

Lecture 10

①

In assessing the performance of a QIT protocol, we need measures of whether an outcome is good or bad, and of the information content of a quantum "message." For these purposes we will use distance measures and entropic quantities.

The two distance measures will mostly use are the trace distance and the fidelity (with some variations).

Defn The trace norm (or 1 , norm) of an operator M is

$$\|M\|_1 \equiv \text{Tr} \left\{ \sqrt{M^\dagger M} \right\}$$

$M^\dagger M$ is always square and positive, so this square root is well defined.

It is not hard to see that $\|M\|_1$ is the sum of the singular values of M . We are mostly interested in the case where $M = M^\dagger$ is Hermitian. In this case,

$$\|M\|_1 = \sum_j |\lambda_j| \quad \text{where } \{\lambda_j\} \text{ are the eigenvalues of } M.$$

What are the properties of the trace norm? (2)

I. Positive definite: $\|M\|_1 \geq 0$, and only equals 0 if $M=0$.

II. Homogeneous: $\|cM\|_1 = |c| \|M\|_1$, for any scalar c .

III. Triangle inequality:

$$\|M+N\|_1 \leq \|M\|_1 + \|N\|_1.$$

IV. Isometric invariance: for any unitary ~~U~~ or isometry U , $\|UMU^\dagger\|_1 = \|M\|_1$.

V. Convexity: for any M, N and $0 \leq x \leq 1$, $\|xM + (1-x)N\|_1 \leq x\|M\|_1 + (1-x)\|N\|_1$.

We can use this norm to define the trace distance: $d(M, N) \equiv \|M - N\|_1$,

$$= \text{Tr} \left\{ \sqrt{(M-N)^\dagger (M-N)} \right\}.$$

This distance obeys the usual properties we want for a distance measure; but when M and N are density matrices, there are additional operational meanings to the trace distance.

$$0 \leq \|\rho - \sigma\|_1 \leq 2.$$

Lemma: The trace distance is twice ③
the largest probability difference the two
states ρ and σ could give for the same outcome
(POVM element) Λ :

$$\|\rho - \sigma\|_1 = 2 \max_{0 \leq \Lambda \leq I} \text{Tr} \{ \Lambda (\rho - \sigma) \}.$$

Proof:

$(\rho - \sigma)$ is Hermitian, so it has real eigenvalues.
Let Π_{\pm} be the projectors onto the spaces spanned
by the eigenvectors of $(\rho - \sigma)$ with pos/neg eigenvalues,
respectively. Then define

$$\alpha_+ = \Pi_+ (\rho - \sigma) \Pi_+, \quad \alpha_- = -\Pi_- (\rho - \sigma) \Pi_-.$$

$$\Rightarrow (\rho - \sigma) = \alpha_+ - \alpha_-.$$

So $\|\rho - \sigma\|_1 = \text{Tr} \{ \alpha_+ \} + \text{Tr} \{ \alpha_- \}$. But

$$\text{Tr} \{ \alpha_+ - \alpha_- \} = \text{Tr} \{ \rho - \sigma \} = \text{Tr} \{ \rho \} - \text{Tr} \{ \sigma \} = 1 - 1 = 0.$$

So $\text{Tr} \{ \alpha_+ \} = \text{Tr} \{ \alpha_- \}$ and

$$\|\rho - \sigma\|_1 = 2 \text{Tr} \{ \alpha_+ \} = 2 \text{Tr} \{ \Pi_+ (\rho - \sigma) \}.$$

It's not hard to see that Π_+ is the Λ that
maximizes the above expression. \square

⊕

The trace distance is closely linked to the following operational question: if we wish to choose a POVM to distinguish between two states ρ_0 and ρ_1 , what is the minimum probability of error? This is

$$P_{\text{err}} = \frac{1}{2} - \frac{1}{4} \|\rho_0 - \rho_1\|_1$$

There are a number of related results; for instance, for any POVM elt. E ,

$$\text{Tr}\{E\rho\} \geq \text{Tr}\{E\sigma\} - \|\rho - \sigma\|_1$$

This implies that nearby states have close meas. probs:

$$\text{Tr}\{E\sigma\} > 1 - \epsilon \quad \& \quad \|\rho - \sigma\|_1 < \epsilon$$

$$\Rightarrow \text{Tr}\{E\rho\} \geq 1 - 2\epsilon.$$

Another very important property is monotonicity under discarding subsystems (partial traces):

$$\|\rho^A - \sigma^A\|_1 \leq \|\rho^{AB} - \sigma^{AB}\|_1$$

$$\rho_A = \text{Tr}_B\{\rho^{AB}\} \quad \sigma_A = \text{Tr}_B\{\sigma^{AB}\}$$

So discarding subsystems makes states less distinguishable.

This property immediately implies that the trace distance is monotonic under CPTP maps: (5)

$$\|N(\rho) - N(\sigma)\|_1 \leq \|\rho - \sigma\|_1, \quad N(\rho) = \sum_F A_k \rho A_k^\dagger$$

Fidelity For 2 pure states $|\psi\rangle$ & $|\phi\rangle$, the quantity

$$F(|\psi\rangle, |\phi\rangle) \equiv |\langle \psi | \phi \rangle|^2$$

is the probability that $|\psi\rangle$ will pass a test for $|\phi\rangle$, or vice versa. Clearly $0 \leq F \leq 1$, and high fidelity corresponds to nearby states.

