

## Lecture 20

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Given the optimal rate (capacity) for entanglement-assisted classical communication, we can derive a whole raft of related results, including the direct coding theorem for quantum capacity.

First, a quick ~~corollary~~ corollary:

Thm ( $\mathcal{Q}$  feedback does not increase the entanglement-assisted classical capacity of a  $\mathcal{Q}$  channel) The classical capacity of a  $\mathcal{Q}$  channel assisted by a (noiseless)  $\mathcal{Q}$  feedback channel is  $I(N)$ , the same as the EAC capacity.

Proof: Direct coding thm.

It is trivial to prove that  $I(N)-S$  is an achievable rate. Bob can locally prepare an unlimited ~~number~~ number of pure entangled states and send half of each to Alice. They can then just use the EAC comm protocol proven before.

## Converse Thm

(2)

What is the most general feedback-assisted protocol A & B can do?

1. Alice chooses a message  $m \in M$ .
2. She applies a map  $E_m^1$  on her systems  $A_1, \dots, A_n$ .
3. She sends  $A_1$  through the channel  $N_{A_1 \rightarrow B_1^r}$  to Bob.
4. Bob applies a map  $F_1^{B_1 \rightarrow B_1 X_1}$  and sends  $X_1$  to Alice.
5. Alice applies a map  $E_m^2$  to  $X_1, A_2, \dots, A_n$  and sends  $A_2$  through  $N_{A_2 \rightarrow B_2}$ .
6. Bob applies  $F_2^{B_1 B_2 \rightarrow B_1 B_2 X_2}$  and sends  $X_2$  to A.
7. A & B iterate 3 & 4 until all of the systems  $A_1, \dots, A_n$  have been sent to B.
8. B measures  $B_1 \dots B_n$  with a POVM  $\{\Lambda_m\}$  to retrieve the message  $m$ !

What rate can they achieve? We can prove a sequence of inequalities:

Suppose that a rate  $C$  is achievable. Then (3)

$$nC = H(M) \quad \leftarrow \text{messages } (= \log |M|)$$

$$= I(M; M') + H(M|M')$$

$M$  = message sent  
 $M'$  = message received

$$\leq I(M; M') + \epsilon nC + 1 \quad \leftarrow \text{Fano's ineq.}$$

$$\leq I(M; B^n) + \epsilon nC + 1 \quad \leftarrow \text{Q data processing}$$

$$\leq I(M; B_n | B^{n-1}) + I(M; B_{n-1} | B^{n-2}) + \dots + I(M; B_1)$$

$$+ \epsilon nC + 1$$

$$\leq n \max_P I(M; B|A)$$

$$+ \epsilon nC + 1$$

$$\leq nI(N) + \epsilon nC + 1 \quad \leftarrow \text{by a series of straightforward steps.}$$

$$\Rightarrow C \leq I(N) + \epsilon C + 1/n$$

$$\rightarrow I(N) \text{ as } \epsilon \rightarrow 0, n \rightarrow \infty.$$

So the quantum feedback channel has the same capacity as the EAC capacity  $C_E$ , and is also single letter.

Having a  $Q$  feedback channel in addition to shared entanglement does not boost capacity. But having a  $Q$  feedback channel does boost capacity over the case w/o entanglement, in contrast to the case of classical feedback, which doesn't help a classical channel.

