

CS 599 - Games, Economics, Networks and Data Analysis - Lecture 2

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1 Best Response, Equilibrium Strategies and Best Response Dynamics

Establishing the notion of best response goes back to the work of von Neumann. In some sense, the best response captures the distributed decision of particles in a system with selfish and diverse interests. His work is followed by the contributions of giants of game and economic theory such as Nash and Arrow-Debrew. This notion is the essence of our understanding of rational agents and is the key step in order to develop the notion of equilibrium of a game. We will define these notions in the context of games related to computer science applications. Namely, we look at two instances of, so called, *congestion games* as explained below.

1.1 Network Routing Game

In our first example, we consider the situation in which our players are interested to route their traffic in a shared network. Namely, we are given a graph $G(V, E)$ where for example an edge e between u and v represents a link between two internet routers.

- There are n users $1, \dots, l$ where user i is interested to route from node s_i to node t_i .
- Each link has a delay which depends on the number of users using that link.
- Every user is selfish and wants to minimize her own total delay (Therefore, the delay is assumed to be additive).

Notice that in the absence of other users, the best strategy in order to meet the objective is to take the shortest path (The path with the shortest

number of hubs, assuming that the links are homogenous.) Here, by a strategy, we mean any of the potential paths that connect s_i to t_i . Now, suppose that each user somehow chooses one of such paths as her strategy. Then, consider $(P_1, P_2, \dots, P_i, \dots, P_n)$ which represents a configuration of strategic choices. For every such configuration, we can look at c_i , the cost function of each user. Namely, we can define $c_i(P_1, P_2, \dots, P_i, \dots, P_n)$ as the cost of player i given a configuration. Given a configuration $\mathcal{P} = (P_1, P_2, \dots, P_i, \dots, P_n)$, the best response of any player i can be defined roughly as P_i that minimizes $c_i(P_1, P_2, \dots, P_i, \dots, P_n)$ fixing all other P_j s. A better response is the one that simply has a lower cost compared to our current strategy. And the equilibrium is roughly defined over best responses where everyone does not have an incentive to deviate.

In order to arrive at the precise definition of equilibrium, consider an asynchronous world, in which at every time, some player i “wakes up” arbitrarily and observes the current configuration, and based on it chooses a strategy in the set of all strategies that minimizes his delay at the moment and sleep. Let \mathcal{P}_j denote the configuration at time j . We define this dynamic process reaches an equilibrium if there is a time $1 < j < \infty$ such that if we wake up any player i at j we have $\mathcal{P}_j = \mathcal{P}_{j-1}$. It is easy to check $\mathcal{P}_k = \mathcal{P}_{j-1}$ for all $k \geq j$.

We call the above-explained process the best-response dynamics. Below is a list of questions that can be raised once we fix the dynamics.

1. Is there an equilibrium?
2. Is the equilibrium strategies unique?
3. Is it possible to reach the equilibrium through the best-reponse dynamics (BRD)?
4. How long does it take to reach an equilibrium if one exists and can be reached through BRD?
5. Is the game one-shot or repeated?

2 Existence of Equilibrium

2.1 Best Response Graphs:

The best response dynamics of a game can be modelled as a directed graph. We illustrate this using our congestion game example. The vertices in the graph are the configurations of the strategic choice, which in our case are just the sets of paths chosen by the players. Observe that in this example, the set of possible configurations is finite and the graph is finite. Given a configuration, one of the players reacts and makes his best response. In particular, if the i^{th} player responds, then the configuration changes from $(P_1, P_2, \dots, P_i, \dots, P_n)^t$ to $(P_1, P_2, \dots, P'_i, \dots, P_n)^t$.

To capture this change, there is an edge from the vertex $(P_1, P_2, \dots, P_i, \dots, P_n)^t$ to $(P_1, P_2, \dots, P'_i, \dots, P_n)^t$ from every $i \in 1, 2, \dots, n$.

2.2 Potential Games:

Our goal in this section is to prove the existence of equilibrium for the congestion game example. We will do so by proving the existence of equilibrium for a class of games called potential games. And then show our congestion game is a potential game.

