

## Lecture 19

### Entanglement-assisted ~~quantum~~<sup>classical</sup> communication

When we were dealing with noiseless resources, we considered the protocol of superdense coding, which allowed us to boost the rate of classical communication using quantum resources.

So it should be no surprise that the same thing can happen using noisy channels.

A pleasant surprise is that there is a single letter formula for the corresponding entanglement-assisted classical capacity!

$$C_E(N) = I(N) = \max_{\phi^{AA'}} I(A; B) \rho, \quad \rho^{AB} = N^{A' \rightarrow B}(\phi^{AA'})$$

↑  
pure.

Suppose A & B share a maximally entangled state

$$|\phi\rangle^{TATB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle^{TA} |ii\rangle^{TB}$$

where d can be as large as we like. Alice chooses a message  $m \in M$  with equal probability.

She encodes this message by applying an encoding map to her half of the state:

$$\cancel{\text{Bob}} \phi^{TATB} \rightarrow E_m^{T_A \rightarrow B^{A^n}}(\phi^{TATB})$$

She then sends her  $n$  systems  $A^n$  through the channel,  $N^{A^n \rightarrow B^n} = (N^{A^1 \rightarrow B})^{\otimes n}$ , to Bob. Bob can measure the systems he receives, together with his half of the entangled state, to determine the message  $m$ , using a POVM  $\{\Lambda_m^{B^n T_B}\}$ .

The probability of error is

$$P_e^{*\max} = \text{Tr} \left\{ (I - \Lambda_m^{B^n T_B}) N^{A^n \rightarrow B^n} (\epsilon_m^{T_A \rightarrow A^n} (\phi^{T_A T_B})) \right\} \leq \epsilon$$

and the communication rate is

$$C = \frac{1}{n} \log_2 |M| + \delta.$$

We put no limits on the entanglement consumed, but the rate of consumption is  $\frac{1}{n} \log_2 d$ .

We now show how to optimally accomplish this protocol.

Thm (Bennett-Shor-Smolin-Thapliyal) The supremum overall achievable rates for entanglement-assisted classical comm is

$$C_E = I(N) = \max_{\phi^{AA'}} I(A; B)_P, P^{AB} = N^{A' \rightarrow B} (\phi^{AA'}).$$

Direct coding thm

Suppose  $A$  &  $B$  share  $n$  copies of an arbitrary pure, bipartite state  $|\phi\rangle^{AB}$ .

The total entanglement of  $(|\phi\rangle^{AB})^{\otimes n}$  is  $nH(A)$ ,<sup>③</sup> and it is possible to prepare these states locally from  $nH(A)$  ebits by a protocol called "entanglement dilution". In the Schmidt form we can write

$$|\phi\rangle^{AB} = \sum_x \sqrt{p_x(x)} |x\rangle^A |x\rangle^B.$$

So  $n$  copies of the state will be

$$|\phi\rangle^{A^n B^n} = \sum_{x^n} \sqrt{p_{x^n}(x^n)} |x^n\rangle^{A^n} |x^n\rangle^{B^n}$$

$$p_{x^n}(x^n) = \prod_{j=1}^n p_x(x_j)$$

$$|x^n\rangle = |x_1\rangle \dots |x_n\rangle.$$

Decompose the state by type subspaces:

$$|\phi\rangle^{A^n B^n} = \sum_{+} \sum_{x^n \in T_+} \sqrt{p(+)} |\Phi_+\rangle^{A^n B^n}$$

where

$$|\Phi_+\rangle^{A^n B^n} = \frac{1}{\sqrt{d_+}} \sum_{x^n \in T_+} |x^n\rangle^{A^n} |x^n\rangle^{B^n}$$

$$p(+) = d_+ \cdot p_{x^n}(x^n) \quad \text{for any } x^n \in T_+$$

$$d_+ = |T_+|.$$

The state  $|\Phi_+\rangle$  is equivalent to a  $d_+$ -dimensional maximally entangled state.