repeatedly  
apply  
chain rule  
&  
data proc

max over all  $P$   
of form

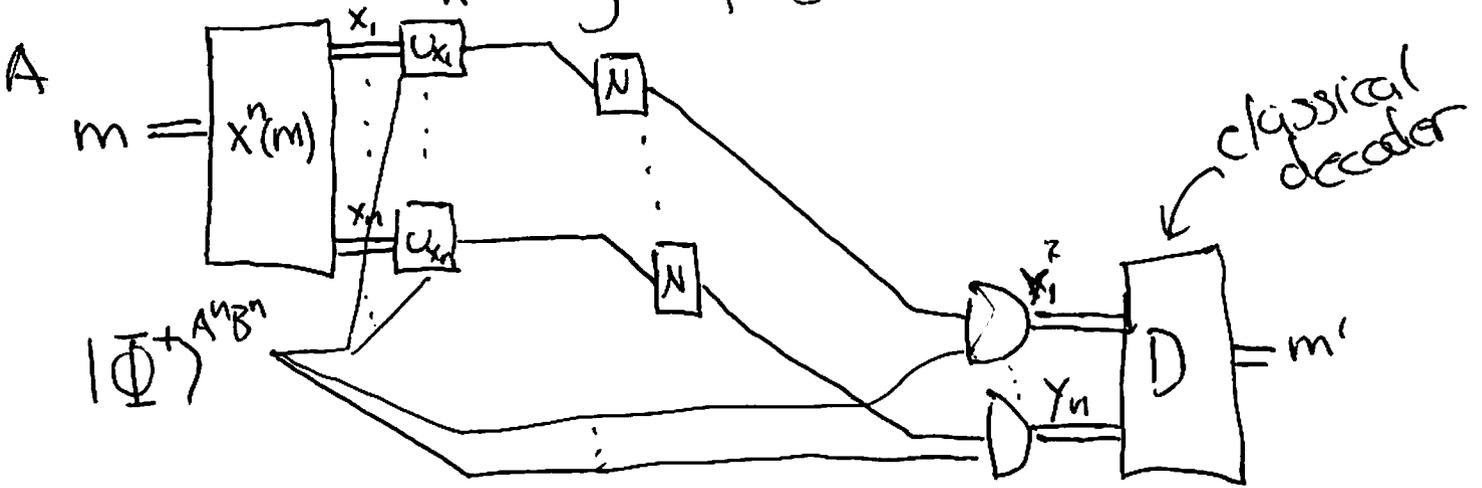
$$P^{MBA} = \sum_m P_M(m) |m\rangle\langle m|$$

$\otimes N^{A \rightarrow B} (P_M^{A|A})$

# Examples of Channels

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For many channels the simple superdense coding strategy is optimal: A & B share  $n$  maximally entangled pairs. A acts separately on each one; Bob measures <sup>each pair</sup> using a projective meas:



$$x_j = I, X, Y, Z$$

Channels where this approach is optimal include:

1. The  $Q$  Erasure channel  
 $C_E = 2(1-\epsilon) \log d_A$

2. The depolarizing channel (for qubits)

$$C_E = 2 + (1-3p/4) \log(1-3p/4) + (3p/4) \log(3/4)$$

3. The dephasing channel  $C_E = 2 - h(p/2)$

But for some channels, the optimal shared state is not maximally mixed, and Bob and Alice must do collective measurements/encodings. The book treats one example at some length: the amplitude-damping channel.

$$N_{AD}(p) = A_0 p A_0^\dagger + A_1 p A_1^\dagger \quad A_0 = |0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1| \quad \textcircled{5}$$

This example is still simple enough to be solved in closed form, as shown in the text; the EAC capacity is

$$C_E = I(N_{AD}) = \max_{p \in [0,1]} h(p) + h((1-\gamma)p) - h(\gamma p).$$

## Entanglement-assisted Q comm

We can turn an entanglement-assisted classical comm protocol into an EA quantum comm protocol just by consuming some extra entanglement and using Q teleportation:

$$2[C \rightarrow C] + [q q] \geq [q \rightarrow q]$$

So since  $I(N)$  is an achievable rate for classical comm (with EA),  $\frac{1}{2}I(N)$  is an achievable rate for EA Q comm.

Moreover, it is easy to see that this is the optimal rate. If we could send qubits at a rate higher than  $\frac{1}{2}I(N)$ , then we could combine with superdense coding

$$[q \rightarrow q] + [q q] \geq 2[C \rightarrow C]$$

to get an EAC rate higher than  $I(N)$

How much entanglement is required for this (6) protocol? For EAC comm we put no restrictions on the amount of entanglement consumed, but it is not hard to see that the protocol we worked out consumes  $nH(A)_\rho$  ebits, where

$$\rho^{AB} \equiv (I^A \otimes N^{A \rightarrow B})(\phi^{AA'})$$

So putting this together gives the resource inequalities

$$\begin{aligned} \langle N \rangle + (H(A)_\rho + \frac{1}{2}I(A; B)_\rho) [q \rightarrow q] \\ \geq I(A; B)_\rho [c \rightarrow c] + \frac{1}{2}I(A; B)_\rho [q \rightarrow q] \\ \geq \frac{1}{2}I(A; B)_\rho [q \rightarrow q]. \end{aligned}$$

