence, the CSNR game reaches an equilibrium after $\mathcal{O}\left(|E|n^{2|E|}\log(n)\right)$ best responses.

References

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