

# Lecture 6

①

Density matrices:

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \quad 0 \leq p_k, \sum_k p_k = 1$$

$$\text{Tr}(\rho) = 1, \rho = \rho^\dagger, \rho \geq 0.$$

We can also construct density matrices as mixtures of other density matrices:

$$\rho = \sum_k p_k \rho_k \leftarrow \text{also a valid density matrix}$$

So the set of density matrices is convex.

Unitary (noiseless) evolution:

$$\rho \longrightarrow U\rho U^\dagger = \sum_k p_k U|\psi_k\rangle\langle\psi_k|U^\dagger.$$

Projective measurement:

$$\{P_\alpha\} \quad P_\alpha = P_\alpha^\dagger = P_\alpha^2, \quad \sum_\alpha P_\alpha = I, \quad P_i P_\alpha = \delta_{i\alpha} P_\alpha.$$

$$\rho \longrightarrow \rho_\alpha = \frac{P_\alpha \rho P_\alpha}{p_\alpha} \quad \text{w/ prob } p_\alpha = \text{Tr}\{P_\alpha \rho P_\alpha\} = \text{Tr}\{P_\alpha \rho\}.$$

## Composite systems:

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I. It is possible for each subsystem to have its own density matrix:

$$\rho = \rho_A \otimes \rho_B$$

Measurements on the subsystems are uncorrelated.

II. More generally, there can be a joint density matrix  $\rho_{AB}$ . If we can write

$$\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k,$$

$\geq 0$

valid density matrices  
on subsystems,

then  $\rho_{AB}$  is separable. Measurements of the subsystems are classically correlated.

Determining if a given matrix  $\rho_{AB}$  is separable is, in general, a hard problem.

III. If  $\rho_{AB}$  is not separable, then it is entangled.

With composite systems, we can introduce the ③  
 concept of reduced or effective states on subsystems,  
 using the Partial Trace:

$$\rho_A \equiv \text{Tr}_B \{ \rho_{AB} \}, \quad \rho_B \equiv \text{Tr}_A \{ \rho_{AB} \}$$

The matrix  $\rho_A$  makes exactly the same predictions  
 as  $\rho_{AB}$  for measurements on ~~the~~ subsystem A  
alone:

$$P_k = \text{Tr} \{ P_k \rho_A \} = \text{Tr} \{ (P_k \otimes I) \rho_{AB} \}.$$

So  $\rho_A$  is like a marginal distribution on A.  
 Likewise,  $\rho_B$ . But note that

$$\rho_{AB} \neq \rho_A \otimes \rho_B$$

in general; they give the same probabilities  
 on the subsystems, but  $\rho_A \otimes \rho_B$  loses any  
 correlations of  $\rho_{AB}$ .

Partial trace:  $\text{Tr}_A \{ O_A \otimes O_B \} = \text{Tr} \{ O_A \} O_B$

$$\text{Tr}_B \{ O_A \otimes O_B \} = \text{Tr} \{ O_B \} O_A$$

Extends to general operators by linearity:

$$\text{Tr}_A \left\{ \sum_{\ell} A_{\ell} \otimes B_{\ell} \right\} = \sum_{\ell} \text{Tr} \{ A_{\ell} \} B_{\ell}.$$

In terms of a basis,  $\rho_{AB} = \sum_{ijkl} P_{(ij)(kl)} |ij\rangle\langle kl|$

$$\Rightarrow \rho_A = \sum_{ijk} P_{(ij)(kl)} |i\rangle\langle k| = \sum_{ik} \left( \sum_j P_{(ij)(kj)} \right) |i\rangle\langle k|$$

$$\rho_B = \sum_{ijl} P_{(ij)(kl)} |j\rangle\langle l| = \sum_{jl} \left( \sum_i P_{(ij)(il)} \right) |j\rangle\langle l|$$

An important fact is that we can define a reduced density matrix even in an entangled pure state (where there is no local state vector): ⑤ ④

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}| \Rightarrow \begin{cases} \rho_A = \text{Tr}_B \{ |\Psi_{AB}\rangle\langle\Psi_{AB}| \} \\ \rho_B = \text{Tr}_A \{ \quad \quad \quad \} \end{cases}$$

Even though the global state is pure, the local state is mixed.