Now we need to specify Alice's encoding operations  $E_m$ . She will apply a unitary of the following form:

$$U \equiv \bigoplus_{x_+} (-1)^{b_+} X(x_+) Z(z_+)$$

$$b_+ = 0, 1$$

$$x_+ = 0, \dots, d_+-1$$

$$z_+ = 0, \dots, d_+-1$$

where  $\& X(x)$  and  $Z(z)$  are the generalized Pauli operators acting on the <sup>type</sup><sub>subspaces</sub>.

These act as follows:

$$X(x) |j\rangle = |j+x \bmod d\rangle, \quad \text{where } \{|j\rangle\} \text{ are a basis.}$$

$$Z(z) |j\rangle = e^{2\pi i z j / d} |j\rangle.$$

Because each  $| \Phi_+ \rangle^{A^n B^n}$  acts like a maximally entangled state, we can use the "transpose trick":

$$(U^{A^n} \otimes I^{B^n}) |\phi\rangle^{A^n B^n} = (I^{A^n} \otimes (U^\top)^{B^n}) |\phi\rangle^{A^n B^n}.$$

For each message  $m$ , Alice chooses ~~to send~~

$$s(m) \equiv (b_+(m), x_+(m), z_+(m))$$

uniformly at random; the encoding unitary is  $U(s(m))$ . After sending through the channel, the state is

$$P_s^{B^n B^n} \equiv (I^{A^n} \otimes (U(s(m))^\top)^{B^n}) N^{A^n \rightarrow B^n} (\Phi^{A^n B^n}) (I^{A^n} \otimes (U(s(m)))^{B^n}).$$

So effectively the unitary commutes with the channel.

Bob now needs to decode the message  $m$  with a collective POVM  $\{\Lambda_m\}$ . Once again, we use the packing lemma. ⑤

A & B choose codewords uniformly at random from the set  $S$ . The codeword projectors are

$$(I^{B^n} \otimes U(S)^{B^n}) \Pi_S^{B^n B^n} (I^{B^n} \otimes U(S)^{* B^n})$$

where  $\Pi_S^{B^n B^n}$  is the typical subspace projector for  $n$  copies of the state

$$P^{B' B} = N^{A \rightarrow B'}(\phi^{AB}).$$

The codespace projector is  $\Pi_S^{B^n} \otimes \Pi_S^{B^n}$ , where  $\Pi_S^{B^n}$  and  $\Pi_S^{B^n}$  are the marginal typical subspace projectors for  $n$  copies of the states

$$P^{B'} \equiv \text{Tr}_{B'} \{ \rho^{B' B} \} \text{ and } P^B = \text{Tr}_{B'} \{ \rho^{B' B} \}.$$

The expected density matrix is

$$\bar{\rho}^{B^n B^n} = \frac{1}{|S|} \sum_{S \in S} (I \otimes U(S)^T) \rho^{B^n B^n} (I \otimes U(S))^*$$

$$= \sum_+ P(+ ) N^{A^n \rightarrow B^n} (\Pi_+^{A^n}) \otimes \Pi_+^{B^n} \leftarrow \begin{matrix} \text{takes} \\ \text{some} \\ \text{work to} \\ \text{show} \end{matrix}$$

The conditions for the packing lemma are:

(6)

$$\textcircled{1} \operatorname{Tr} \left\{ \left( \Pi_S^{B'^n} \otimes \Pi_S^{B^n} \right) \left( U(S)^{B^n} \rho^{B'^n B^n} U^*(S)^{B^n} \right) \right\} \geq 1 - \epsilon$$

$$\textcircled{2} \operatorname{Tr} \left\{ \left( U(S)^{B^n} \overline{\Pi}_S^{B'^n B^n} U^*(S)^{B^n} \right) \rho_S^{B'^n B^n} \right\} \geq 1 - \epsilon$$

$$\textcircled{3} \operatorname{Tr} \left\{ U(S)^{B^n} \overline{\Pi}_S^{B'^n B^n} U^*(S)^{B^n} \right\} \leq 2^n (H(B'B)_{\rho} + c\delta)$$

$$\textcircled{4} (\Pi_S^{B'^n} \otimes \Pi_S^{B^n}) \bar{\rho}^{B'^n B^n} (\Pi_S^{B'^n} \otimes \Pi_S^{B^n}) \xrightarrow[n \rightarrow \infty]{\eta \rightarrow 0 \text{ as } S \rightarrow 0} \\ \leq 2^{-n(H(B')_{\rho} + H(B)_{\rho} - \eta(n, \delta) - c\delta)} (\Pi_S^{B'^n} \otimes \Pi_S^{B^n})$$

