

Lecture 4

①

Quantum measurement - Postulate 3:

In a quantum measurement, the different possible measurement outcomes (labeled 1, 2, ..., m) are each associated with a projector P_j , where the projectors form an orthogonal decomposition of the identity:

$$P_i = P_i^\dagger, \quad P_i P_j = \delta_{ij} P_i, \quad \sum_{i=1}^m P_i = I$$

The ~~probabilities~~ probabilities p_1, \dots, p_m of the outcomes are given by the Born rule

$$P_j = \langle \Psi | P_j | \Psi \rangle \equiv \langle P_j \rangle_\Psi$$

and if the measurement is ideal (non-demolition) the state of the system changes:

$$|\Psi\rangle \rightarrow |\Psi_j\rangle = P_j |\Psi\rangle / \sqrt{P_j}$$

We can associate these projectors with the eigenspaces of a Hermitian operator

$$O = O^\dagger = \sum_{j=1}^m \gamma_j P_j$$

↑ projectors onto
eigenspaces
↓ real eigenvalues,
all distinct

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The eigenvalues $\{\lambda_j\}$ represent the values of some physical quantity being measured. Such a Hermitian operator \hat{O} is called an observable.

If $|\psi\rangle$ is an eigenstate of \hat{O} , then one measurement outcome (say λ) has prob. 1, and all the others 0. We say that $|\psi\rangle$ has a "well-defined" value of \hat{O} . Because most Hermitian matrices do not commute, $|\psi\rangle$ cannot simultaneously have well-defined values of every observable. Noncommuting observables are called complementary.

The expectation value of an observable is the average over all measurement outcomes:

$$\sum_j p_j \lambda_j = \sum_j \lambda_j \langle \psi | P_j | \psi \rangle = \sum_j \langle \psi | \lambda_j P_j | \psi \rangle \\ = \langle \psi | \left(\sum_j \lambda_j P_j \right) | \psi \rangle = \langle \psi | \hat{O} | \psi \rangle \equiv \langle \hat{O} \rangle_{\psi}.$$

The variance of an observable $\langle \Delta \hat{O}^2 \rangle$ is

$$\langle \psi | (\hat{O} - \langle \hat{O} \rangle)^2 | \psi \rangle = \langle \psi | (\hat{O}^2 - 2\hat{O}\langle \hat{O} \rangle + \langle \hat{O} \rangle^2) | \psi \rangle \\ = \underbrace{\langle \psi | \hat{O}^2 | \psi \rangle}_{\langle \hat{O}^2 \rangle} - 2 \underbrace{\langle \psi | \hat{O} | \psi \rangle}_{\langle \hat{O} \rangle} \langle \hat{O} \rangle + \underbrace{\langle \psi | \psi \rangle}_{1} \langle \hat{O} \rangle^2 \\ = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2.$$

Complementary observables obey uncertainty principles. These give lower bounds on the variances. The most famous is Heisenberg's:

$$\langle \Delta X^2 \rangle \langle \Delta P^2 \rangle \geq \hbar^2/4.$$

Similar bounds can be written for any pair of complementary variables (though the r.h.s. may depend on the state).

Two observables that commute are called compatible. If two observables have the same spectral decomposition (but different eigenvalues) they are equivalent.

In this class we rarely care about the eigenvalues $\{\lambda_j\}$. So we will usually specify a measurement just in terms of the projectors $\{P_j\}$.

Note an important special case. If $\{|\Phi_j\rangle\}$ is an orthonormal basis, then the projectors $\{P_j = |\Phi_j\rangle\langle\Phi_j|\}$ form a measurement. We can write any state $|Y\rangle$ in terms of this basis:

$$|Y\rangle = \sum_j \alpha_j |\Phi_j\rangle \Rightarrow P_j = |\alpha_j|^2, |Y_j\rangle = |\Phi_j\rangle.$$

Such a measurement — where all the projectors are rank 1 — is called complete.

As we showed last time in the qubit case,
 we can find the amplitudes of a state
 by taking inner products with the basis
 vectors:

$$\begin{aligned}
 |\psi\rangle &= \sum_j \alpha_j |\phi_j\rangle = \sum_j (\underbrace{\langle\phi_j|\psi\rangle}_{\alpha_j}) |\phi_j\rangle \\
 &= \sum_j \underbrace{|\phi_j\rangle\langle\phi_j|}_{\text{projector } P_j} |\psi\rangle \\
 &= \underbrace{\left(\sum_j |\phi_j\rangle\langle\phi_j|\right)}_{\text{This is the identity I.}} |\psi\rangle
 \end{aligned}$$

This is a special case of an earlier assumption that in a spectral decomposition $\sum P_j = I$. These are called ~~decom~~ orthogonal decompositions of the identity.