There is an important connection to the Schmidt decomposition:

$$|\Psi_{AB}\rangle = \sum_j \sqrt{\lambda_j} |\hat{j}\rangle_A |\tilde{j}\rangle_B$$

$$\Rightarrow |\Psi_{AB}\rangle\langle\Psi_{AB}| = \sum_{ij} \sqrt{\lambda_i \lambda_j} |\hat{i}\rangle_A \langle\hat{j}|_A \otimes |\tilde{i}\rangle_B \langle\tilde{j}|_B$$

$$\Rightarrow \rho_A = \sum_{ij} \sqrt{\lambda_i \lambda_j} |\hat{i}\rangle_A \langle\hat{j}|_A \underbrace{\text{Tr} \{ |\tilde{i}\rangle_B \langle\tilde{j}|_B \}}_{\substack{\rightarrow \langle\tilde{j}|\tilde{i}\rangle_B = \delta_{ij}}} = \sum_i \lambda_i |\hat{i}\rangle_A \langle\hat{i}|_A$$

$$\rho_B = \sum_i \lambda_i |\tilde{i}\rangle_B \langle\tilde{i}|_B \quad \leftarrow \text{diagonal decomposition!}$$

So the Schmidt coefficients  $\{\lambda_j\}$  are the eigenvalues of both  $\rho_A$  and  $\rho_B$ , and the Schmidt bases  $\{|\hat{i}\rangle_A\}$  and  $\{|\tilde{i}\rangle_B\}$  are their respective eigenbases.

Representing classical prob. theory with density matrices. ⑤

Here, we just require the density matrix, and any projectors and/or observables, to all be diagonal in the same standard basis  $\{|x\rangle\}_{x \in X}$

$$\rho_{\mathcal{X}}(x) \rightarrow \rho = \sum_x \rho_{\mathcal{X}}(x) |x\rangle\langle x|$$

$$\mathcal{X} \rightarrow \sum_x x |x\rangle\langle x| = \sum_x x P_x$$

So the expectation becomes

$$\mathbb{E}[\mathcal{X}] = \text{Tr} \{ \mathcal{X} \rho \} = \sum_x \rho_{\mathcal{X}}(x) x$$

and the probability of measurement outcome  $x$  becomes

$$p_x = \text{Tr} \{ P_x \rho \} = \rho_{\mathcal{X}}(x).$$

We can more generally define indicator functions, which are just projectors:

$$\mathbb{I}_A = \sum_{x \in A} P_x = \sum_{x \in A} |x\rangle\langle x|.$$

This lets us define intersections and unions of subsets in the usual way.

## Generalized measurements

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In a projective measurement, the outcomes are specified by a set of orthogonal projectors  $\{P_j\}$ .

A generalized measurement uses a set of general measurement operators  $\{M_j\}$ . The only restriction is

$$\sum_j M_j^\dagger M_j = I$$

The outcome probabilities are  $p_j = \langle \psi | M_j^\dagger M_j | \psi \rangle$

or  $p_j = \text{Tr}\{M_j \rho M_j^\dagger\} = \text{Tr}\{M_j^\dagger M_j \rho\}$ .

After measurement, the state becomes

$$|\psi\rangle \rightarrow M_j |\psi\rangle / \sqrt{p_j}$$

$$\rho \rightarrow \rho_j = \frac{M_j \rho M_j^\dagger}{p_j}$$

In many cases (e.g., when a measurement is destructive), we only care about the probabilities.

In that case it suffices to specify the POVM elements  $E_j = M_j^\dagger M_j$ .  $p_j = \text{Tr}\{E_j \rho\}$ .

It is always possible to do a generalized measurement indirectly, using a unitary interaction with an ancilla, followed by a projective meas:

$$U |\psi\rangle |0\rangle = \sum_j M_j |\psi\rangle |j\rangle \rightarrow |\psi_j\rangle |j\rangle \text{ w/ prob } p_j.$$

In this case

$$M_j = \langle j | U (I \otimes |0\rangle\langle 0|)$$

# CPTP maps

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Unitary maps are not the most general linear evolution that takes ~~z~~ density matrices to density matrices. Those are completely positive, trace-preserving maps:  $\rho \rightarrow \mathcal{N}(\rho)$ .

① A positive map takes positive operators to positive operators.

②  $\mathcal{N}$  is completely positive if it is positive and  $\mathcal{N} \otimes \mathcal{I}_d$  is also positive for all  $d$ .

