

Lecture 13

①

Quantum Information Inequalities

I. The Fundamental Q Relative Entropy

This inequality goes beyond the classical result — which states that $D(p||q) \geq 0$, which we have already proven in the Q case — to show that quantum relative entropy is monotonic under CPTP maps:

$$\underline{\text{Thm}} \quad D(\rho||\sigma) \geq D(N(\rho)||N(\sigma)) \quad \forall \rho, \sigma, N.$$

Proof: We sketch this briefly.

$$\begin{aligned} D(\rho||\sigma) &= D(\rho||\sigma) + D(|0\rangle\langle 0|^E || |0\rangle\langle 0|^E) \\ &= D(\rho \otimes |0\rangle\langle 0|^E || \sigma \otimes |0\rangle\langle 0|^E) \\ &= D(U(\rho \otimes |0\rangle\langle 0|^E)U^\dagger || U(\sigma \otimes |0\rangle\langle 0|^E)U^\dagger) \\ &\geq D(N(\rho)||N(\sigma)) \end{aligned}$$

The last step follows from the (far from trivial) monotonicity under partial traces:

$$D(\rho^{AB} || \sigma^{AB}) \geq D(\rho^A || \sigma^A).$$

This is proven in appendix B of the textbook.

□

one of the important corollaries of this theorem is strong subadditivity:

$$H(AB) + H(BC) \geq H(ABC) + H(B)$$

which is equivalent to $I(A; B|C) \geq 0$. It follows from

$$I(A; B|C) = D(\rho^{ABC} \| \rho^A \otimes \rho^{BC}) \geq D(\rho^{AB} \| \rho^A \otimes \rho^B),$$

$$\therefore I(A; B) \geq 0.$$

Quantum relative entropy provides a useful upper bound on the trace distance. This is the Q Pinsker Inequality:

$$\frac{1}{2 \ln 2} (\|\rho - \sigma\|_1)^2 \leq D(\rho \| \sigma).$$

II. The Q Data Processing Inequality.

This is an inequality for coherent information. It shows that passing part of a quantum system through a succession of CPTP maps can only reduce the quantum correlations.

Then consider an initial pure state $\phi^{AB} = |\phi\rangle\langle\phi|$, and two subsequent states derived from it, ③

$$\rho^{AB_1} \equiv \mathcal{N}_1^{B \rightarrow B_1}(\phi^{AB})$$

$$\sigma^{AB_2} \equiv \mathcal{N}_2^{B_1 \rightarrow B_2}(\rho^{AB_1})$$

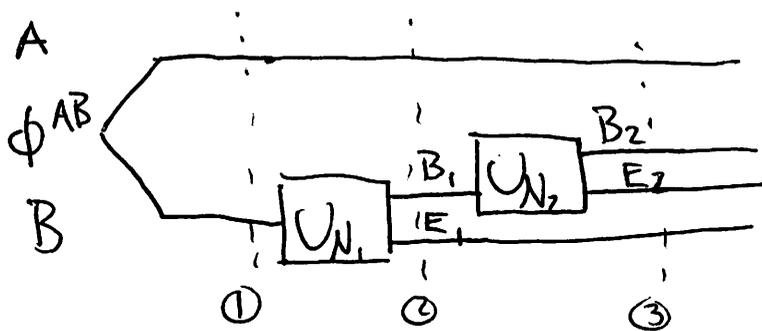
(This is a Q Markov Chain.) Then

$$I(A>B)_\phi \geq I(A>B_1)_\rho \geq I(A>B_2)_\sigma.$$

Proof: Since ϕ^{AB} is pure,

$$I(A>B)_\phi \equiv H(B)_\phi - H(AB)_\phi = H(B)_\phi = H(A)_\phi.$$

We can represent the succession of maps by ~~isometries~~ isometries; the circuit looks like this:



At the pt ① the state is ϕ ; at the point ② the state is ψ , where $\rho = \text{tr}_{E_1}\{\psi\}$. At point ③ the state is χ , where $\sigma = \text{tr}_{E_1, E_2}\{\chi\}$.

④

$$I(A>B_1)_\rho = I(A>B_1)_\psi$$

$$= H(B_1)_\psi - H(AB_1)_\psi = H(AE_1)_\psi - H(E_1)_\psi$$

where in the last step we've used the fact that ψ is pure. Now $H(A)_\phi = H(A)_\psi \geq H(AE_1)_\psi - H(E_1)_\psi$, which follows because $I(A; E_1)_\psi \geq 0$. So

$$I(A>B)_\phi \geq I(A>B_1)_\psi.$$

At the next step ③ we have

$$I(A>B_2)_\sigma = I(A>B_2)_\chi$$

$$= H(B_2)_\chi - H(AB_2)_\chi$$

$$= H(AE_1E_2)_\chi - H(E_1E_2)_\chi$$

$$= H(AE_2|E_1) - H(E_2|E_1)_\chi.$$

If we compare this to $I(A>B_1)_\rho$ we see

$$I(A>B_1)_\rho = H(AE_1)_\psi - H(E_1)_\psi$$

$$= H(AE_1)_\chi - H(E_1)_\chi = H(A|E_1)_\chi.$$

Again we can conclude that

$$H(A|E_1)_\chi \geq H(AE_2|E_1)_\chi - H(E_2|E_1)_\chi$$

by the fact that $I(A; E_2|E_1) \geq 0$. \square

There is also a data processing inequality for Q mutual information:

$$I(A; B)_\phi \geq I(A; B_1)_\rho \geq I(A; B_2)_\sigma.$$

We get this just by adding $H(A)$ to $I(A>B)$.

Another useful corollary of this thm is that ⑤
 the Holevo ~~bound~~ information is an upper bound
 on the accessible information:

$$\chi(\mathcal{E}) = H(p) - \sum_x p(x) H(p_x) \geq I_{\text{acc}}(\mathcal{E}) = \max_{\{A, Y\}} I_{\mathcal{E}}(X; Y)$$

(This is a homework problem!)

III. The Alicki-Fannes Inequality

The proof of this is complicated, so I'm just
 going to state the result and its implications.

Thm For two states ρ^{AB} and σ^{AB} with

$$\|\rho^{AB} - \sigma^{AB}\|_1 \leq \epsilon,$$

$$|H(A|B)_\rho - H(A|B)_\sigma| \leq 4\epsilon \log d_A + 2h(\epsilon).$$

This result proves the continuity of Q entropy
 and conditional Q entropy. As $\epsilon \rightarrow 0$ the entropies
 also approach each other.

This inequality is stronger than, and
 implies, the earlier Fannes inequality:

$$|H(p) - H(\sigma)| \leq 2\epsilon \log d + 2h(\epsilon)$$

for any $\|p - \sigma\|_1 \leq \epsilon$.