

Lecture 7 - Auctions and Mechanism Design

*Prof. Shang-hua Teng**Scribes: Tomer Levinboim and Scott Alfeld*

An Illustrative Example

We begin with a specific example of buying a house in a realstate auction. There is one house being auctioned, and eight parties interested in purchasing it. The auction process is as follows:

- I All eight parties submit sealed bids to the auctioneer.
- II The auctioneer ignores all bids, and asks for new bids.
- III All eight parties enter new sealed bids to the auctioneer.

The auctioneer then must determine two things – which bidder wins, and how much they pay for the house. In one notable case of such an auction, the winner was Shang-Hua Teng (who had the highest bid), and the asking price was his bid.

However, being the crafty soul our professor is, he made a keen observation. If he, as the highest bidder were simply to refuse to buy the house, the best the auctioneer could expect to receive is the second-highest bid. The auctioneer was therefore being unfair in asking the winner to pay his own bid, and through some machine learning techniques and other acts of mathematical ingenuity, our professor was able to save himself \$80,000. We will see more of this issue about a fair auctioneer in a later section, but first we describe auctions in more detail.

Auctions in General

For now, we will assume single-item Auctions. That is to say we have one item being auctioned, and several bidders aiming to buy that item.

We begin with some notation. We say that every bidder has a valuation for the item being auctioned. This is a measure of how much that bidder wants the item, or in other words, the maximum amount they would be willing to pay to get the item. We define the utility of a bidder to be their valuation minus how much they are charged, and we note that every bidder aims to maximize her own utility. Formally, we have the following terms:

- n – The number of bidders.
- $B = \{b_i\}$ – The set of bids, where b_i is the bid of the i -th player. Unless stated otherwise, we shall impose an ordering on the bids $b_1 > b_2 > \dots b_n$.
- $V = \{v_i\}$ – The set of valuations, where v_i is the valuation of the i -th player.
- $P = \{p_i\}$ – The prices, where p_i is the price that the i -th player is charged.
- $u_i = v_i - p_i$ – The utility of the i -th player.

The mechanism of an auction consists of the following:

- The bidding procedure.
- $\mathcal{P}(B)$, the pricing function, which determines P .
- $\mathcal{A}(B)$, the allocation function, which determines who receives the item.

Note that we will assume \mathcal{P} and \mathcal{A} are deterministic functions.

English Auction

Perhaps the most stereotypical auction is an open-outcry auction, which is a form of an English Auction. In an English Auction, there is a “going price”, which increases over time. The going price may be increased steadily by the auctioneer, or it may be determined by bidders openly exclaiming their bids. We will consider the former method, wherein the going price begins at 0:

1. While more than one bidder is willing to pay the price:
 - The going price increases.
 - All bidders say whether they are willing or unwilling to pay the going price.
2. The item is awarded to the one bidder willing to pay the going price.
3. All bidders that did not win the item pay 0.
4. The winner pays the final going price.

Notice that whenever a player drops out of the auction, his true valuation is revealed. Players have an incentive to remain in the game as long as the going price is less than their own valuation. It is therefore not difficult to see that the winner would be the player with the highest valuation, i.e., player 1. Since player 2 drops out of the auction as soon as the going price is at least v_2 (and since he is the last one to drop out) we conclude that player 1 should pay v_2 and that his utility should be $u_1 = v_1 - v_2 > 0$.

Vickrey Auction

Also known as a second-price auction, the Vickrey Auction[4] procedure was developed by the economics professor William Vickrey and studied by Clarke and Groves. The bidding process in a Vickrey Auction is as follows:

1. The bidders submit sealed bids to the auctioneer.
2. The auctioneer determines the allocation and pricing as follows:
 - (a) Allocation: The highest bidder (player 1) wins the item.
 - (b) Pricing: The winner pays the second highest bid (b_2). All other players pay 0.

Note the following properties of the Vickrey Auction:

- As proved by Vickrey, although the Vickrey auction is a one-shot auction, it is mathematically equivalent to the English auction (i.e., they yield the same results).
- Rational players are encouraged to be truthful (see the following section).
- The allocation produced maximizes social welfare in the sense that the bidder with the highest valuation indeed gets the item (assuming rational bidders).