Any CP map can be written as a Kraus decomposition:

$$\rho \xrightarrow{\mathcal{N}} \sum_k A_k \rho A_k^\dagger$$

The  $\{A_k\}$  are Kraus operators

For any CP map there are infinitely many Kraus decomp.s:

$$\mathcal{N}(\rho) = \sum_k A_k \rho A_k^\dagger = \sum_\ell B_\ell \rho B_\ell^\dagger \quad B_\ell = \sum_k U_{\ell k} A_k$$

unitary matrix

To be trace-preserving, we need

$$\sum_k A_k^\dagger A_k = I$$

This is the same condition as a generalized measurement! We can think of a CPTP map as a generalized meas. where we are ignorant of the outcome. This also means that any CPTP map can be done by a unitary w/ an ancillary system followed by a partial trace:

$$\rho \rightarrow \mathcal{N}(\rho) = \text{Tr}_{\text{anc}} \left\{ U(\rho \otimes |0\rangle\langle 0|) U^\dagger \right\}$$

$$U|\psi\rangle|0\rangle = \sum_k A_k|\psi\rangle|k\rangle$$

The ancilla is often called the environment ⑧  
of the reference system. The process of being  
coupled to the environment is called decoherence.

Q Channels - In QIT we will often refer  
to a CPTP map as a quantum channel.

Here are some examples: Unitary  $\rho \rightarrow U\rho U^\dagger$ .

Bit-flip channel:  $\rho \rightarrow (1-p)\rho + X\rho X$ .

Phase-flip channel:  $\rho \rightarrow (1-p)\rho + Z\rho Z$ .

Pauli channel:  $\rho \rightarrow (1-p_x - p_y - p_z)\rho + p_x X\rho X$   
 $+ p_y Y\rho Y + p_z Z\rho Z$ .

When  $p_x = p_y = p_z = p/3$  this is the  
depolarizing channel. This channel is  
equivalent to removing the qubit and  
replacing it with a maximally mixed state  
with some probability  $q$ :

$$\frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) = \pi = I/2 \quad \forall \rho, \text{ so}$$

$$\rho \rightarrow (1-q)\rho + q\pi = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$
$$q = \frac{4}{3}p.$$

Note that Kraus operators need not be squares.  
We can map between spaces of different  
dimensions.

e.g., Partial trace  $\rho_{AB} \rightarrow \rho_A = \sum_k A_k \rho_{AB} A_k^\dagger$  ⑨  
 $A_k = I \otimes |k\rangle\langle k|$ . where  $\{|k\rangle_B\}_k$  is a basis.

Erasure channel  $\rho \rightarrow (1-p)\rho + p|e\rangle\langle e|$   
 where  $|e\rangle$  is an extra state orthogonal to  $|0\rangle$  and  $|1\rangle$ . (Erasures can thus be detected.)

Classical-quantum channel First measure  $\rho$  in an orthonormal basis  $|k\rangle$ , then output a state  $\sigma_k$  conditioned on the outcome:  
 $\rho \rightarrow \sum_k |k\rangle\langle k| \rho |k\rangle\langle k| \otimes \sigma_k$  ; output state  
 (This is an entanglement-breaking channel.)

Quantum Instrument This is partway between a generalized meas and a CPTP map—like a meas. with partial info about the outcome:

$$\rho \rightarrow \sum_j \mathcal{E}_j(\rho) \otimes |j\rangle\langle j|$$

↑ transformed state
↑ classical outcome

$$\mathcal{E}_j(\rho) = \sum_{k=1}^{m_j} A_{jk} \rho A_{jk}^\dagger, \quad \sum_j \sum_k A_{jk}^\dagger A_{jk} = I$$

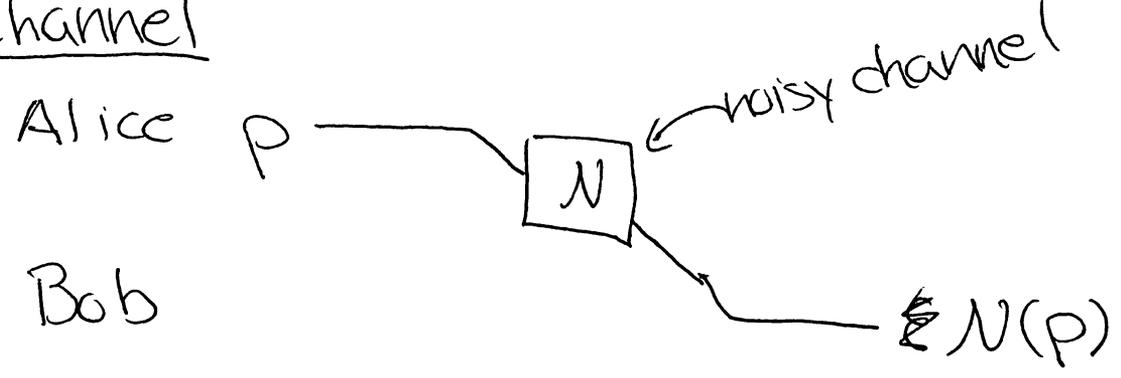
Conditional quantum channels take both a classical and quantum input and produce a Q output:

$$|m\rangle\langle m| \otimes \rho \rightarrow \mathcal{E}_m(\rho), \quad \mathcal{E}_m(\rho) = \sum_k A_{mk} \rho A_{mk}^\dagger$$