Consider a function Φ that maps configurations in a game to a real number. In our example Φ maps (P_1, P_2, \dots, P_n) to the reals. Now consider an edge in the dynamic response graph from $(P_1, P_2, \dots, P_i, \dots, P_n)^t$ to $(P_1, P_2, \dots, P'_i, \dots, P_n)^t$. Call this change $A \rightarrow B_i$. We now define two functions.

$$\Delta\Phi(A, B_i) := \Phi(B_i) - \Phi(A)$$

$$\Delta\mu_i(A, B_i) := \mu_i(B_i) - \mu_i(A)$$

Recall that μ_i is the utility function of the i^{th} player. Note that the function $\Delta\mu_i(A, B_i)$ is defined for that i that made the best response in $A \rightarrow B_i$. A game is a potential game if there exists a function Φ such that, for all A and for all i , $\Delta\Phi(A, B_i)$ and $\Delta\mu_i(A, B_i)$ have the same sign.

2.3 Existence of Equilibrium for Potential Games

We now argue that potential games have an equilibrium that can be achieved through best response dynamics. Consider the best response graph of a potential game. Observe that for every edge (A, B_i) in the graph, $\Delta\mu_i(A, B_i)$ is strictly positive. Which implies that $\Phi(B_i) > \Phi(A)$. That is for all edges (A, B) in the graph, $\Phi(B) > \Phi(A)$.

Assume there exists a cycle in the graph. Then clearly there exists an edge (A, B) in the cycle such that $\Phi(B) \leq \Phi(A)$. This is a contradiction. Thus our graph is a finite directed acyclic graph. Such graphs always have a sink node (from previous class). Clearly, such a sink node represents an equilibrium.

2.4 Alternate Characterization of Potential Games

Potential games are the games whose best response graph is a directed acyclic graph (DAGs).

From the previous section, it is clear that the best response graphs of potential games are DAGs. The converse is left as an exercise. Hint: Use topological ordering on DAGs, to construct a potential function.

2.5 Equilibrium for the Congestion Game Example

All that remains to be shown is that our example has a potential function. Rosenthal defines a family of potential functions for a class of congestion games [1]. First, we define the notion of edge congestion n_e for our example. Given a configuration, the number of paths sharing an edge is called as the edge connectivity of that edge. Formally,

$$n_e(P_1, P_2, \dots, P_n) = |\{P_i | e \in P_i\}|$$

The potential function is defined as follows

$$\Phi(P_1, P_2, \dots, P_n) = \sum_{e \in E} (0 + 1 + 2 + \dots + n_e)$$

We claim that Φ is indeed a potential function. Consider the transition $A = (P_1, \dots, P_i, \dots, P_n)^t \rightarrow B = (P_1, \dots, P'_i, \dots, P_n)^t$. The only edges whose congestion change are the ones in only one of either P_i or P'_i . Call the set of edges that are in P_i but not in P'_i as Δ_{out} . And the set of edges in P'_i and not in P_i as Δ_{in} . The congestion of the edges in Δ_{out} drop by 1 and those in Δ_{in} increase by 1. Thus

$$\begin{aligned} \Phi(B) - \Phi(A) &= \sum_{e \in \Delta_{in}} n_e - \sum_{e \in \Delta_{out}} n_e \\ &= \sum_{e \in P'_i} n_e - \sum_{e \in P_i} n_e = \Delta\mu_i \end{aligned}$$

Thus $\Delta\Phi$ turns out to be exactly equal to $\Delta\mu_i$ (and hence in sign). Thus our example is a potential game and hence has an equilibrium.

3 Quiz 2

Let $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ where $x_i \in \{0, 1\}, y_i \in \{0, 1\}, \forall i$ be variables. Consider the set of constraints: $y_i = x_{i+1}, 1 \leq i < n$ and $x_1 = 0, y_n = 1$. Prove that for every solution,

- (i) $\exists i$ such that $x_i \neq y_i$
- (ii) $|\{i | x_i \neq y_i\}|$ is odd

Solution: As zero is not odd, (ii) implies (i).

$$\sum_i x_i + y_i = x_0 + y_n = 1 \pmod{2}$$

$\Rightarrow |\{i | x_i \neq y_i\}|$ is odd.

References

- [1] R. W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.