If we compare a pure state $|\psi\rangle$ to a mixed state ρ , this generalizes naturally:

$$F(|\psi\rangle, \rho) \equiv \langle \psi | \rho | \psi \rangle$$

← For instance, we could compare a pure input to the noisy output of a channel.

(expected fidelity).

What about comparing 2 mixed states?

An idea due to Uhlmann is to purify both states and use the pure state fidelity. Since purifications are not unique, we choose the purifications that maximize the overlap:

$$\begin{aligned} \rho &\rightarrow |\psi^{AR}\rangle \\ \sigma &\rightarrow |\phi^{AR}\rangle \end{aligned} \quad F(\rho, \sigma) = \max_{U_R} |\langle \psi^{AR} | I \otimes U_R | \phi^{AR} \rangle|^2$$

It turns out that this maximization can be done exactly, and (6)

$$F(\rho, \sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1^2 = \text{Tr} \left\{ \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \right\}^2.$$

Fidelity also has a number of important properties: $F(\rho, \sigma) = F(\sigma, \rho)$, $0 \leq F \leq 1$.

I. Multiplicativity: $F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2)$
 $= F(\rho_1, \sigma_1) F(\rho_2, \sigma_2)$

II. Joint concavity:

$$F\left(\sum_x p(x) \rho_x, \sum_x p(x) \sigma_x\right) \geq \sum_x p(x) F(\rho_x, \sigma_x)$$

III. Concavity: $0 \leq x \leq 1$

$$F(x\rho_1 + (1-x)\rho_2, \sigma) \geq x F(\rho_1, \sigma) + (1-x) F(\rho_2, \sigma).$$

IV. Monotonicity under partial traces

$$F(\rho^{AB}, \sigma^{AB}) \leq F(\rho^A, \sigma^A)$$

V. Monotonicity under CPTP maps

$$F(\rho, \sigma) \leq F(N(\rho), N(\sigma)).$$

There is a relationship between fidelity and trace distance:

(7)

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}$$

This allows us to relate fidelity to distinguishability (on the one hand) and how well one state can substitute for another (on the other). In particular, it implies the following:

A. $\|\rho - \sigma\|_1 \leq \epsilon \Rightarrow F(\rho, \sigma) \geq 1 - \epsilon$

B. $F(\rho, \sigma) \geq 1 - \epsilon \Rightarrow \|\rho - \sigma\|_1 \leq 2\sqrt{\epsilon}$

"Gentle" measurement

IF $\{\Lambda\rho\} \geq 1 - \epsilon$

$0 \leq \Lambda \leq I$,

then the post-measurement state is close to the original:

$$\rho' \equiv \frac{\sqrt{\Lambda} \rho \sqrt{\Lambda}}{\text{Tr}\{\Lambda\rho\}}, \quad \|\rho - \rho'\|_1 \leq 2\sqrt{\epsilon}$$

More generally,

$$\text{Tr}\{\Lambda\rho\} \geq 1 - \epsilon \Rightarrow \|\rho - \sqrt{\Lambda} \rho \sqrt{\Lambda}\|_1 \leq 2\sqrt{\epsilon}$$

Channel Fidelity

(8)

An important practical question is how much a channel preserves the quantum states that pass through it. We can quantify this by calculating the fidelity of the input and output states:

$$F_{\min}(N) \equiv \min_{|\psi\rangle} F(|\psi\rangle, N(|\psi\rangle\langle\psi|)) \\ = \min_{|\psi\rangle} \langle\psi| N(|\psi\rangle\langle\psi|) |\psi\rangle.$$

This minimum fidelity is unfortunately not that easy to calculate in practice, because it is hard to minimize over large spaces.

Two other measures are more useful:

Expected (or average) fidelity:

$$\bar{F}(N) \equiv \int F(U|\psi\rangle, N(U|\psi\rangle\langle\psi|U^\dagger)) dU \\ = \int \langle\psi| U^\dagger N(U|\psi\rangle\langle\psi|U^\dagger) U |\psi\rangle dU$$

The Haar measure over all unitaries

Entanglement fidelity

Let N act on \mathcal{H}_A , and let $|\psi\rangle$ be a maximally entangled state on $\mathcal{H}_A \otimes \mathcal{H}_A$. ⑨

Then

$$F_e(N) \equiv \langle \psi | (N \otimes \mathbb{I})(|\psi\rangle\langle\psi|) | \psi \rangle$$

Remarkably, this can be evaluated in closed form: $N(\rho) = \sum_k A_k \rho A_k^\dagger$

$$\Rightarrow F_e(N) = \frac{1}{d} \sum_k |\text{Tr}\{A_k\}|^2$$

There is a simple relationship between expected and entanglement fidelity:

$$\bar{F}(N) = \frac{d \cdot F_e(N) + 1}{d+1}$$

So both these quantities can be calculated efficiently.