

Lecture 14

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We are going to study the idea of weak typicality (or weakly typical sequences), and then apply this to products of Q states to give the idea of typical subspaces. Later we will return and define strong typicality and the method of types. These are the tools we will use to prove asymptotic results in QIT .

To start, recall the defn of sample entropy:

$$\bar{H}(x^n) \equiv -\frac{1}{n} \log(p_{X^n}(x^n)).$$

(This is the information content of the string x^n divided by n .)

Strings with symbols occurring with a frequency close to their underlying probabilities $p_X(x)$ will have sample entropy close to $H(X)$.

Defn A sequence x^n is δ -typical if

$$|\bar{H}(x^n) - H(X)| \leq \delta.$$

Defn The δ -typical set $T_\delta^{X^n}$ is the set of all δ -typical sequences x^n :

$$T_\delta^{X^n} \equiv \{x^n : |\bar{H}(x^n) - H(X)| \leq \delta\}.$$

Properties of the typical set

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1. Unit probability

$\forall \epsilon > 0, \Pr\{\mathcal{X}^n \in T_\delta^{\mathcal{X}^n}\} \geq 1 - \epsilon$ for large enough n .

2. Exponentially small cardinality

$$|T_\delta^{\mathcal{X}^n}| \leq 2^{n(H(\mathcal{X}) + \delta)}$$

This means the fraction of strings that are typical is $|T_\delta^{\mathcal{X}^n}| / |\{\mathcal{X}^n\}| \leq 2^{-n(\log|\mathcal{X}| - H(\mathcal{X}) - \delta)}$

There is also a lower bound for large n :

$$\forall \epsilon > 0, |T_\delta^{\mathcal{X}^n}| \geq (1 - \epsilon) 2^{n(H(\mathcal{X}) - \delta)}$$

for large enough n .

3. Equipartition

If $x^n \in T_\delta^{\mathcal{X}^n}$ then

$$2^{-n(H(\mathcal{X}) + \delta)} \leq p_{\mathcal{X}^n}(x^n) \leq 2^{-n(H(\mathcal{X}) - \delta)}$$

So all typical sequences are (roughly) equally probable, when n is large enough.

These three properties are straightforward to prove: (3)

$$3. \quad \left| -\frac{1}{n} \log p_{\mathbb{X}^n}(x^n) - H(\mathbb{X}) \right| \leq \delta$$

$$\Rightarrow H(\mathbb{X}) - \delta \leq -\frac{1}{n} \log p_{\mathbb{X}^n}(x^n) \leq H(\mathbb{X}) + \delta$$

$$\Rightarrow e^{-n(H(\mathbb{X})+\delta)} \leq p_{\mathbb{X}^n}(x^n) \leq e^{-n(H(\mathbb{X})-\delta)}$$

1. We can define a random variable which is the sample entropy of \mathbb{X}^n : $-\frac{1}{n} \log p_{\mathbb{X}^n}(\mathbb{X}^n)$.

~~By the law of large numbers~~

~~(weak)~~

Because \mathbb{X}^n is i.i.d.,

$$-\frac{1}{n} \log p_{\mathbb{X}^n}(\mathbb{X}^n) = \frac{1}{n} \sum_j -\log p_{\mathbb{X}}(\mathbb{X}_j)$$

$$\stackrel{\text{(weak)}}{=} \frac{1}{n} \sum_j i(\mathbb{X}_j) \equiv \overline{i(\mathbb{X})}$$

By the law of large numbers

$$P_{\mathbb{X}} \left\{ \left| \overline{i(\mathbb{X})} - \mathbb{E}[i(\mathbb{X})] \right| < \delta \right\} > 1 - \epsilon$$

for any $\epsilon > 0, \delta > 0$, and sufficiently large n .

But since $\mathbb{E}[i(\mathbb{X})] = H(\mathbb{X})$, this is just the probability that $x^n \in T_{\delta}^{\mathbb{X}^n}$ for large enough n .

2. We know the total probability of $T_\delta^{\mathcal{X}^n} \leq 1$, and that the minimum probability of a string is $2^{-n(H(\mathcal{X})+\delta)}$. So the cardinality is upper bounded by $1/2^{-n(H(\mathcal{X})+\delta)}$ ④

$$\Rightarrow |T_\delta^{\mathcal{X}^n}| \leq 2^{n(H(\mathcal{X})+\delta)}$$

For the lower bound, the total Prob is $\geq 1-\epsilon$ and the maximum prob of a string is $2^{-n(H(\mathcal{X})-\delta)}$, so the cardinality is at least

$$|T_\delta^{\mathcal{X}^n}| \geq (1-\epsilon) 2^{n(H(\mathcal{X})-\delta)}$$

Joint typicality

Suppose we have n independent realizations of two random vars \mathcal{X} and \mathcal{Y} (which may or may not be correlated). Then just as with a single random var we can define a sample entropy

$$\bar{H}(x^n, y^n) \equiv -\frac{1}{n} \log P_{\mathcal{X}^n \mathcal{Y}^n}(x^n, y^n)$$

$$\text{where } P_{\mathcal{X}^n \mathcal{Y}^n}(x^n, y^n) = \prod_{j=1}^n P_{\mathcal{X} \mathcal{Y}}(x_j, y_j).$$

A pair of sequences x^n, y^n is jointly ^{weakly} typical if (5)

$$|\bar{H}(x^n, y^n) - H(xY)| \leq \delta$$

and the set of all δ -jointly typical pairs of sequences is the jointly typical set

$$T_\delta^{X^n Y^n} \equiv \{x^n, y^n \mid \text{jointly typical}\}$$

This set also has properties like regular weakly typical sets:

1. Unit prob: $\forall \epsilon > 0 \Pr\{X^n Y^n \in T_\delta^{X^n Y^n}\} \geq 1 - \epsilon$
for sufficiently large n .

2. Exponentially small cardinality

$$|T_\delta^{X^n Y^n}| \leq 2^{n(H(xY) + \delta)}$$

3. Equipartition

$$2^{-n(H(xY) + \delta)} \leq p_{X^n Y^n}(x^n, y^n) \leq 2^{-n(H(xY) - \delta)}$$

The proofs of these follow from those we have seen already — we can think of $x_j y_j$ as a symbol in a larger alphabet.