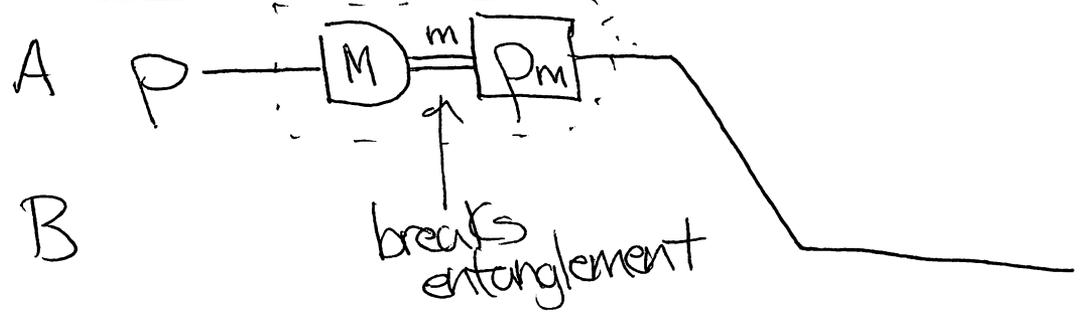
$$\sum_k A_{mk}^\dagger A_{mk} = I \quad \forall m. \quad \text{Kraus operators: } \{|m\rangle\langle m| \otimes A_{mk}\}$$

Graphically, we can include Q channels in circuit diagrams.

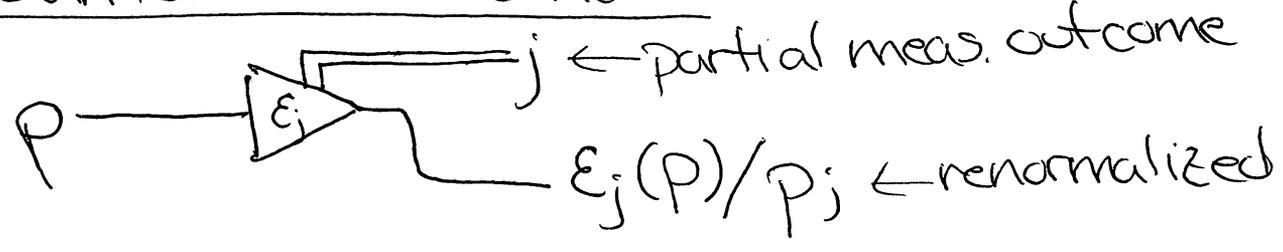
Channel



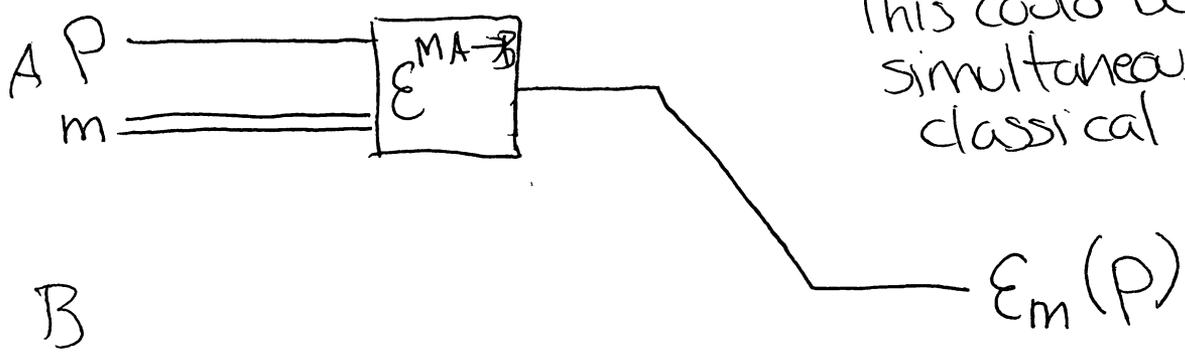
Classical-Quantum



Quantum Instrument



Conditional Q Channel



This could be used to simultaneously encode classical + Q into.