Truthfulness

Definition An auction mechanism is *truthful* if a bidder’s best option is to bid their valuation (set $b_i := v_i$), no matter what the behavior of the other bidders.

An alternative definition for truthfulness might be “under the assumption that all other players are rational / bid according to their valuations, player P ’s best option is to bid their own valuation”. We will be using the first (stronger) definition.

Claim 1. *The Vickrey auction is truthful.*

Proof. Consider player i and let $B_i = \max_{j \neq i} b_j$. We need to show that setting $b_i := v_i$ maximizes the i th player's utility. We separate to two disjoint cases:

1. If $v_i > B_i$ then

- for $b_i > v_i$ - It holds that the utility $u_i < 0$.
- for $B_i < b_i \leq v_i$ - The i th player wins and $u_i = v_i - B_i > 0$.
- for $b_i \leq B_i$ - $u_i = 0$.

Therefore setting $b_i := v_i$ (i.e., being truthful) is an optimal strategy (note that $b_i := B_i + \epsilon$ is no better than $v_i = b_i$).

2. Otherwise, $v_i \leq B_i$ and therefore

- for $b_i < B_i$ - The i th player will lose and $u_i = 0$.
- for $b_i \geq B_i$ - It holds that $u_i = v_i - B_i \leq 0$.

Again, setting $b_i := v_i$ is an optimal strategy.

It follows that the truthful strategy is no worse than all other strategies and therefore maximizes the player's utility. \square

Variations of Vickrey

Variant I

In the first variant, there are now m (identical and indivisible) items to be auctioned. We modify the allocation and pricing function as follows:

- Allocation: allocate the m items to the m highest bidders.
- Pricing: Let S denote the set of the winning players. Each of the m winners is charged according to the $(m-1)$ 'th highest bid (i.e., $\forall i \in S, p_i = \max_{j \notin S} b_j$). The rest pay 0.

It is not difficult to see that under this auction mechanism is truthful. The proof is similar to that of Claim 1.

Variant II

The second variant takes the same form as Variant I, however for the pricing function we have the following:

- Pricing: The m winners are simply $1, \dots, m$ and the i th winner pays $p_i = b_{i-1}$. The rest are charged 0.

We argue that this scheme is not truthful - for the sake of brevity, suppose $m = 2$ and consider the winners player 1 and player 2 who are charged b_2 and b_3 (resp.) It is clear that if player 1 had set $b_1 := b_3 + \epsilon$, he would have been ranked the second highest bidder and thus would have been charged only b_3 . Therefore lying is a better strategy.

Variant III

Consider the original Vickrey Auction, where we modify the pricing function as follows:

- Pricing: the winner gets the item for free (pays 0). The other players pay $-b_1$ (i.e., each is compensated b_1 for not getting the item)

In the following section we prove this variant is truthful.

Groves' Theorem

Stated here without proof, we present the following Theorem[2].

Theorem 1 (Groves). *Suppose we have a truthful auction with allocation function \mathcal{A} and pricing function \mathcal{P} . Consider n arbitrary functions $h_i : B_{-i} \rightarrow \mathcal{R}$, where B_{-i} is the set of all bids other than the i -th player's. If we can change the pricing function in the following way:*

$$p'_i = p_i - h_i(B_{-i})$$

and leave \mathcal{A} unchanged, the resulting auction is still truthful.

We apply Groves' Theorem to prove the following Claim:

Claim 2. *Variant III is truthful.*

Proof. For all i define the function h_i to be the function that returns the maximum value in B_{-i} . We use these functions on the original Vickrey auction mechanism to obtain a new pricing function and leave the allocation function unchanged. Clearly by Grove's Theorem the resulting auction is still truthful. Now, since $p_1 = b_2$ and $h_1(B_{-1}) = b_2$ we have that $p'_1 = 0$ and similarly we can verify that for all $i \neq 1$, $p'_i = -b_1$ - that is, the result auction is exactly Variant III. This completes the proof. \square

Combinatorial Auctions

We now turn to a more complicated form of auction wherein there are m auctioned items, and each bidder's valuation is a mapping from a subset (or "bundle") of the items to a monetary value. That is to say player i has a valuation function $v_i : 2^G \rightarrow \mathbb{R}$ where G is the set of goods. For example, suppose B='Bread', P='Peanut Butter', and J='Jelly' are for sale. One bidder might have a valuation of:

- $\{B,P,J\} = 40$
- $\{B,P\} = 25$
- $\{B,J\} = 10$
- anything else = 0 (Well, what would you do without the bread?)