Postulate 4: Composite Systems

4

For a composite system—comprising 2 or more subsystems—the joint Hilbert space is the tensor product of the Hilbert spaces of the subsystems.

$$d_1 = \dim \mathcal{H}_1, \quad d_2 = \dim \mathcal{H}_2 \Rightarrow \dim \mathcal{H}_1 \otimes \mathcal{H}_2 = d_1 d_2.$$

If we have a basis $\{|\psi_i\rangle\}$ for \mathcal{H}_1 , and a basis $\{|\phi_j\rangle\}$ for \mathcal{H}_2 , then $\{|\psi_i\rangle \otimes |\phi_j\rangle\}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

2 indices i & j!

For qubits, we can use the $\{|0\rangle, |1\rangle\}$ basis.

2 qubits have 4 basis vectors:

$$\begin{aligned} &|0\rangle \otimes |0\rangle, \text{ or } |0\rangle |0\rangle, \text{ or } |00\rangle \\ &|0\rangle \otimes |1\rangle, \text{ or } |0\rangle |1\rangle, \text{ or } |01\rangle \\ &|1\rangle \otimes |0\rangle, \text{ or } |1\rangle |0\rangle, \text{ or } |10\rangle \\ &|1\rangle \otimes |1\rangle, \text{ or } |1\rangle |1\rangle, \text{ or } |11\rangle \end{aligned} \quad \left. \begin{array}{l} \text{or} \\ \text{ } \\ \text{ } \end{array} \right\} \text{ where } x=0, 1, 2, 3. \quad \text{Here we've combined i & j into a single index:} \\ &\text{implied symbol} \quad |i\rangle \rightarrow |2i\rangle. \end{aligned}$$

Similarly, For 3 qubits there are 8 basis vectors:

$$|000\rangle, |001\rangle, \dots, |111\rangle$$

or

$$|0\rangle, |1\rangle, \dots, |7\rangle. \quad 0 \leq x \leq 2^k - 1 \quad \text{for } k \text{ qubits}$$

If subsystem 1 is in state $|1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ ⁽⁵⁾
 and " " 2 " " " $|1\rangle_2 = \alpha_2|0\rangle + \beta_2|1\rangle$,
 then the joint system is in state $|1\rangle \otimes |1\rangle_2$
 $= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \cancel{\beta_1\alpha_2}|10\rangle + \beta_1\beta_2|11\rangle$.
 These are product states.
 But note that most states are not products:

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle.$$

A state of a composite system that is
not a product is called entangled.

Entanglement is a central concept of
 this course. The first thing to note is
 that, if a system is in an entangled state
 then measurements on the 2 subsystems are
 generally correlated. (In product states they
 are independent.) In fact, as we will
 see shortly, they are in a certain sense
 more strongly correlated than is possible
 classically.

Let's see how postulates 2 and 3
 apply to composite systems.

An entangled state means that the subsystems

don't have their own state vectors.

Evolution Reversible evolution is still described by unitary transformations. If $| \Psi \rangle$ is a state of 2 qubits, then

$$| \Psi \rangle \rightarrow U | \Psi \rangle$$

$$\text{4x4 unitary } UU^T = U^T U = I.$$

What if we apply a unitary transformation just to one subsystem? On the joint system this is written as $U \otimes I$ or $I \otimes U$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



$$\Rightarrow U \otimes I = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix} = \begin{pmatrix} aa & ab & 0 & 0 \\ ca & cb & 0 & 0 \\ 0 & 0 & dd & 0 \\ 0 & 0 & 0 & dd \end{pmatrix},$$

$$I \otimes U = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ cd & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & cd & 0 \end{pmatrix}.$$

If a joint unitary U can be factorized

$$U = U_A \otimes U_B$$

then A & B are evolving independently.

Otherwise, A & B are interacting. Physically, most interactions are local: that is, the subsystems must be close together in space.

Note that, just like states, most operators are not products: $\hat{O} \neq \hat{A} \otimes \hat{B}$. But any operator can be written as a linear combination of products: $\hat{O} = \sum_k A_k \otimes B_k$.

Measurement

IF \hat{O} is an observable on one subsystem, then the observable on the joint system is $\hat{O} \otimes I$ or $I \otimes \hat{O}$. The projectors take the form $P_j \otimes I$ or $I \otimes P_j$, respectively.

It is also possible to do joint measurements on more than one subsystem at a time. Then the joint observable acts nontrivially on both subsystems.

E.g., $\hat{O} = X \otimes X$. The eigenvalues are ± 1 , and the eigenvectors are

$$+1 \left\{ \begin{array}{l} |++\rangle \\ |--\rangle \end{array} \right. , \quad -1 \left\{ \begin{array}{l} |+-\rangle \\ |-+\rangle \end{array} \right.$$

Note that even though this observable is a product, this is not a product of two measurements. I.e., it is possible to measure $X \otimes X$ w/o separately measuring X on each subsystem.

Generally, to do a joint measurement one must bring the subsystems physically together. But there are tricks using entangled ancilla's that allow one to do joint measurements on physically separated subsystems (as we shall see).

