

Lecture 9

Noiseless rate regions —

Consider a protocol like Q teleportation :

$$[q\bar{q}] + 2[c \rightarrow c] \geq [q \rightarrow q].$$

We can adopt a convention where we distinguish resources produced from resources consumed by the sign of their rates; i.e., Q teleportation becomes

$$0 \geq -2[c \rightarrow c] + [q \rightarrow q] - [q\bar{q}]$$

↑ resources consumed ↗ resources produced

We can then represent any protocol that interconverts among these 3 noiseless unit resources as a rate triplet (C, Q, E) :

$$0 \geq C[c \rightarrow c] + Q[q \rightarrow q] + E[q\bar{q}]$$

What set of points is achievable, by the laws of QM?

Our 3 unit protocols correspond to vectors:

Q teleportation: $(-2, 1, -1)$

Superdense coding: $(2, -1, -1)$

Entanglement distribution: $(0, -1, 1)$

We will now prove that any achievable protocol (C, Q, E) can be done by a combination of these 3 protocols.

Thm The unit resource capacity region

$$0 \geq C[C \rightarrow C] + Q[Q \rightarrow Q] + E[E \rightarrow E]$$

consists of all points (C, Q, E) of the form

$$(C, Q, E) = \underbrace{\alpha X_{TP}}_{(C, Q, E)} + \underbrace{\beta X_{SD}}_{(C, Q, E)} + \underbrace{\gamma X_{ED}}_{(C, Q, E)},$$

where $\alpha, \beta, \gamma \geq 0$. (This is a cone in a real 3D space.)

Like most IT results, this has 2 parts:
 the Direct Coding Thm (achievability) and
 the Converse Thm (optimality).

I. Direct coding.

Suppose $(C, Q, E) = \alpha X_{TP} + \beta X_{SD} + \gamma X_{ED}$ for some $\alpha, \beta, \gamma \geq 0$. Then consider the following procedure: carry out TP N_1 times, SD N_2 times, and ED N_3 times, where we let $N_1, N_2, N_3 \rightarrow \infty$ s.t.

$$N_1/(N_1 + N_2 + N_3) \rightarrow \alpha$$

$$N_2/(N_1 + N_2 + N_3) \rightarrow \beta$$

$$N_3/(N_1 + N_2 + N_3) \rightarrow \gamma$$

Note: if α, β, γ are all integers, we don't need to go to the asymptotic limit.

2 quick points:

A. Here I have chosen $\alpha + \beta + \gamma = 1$. Of course, this is not required. But obviously, if (C, Q, E) is achievable, so is $(\alpha C, \alpha Q, \alpha E)$ for any $\alpha > 0$.

B. Resources produced by one protocol may be ③ consumed by a later protocol; and we may need to assume some initial resources that are consumed, but later restored (catalytic resources). As $N \rightarrow \infty$ these are negligible.

What does this achievable region look like? We can write it as a matrix eqn:

$$\begin{pmatrix} C \\ Q \\ E \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{where } \alpha, \beta, \gamma \geq 0.$$

This matrix is invertible:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ -1/2 & -1 & 0 \end{pmatrix} \begin{pmatrix} C \\ Q \\ E \end{pmatrix}$$

The requirement that $\alpha, \beta, \gamma \geq 0$ then implies the following inequalities (one per row above):

$C + Q + E \leq 0$	← some resource must be consumed
$Q + E \leq 0$	
$C + 2Q \leq 0$	

For the converse thm, we will show that any allowed protocol must lie in the above region.

II. Converse Thm.

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To prove the converse thm, we must use the following 3 results from QM:

- ① Classical communication, by itself, cannot produce either entanglement or quantum communication (or both).
- ② Entanglement, by itself, cannot produce either classical or quantum communication.
- ③ Quantum communication, by itself, can produce at most an equal amount of classical communication (Holevo's thm.)

① & ② are both pretty obvious. ③ is not. We will prove it later in the class, but the intuition is relatively straightforward: for a d -dimensional Hilbert space, there is no POVM with more than d outcomes such that, for each outcome \exists a state s.t. that outcome occurs with probability close to 1.

We break the thm down into 8 cases. For each point (C, Q, E) , resources that are consumed have negative rates, resources that are produced have ~~g~~ positive rates. Thus, we can divide the set of points into 8 octants by the signs of the 3 rates: (\pm, \pm, \pm) .

Here are the 8 cases:

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- i) $(+, +, +)$. This octant is empty—we cannot create resources out of nothing.
- ii) $(+, +, -)$. This octant is empty—entanglement alone cannot create C and/or Q comm. (see ②).
- iii) $(+, -, +)$. This octant uses qubit channels to produce classical comm & entanglement.
Suppose (C, Q, E) is achievable, with $C, E \geq 0$ and $Q \leq 0$. Then the point $(C+2E, Q-E, 0)$ is also achievable: we just apply superdense coding E times. This new protocol involves only C & Q comm; by ③ above, we must have

$$C + 2E \leq |Q| + E \Rightarrow \boxed{C + E \leq |Q|}.$$

This, together with $C \geq 0$ and $E \geq 0$ implies

$$\cancel{\textcircled{2}} \quad C + Q + E \leq 0$$

$$Q + E \leq 0$$

$$C + 2Q \leq 0$$

So all allowed protocols are inside the region.

- iv) $(+, -, -)$ These protocols use Q qubit channels and E ebits to simulate C classical bit channels. The achievability of (C, Q, E) with $C \geq 0$ and $E, Q \leq 0$ implies the achievability of two other points:

⑥.

a) $(0, Q + C/2, E - C/2)$ — use the classical bits to do teleportation.

From ② we must have $Q + C/2 \leq 0$, because entanglement alone cannot generate quantum communications.

b) $(C, Q + E, 0)$ — we can replace the ebits w/ qubit channels by entanglement distribution

From ③ we must have $C \leq |Q| + |E|$.

Combining these gives

$$\boxed{\begin{aligned} C + Q + E &\leq 0 \\ Q + E &\leq 0 \\ C + 2Q &\leq 0. \end{aligned}}$$

So all allowed protocols in this ~~quadrant~~ octant are inside the region.

v) $(-++)$. This octant is empty by ①.

vi) $(-+-)$. Here we use classical comm and entanglement to simulate quantum comm.

If (C, Q, E) is achievable for $C, E \leq 0, Q \geq 0$, then the following points are also achievable:

a) $(C, 0, Q + E)$ — use the qubit channels for E.D.
Since classical comm cannot create entang.
we must have $Q + E \leq 0$.

b) $(C+2Q, 0, E-Q)$ — use the qubit channels for ⑦ superdense coding.

Since entang. cannot produce classical comm, must have $C+2Q \leq 0$. Putting it together we get

$$C+Q+E \leq 0$$
$$Q+E \leq 0$$
$$C+2Q \leq 0$$

So allowed protocols in this octant are all inside the region.

vii) $(-, -+)$. Use qubit channels and classical comm to create entanglement.

IF (C, Q, E) is achievable for $C, Q \leq 0, E \geq 0$, then obviously $C+2Q \leq 0$. The point $(C-2E, Q+E, 0)$ must also be achievable — use the entanglement for ~~super~~ TP. But by ① this implies that $Q+E \leq 0$, since classical comm cannot create Q comm. So

$$C+2Q \leq 0$$
$$Q+E \leq 0$$
$$C+Q+E \leq 0.$$

viii) $(--, --)$ These protocols consume but do not create resources — all are possible, and all are inside the region.

QED!