Where single item auctions aim to assure that the bidder with the highest valuation wins, combinatorial auctions aim for a similar optimization - we aim to design a mechanism that maximize the sum of the valuations for all bidders who received a bundle. Formally, maximizing the total social welfare is obtained by maximizing

$$\sum_i v_i(k_i)$$

where k_i is the bundle awarded to the i -th player. Note that we may instead wish to maximize quantities other than social welfare. For example, a business (taking the role of the auctioneer) may instead be interested in maximizing revenue. Throughout these notes, we focus only on the goal of maximizing social welfare.

A more concrete example (albeit a simplified example of Combinatorial Auction where only singletons may have non-zero valuation) is the following variant of Google's AdWords auction: The auctioned items are the m available ad-positions on a Google search result page and each of the bidders would like to position his ad in a single slot (meaning we can treat the valuation of the i th player as $v_i : G \rightarrow \mathbb{R}$).

We can formulate this problem as the maximum weighted matching problem in a (complete) bipartite graph, where the vertices of the bipartite graph are the bidders and ad-positions, and the weight of the edge between a player and an ad-position is simply the bid offered by the player for this position (See Figure 1).

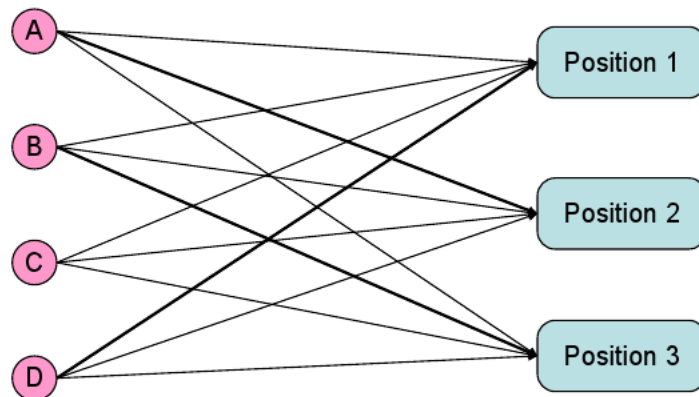


Figure 1: In this instance of the simplified Adwords auction 3 ad-positions are being auctioned to 4 players $\{A, B, C, D\}$. The weight of each edge (depicted by thickness) is the bid assigned by the player to that position. Note that a solution that maximizes social welfare is a maximal weighted matching.

1 VCG Auction

We now examine a generalization of the Vickrey Auction to combinatorial auctions. A Vickrey-Clarke-Groves auction[2, 1] provides a pricing function with the intuition being that you should pay only what your cost to society. Let

$$T(B, G) = \sum_i v_i(k_i)$$

be the total gain of a group of bidders when bundles are assigned through allocation function \mathcal{A} . Intuitively, the price the i -th player is charged the difference between the total gain of all the other bidders, and the total gain that would have occurred if the i -th player never existed. More formally:

$$p_i = \sum_{j \neq i} v_j(k_j) - T(B \setminus \{b_i\}, G)$$

Consider how much better off society would be had the winner not participated in the game in the first place. In this new game player 2 (assuming rational bidders) gets the item while the rest are unaffected in which case society gains v_2 , which is exactly the amount player 1 paid in the original game. We may therefore view the outcome of the original game as player 1 paying for the “harm” he causes to society by participating. This is referred to as the *Vickrey-Clarke-Groves (VCG) principle*, after the work of Clarke and Groves[3].

References

- [1] Edward H. Clarke. Multipart pricing of public goods. *Public Choice*, 11(1), September 1971.
- [2] Theodore Groves. Incentives in teams. *Econometrica*, 41(4):617–31, July 1973.
- [3] Noam Nisan. Tim Roughgarden. Eva Tardos. and Vijay V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, 2007.
- [4] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, 16(1):8–37, 1961.