Quantum circuits

Q circuits are a handy graphical representation of a quantum protocol, analogous to logical circuit diagrams. Qubits are represented as wires, with time going left to right:

$| \Psi_0 \rangle$ —————

Classical info is written w/ double or thick lines.

$| \Psi_1 \rangle$ —————
+ —————

$x = \overline{\square} = \pm 1$

There are three main elements to a circuit.

First are state preparations:

$| \Psi \rangle$ —————

$\lvert \Psi \rangle$

$| \Psi_+ \rangle$ ←————

$x = \overline{\square} = | \Psi_+ \rangle$

Next, there are unitary transformations.

$\lvert H \rangle$

—————
—————○—————

————— : U : —————

$\lvert X \rangle$

—————
—————○—————

————— + U + —————

$\lvert R_{\theta} \rangle$

—————
—————○—————

one-bit

2-bit

subcircuits or
unitary blocks

Finally, there are measurements:

$\lvert Z \rangle$

————— Z —————

Sometimes we explicitly include the classical output

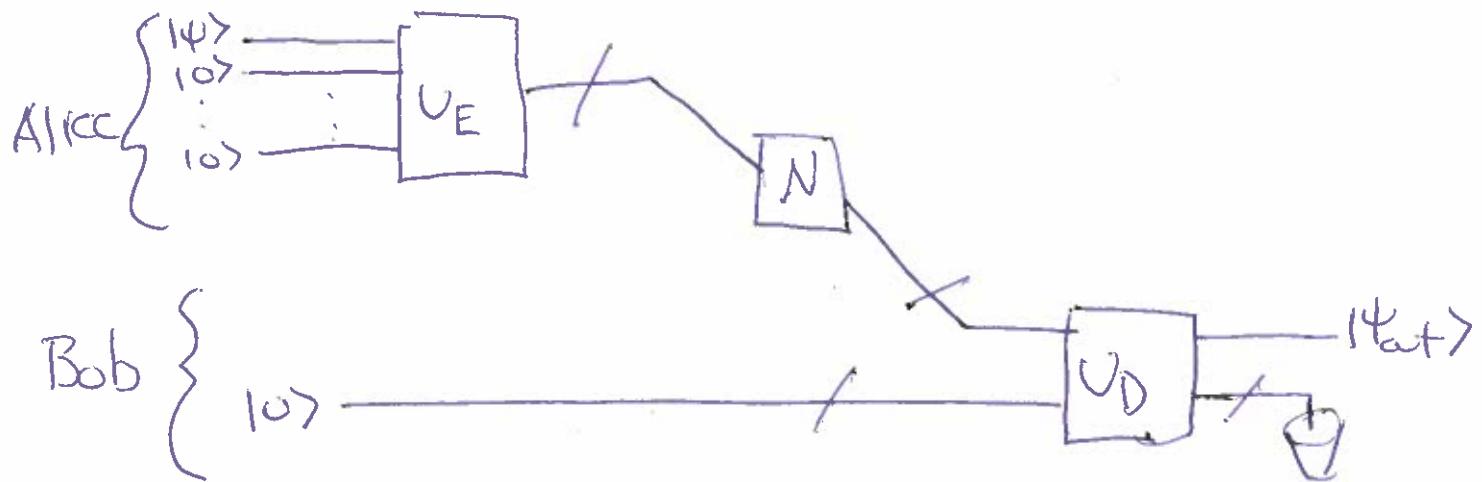
$\lvert A \rangle$

————— Z —————

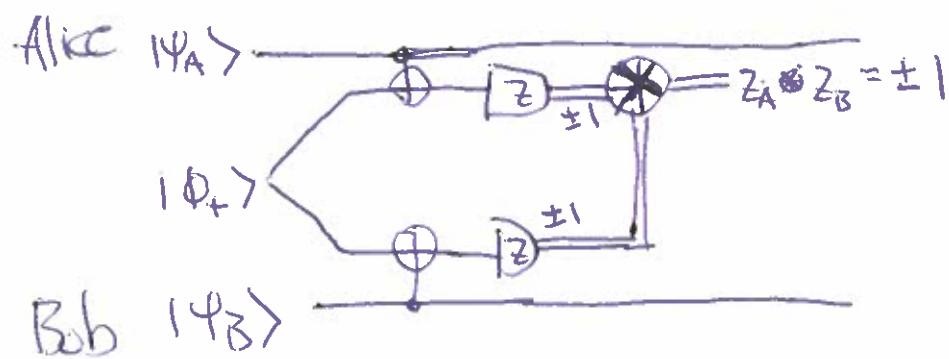
$z = \pm 1$

(9)

When we have multiple, spatially separated parties we generally segregate them vertically:



So, for example, to remotely measure $Z_A Z_B$ we can use the following circuit:



where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a maximally-entangled pair or ebit.

We'll see more tricks with entanglement next time, and throughout the course.