\textcircled{2} & \textcircled{3} Follow straightforwardly from properties of the typical subspace. \textcircled{1} follows, with a bit more work, from the <sup>Properties of the</sup> marginal typical subspaces.

\textcircled{4} Is the most nontrivial to show, but the derivation is in the book.

From the derandomized packing lemma, the probability of error is <sup>there exists a code s.t.</sup>

$$P_C^* \leq 4(\epsilon + 2\sqrt{\epsilon}) + 8 \left( \frac{D}{d|M|} \right)^{-1}$$

$$= 4(\epsilon + 2\sqrt{\epsilon}) + 8|M| 2^{n(H(B'B)_{\rho} + c\delta)} 2^{-n(H(B')_{\rho} + H(B)_{\rho} - \eta - c\delta)}$$

$$= 4(\epsilon + 2\sqrt{\epsilon}) + 8|M| 2^{-n(I(B'; B')_{\rho} - \eta(n, \delta) - 2c\delta)}$$

So A & B can communicate with low error is

$$\frac{1}{n} \log_2 |M| = I(B'; B)_{\rho} - \eta(n, \delta) - 3c\delta$$

$$\Rightarrow C_E = \max_{\Phi^{AA'}} I(B'; B)_{\rho} \equiv I(N).$$

□

## Converse Thm

We sketch this briefly. Again, we upper bound classical communication by common randomness distribution. A & B share an entangled state  $|\phi\rangle^{T_A T_B}$ , and A wants to share the state

$$\bar{\Phi}^{MM'} = \sum_{M' M} |M\rangle\langle M|^M \otimes |M\rangle\langle M|^{M'}$$

Alice encodes, sends the states through the channel, and Bob decodes, producing the state

$$\omega^{MM'} \equiv \underbrace{\omega}_{\text{decode}} \underbrace{\xrightarrow{B^n T_B \rightarrow M}}_{\text{channel}} \underbrace{(N^{A^n \rightarrow B^n} (\varepsilon^{M' T_A \rightarrow A^n} (\bar{\Phi}^{MM'} \otimes \phi^{T_A T_B})))}_{\text{encode}}$$

Suppose C is an achievable rate, so  $\|\omega^{MM'} - \bar{\Phi}^{MM'}\|_1 \leq \epsilon$ .

$$\begin{aligned}
 nC &= I(M; \bar{\Phi}) \quad \text{Alicki-Fannes} \\
 &\leq I(M; \bar{\Phi}) \omega + n\epsilon' \quad \text{Data processing} \\
 &\leq I(M; B^n T_B) \omega + n\epsilon' \quad \text{Chain rule} \\
 &= I(T_B M; B^n) \omega + I(M; T_B) \omega \\
 &\quad - I(B^n; T_B) \omega + n\epsilon' \quad \text{Product between } M \text{ & } T_B \\
 &= I(T_B M; B^n) \omega - I(B^n; T_B) \omega + n\epsilon' \quad \text{Positivity of } I \\
 &\leq I(T_B M; B^n) \omega + n\epsilon' \\
 &\leq \max_{P^{XAA'^n}} I(AX; B^n) \rho + n\epsilon' \quad \text{Max achieved by pure inputs} \\
 &= \max_{\phi^{AA'^n}} I(A; B^n) \omega + n\epsilon' \\
 &= nI(N) + n\epsilon'. \quad \square
 \end{aligned}$$

⑥

Next time we will look briefly at this result for a few channels, then see how we can derive from this capacities for entanglement-assisted quantum capacity, and <sup>ultimately</sup> quantum capacity.

We'll also briefly discuss feedback channels.