The set of conditionally typical sequences ⑦ is a subset of the set of typical sequences:

$$T_{\delta}^{Y^n | X^n} \subseteq T_{\delta}^{Y^n}$$

They are only the same if X and Y are independent.

(Note, through all this we must only choose typical sequences x^n to condition on.)

Properties:

1. Unit prob:

$$\forall \epsilon > 0, \mathbb{E}_{X^n} \left\{ \Pr_{Y^n | X^n} \left[Y^n \in T_{\delta}^{Y^n | X^n} \right] \right\} \geq 1 - \epsilon$$

for large enough n .

2. Exponentially small cardinality

$$|T_{\delta}^{Y^n | X^n}| \leq 2^{n(H(Y|X) + \delta)}$$

Lower bound

$$\forall \epsilon > 0, \mathbb{E}_{X^n} \left[|T_{\delta}^{Y^n | X^n}| \right] \geq (1 - \epsilon) 2^{n(H(Y|X) - \delta)}$$

For large enough n .

3. Equipartition

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$$2^{-n(H(Y|X)+\delta)} \leq P_{Y^n|X^n}(y^n|x^n) \leq 2^{-n(H(Y|X)-\delta)}$$

for any conditionally typical sequence $y^n|x^n$.

Quantum weak typicality: the typical subspace.

Let us now define a Q information source as a source that (independently) emits Q systems in a state $|\psi_y\rangle$ with prob $P_Y(y)$. Each individual system can be described by a density matrix

$$\rho = \sum_y P_Y(y) |\psi_y\rangle\langle\psi_y|$$

If the source emits n systems, their joint state is $\rho \otimes \rho \otimes \dots \otimes \rho \equiv \rho^{\otimes n}$.

For the single state ρ we can diagonalize in the eigenbasis:

$$\rho = \sum_x P_X(x) |x\rangle\langle x|$$

$$\langle x|x'\rangle = \delta_{xx'}$$

The von Neumann entropy is ①

$$H(\rho) = -\text{Tr}\{\rho \log \rho\} = -\sum_x p_{\mathcal{X}}(x) \log p_{\mathcal{X}}(x) \\ = H(X).$$

If we diagonalize the whole state $\rho^{\otimes n}$ we get

$$\rho^{\otimes n} = \sum_{x^n} p_{\mathcal{X}^n}(x^n) |x^n\rangle\langle x^n|$$

where

$$p_{\mathcal{X}^n}(x^n) = \prod_{j=1}^n p_{\mathcal{X}}(x_j), \quad |x^n\rangle = |x_1\rangle \otimes \dots \otimes |x_n\rangle.$$

Since $p_{\mathcal{X}^n}(x^n)$ is a classical probability for an i.i.d. sequence, it will also be dominated by typical sequences:

Defn The δ -typical subspace $\mathcal{T}_{\delta}^{\mathcal{X}^n}$ is a subspace of $\mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H} \equiv \mathcal{H}^{\otimes n}$ defined by

$$\mathcal{T}_{\delta}^{\mathcal{X}^n} \equiv \text{span} \left\{ |x^n\rangle \mid x^n \in \mathcal{T}_{\delta}^{\mathcal{X}^n} \right\}$$

↑ the x^n are
classical typical
sequences.

We can define a projector onto this subspace, the typical projector: ⑩

$$\Pi_{\delta}^{\mathbb{X}^n} \equiv \sum_{x^n \in T_{\delta}^{\mathbb{X}^n}} |x^n\rangle\langle x^n|.$$

Using this, we can ~~define~~ ^{describe} the properties of the typical subspace, which essentially follow from the classical properties:

1. $\forall \epsilon > 0, \text{Tr}\{\Pi_{\delta}^{\mathbb{X}^n} \rho^{\otimes n}\} \geq 1 - \epsilon$ (Unit prob.)
for large enough n .

2. Exponentially small dimension

$$\text{Tr}\{\Pi_{\delta}^{\mathbb{X}^n}\} \leq 2^{n(H(\mathbb{X}) + \delta)}$$

lower bound: $\text{Tr}\{\Pi_{\delta}^{\mathbb{X}^n}\} \geq (1 - \epsilon) 2^{n(H(\mathbb{X}) - \delta)}$
for large enough n .

3. Equipartition

$$2^{-n(H(\mathbb{X}) + \delta)} \Pi_{\delta}^{\mathbb{X}^n} \leq \Pi_{\delta}^{\mathbb{X}^n} (\rho^{\otimes n}) \Pi_{\delta}^{\mathbb{X}^n} \leq 2^{-n(H(\mathbb{X}) - \delta)} \Pi_{\delta}^{\mathbb{X}^n}$$

And two important quantum properties:

① $[\Pi_{\delta}^{\mathbb{X}^n}, \rho^{\otimes n}] = 0 \leftarrow$ they commute!

② $\|\rho^{\otimes n} - \Pi_{\delta}^{\mathbb{X}^n} (\rho^{\otimes n}) \Pi_{\delta}^{\mathbb{X}^n}\|_1 \leq 2\sqrt{\epsilon}$.