angle bracket denotes a noisy resource

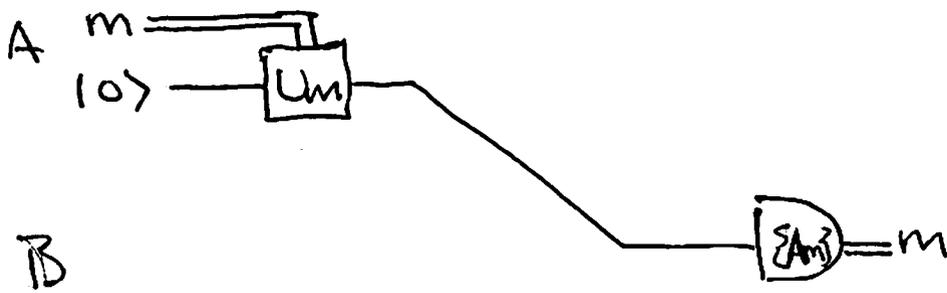
This now raises an interesting question: is this the best we can do? Or is it possible to achieve the same rate while consuming less entanglement?

It turns out that we can consume fewer ebits by making the protocol coherent.

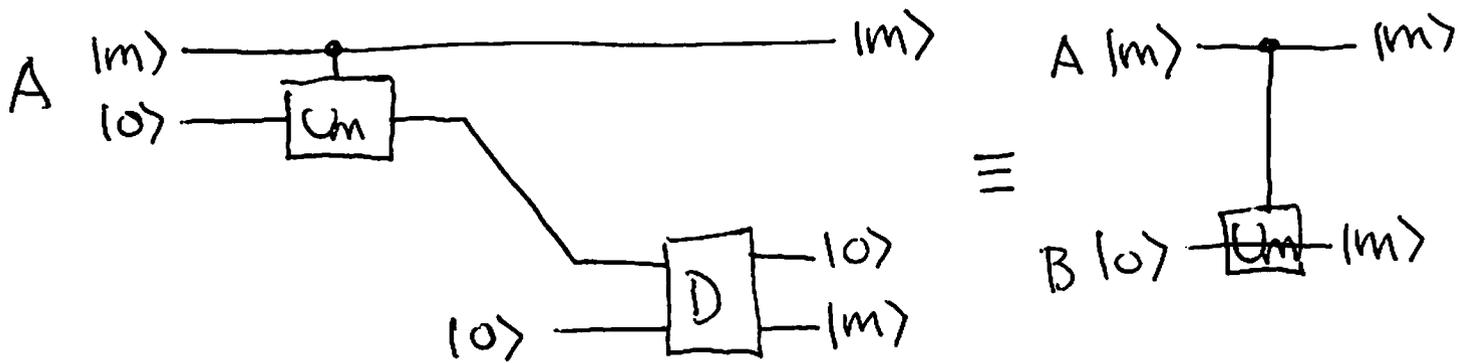
# Coherent communication (Chapter 7)

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Consider the following circuit for classical communication:



We can make this coherent by making the random variable  $M$  into a quantum system, turning the classical control into  $Q$  control, and replacing the final measurement with a coherent measurement:



This protocol maps a superposition state of Alice's into an entangled state:

$$|\psi\rangle^A = \sum_m \alpha_m |m\rangle^A \longrightarrow \sum_m \alpha_m |m\rangle^A |m\rangle^B \equiv |\psi'\rangle^{AB}$$

Such a map is called a coherent bit (or cobit) channel.

We denote the cobit resource as  $[q \rightarrow qq]$ . ⑧  
A cobit is stronger than  $[qq]$  and  $[c \rightarrow c]$ :

$$[q \rightarrow qq] \geq [qq] \quad : \quad \frac{1}{\sqrt{2}}(|0\rangle^A + |1\rangle^A) \rightarrow |\psi_+\rangle^{AB} \\ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$[q \rightarrow qq] \geq [c \rightarrow c] \quad |x\rangle^A \rightarrow |x\rangle^A |x\rangle^B$$

But it is weaker than  $[q \rightarrow q]$ :

$$[q \rightarrow q] \geq [q \rightarrow qq] \quad \text{--- see previous circuit}$$

$$[q \rightarrow qq] \not\geq [q \rightarrow q] \quad \text{--- a cobit by itself cannot transfer an arbitrary state } |\psi\rangle$$

The cobit is not a natural resource; there are no real cobit channels. But it is a useful tool, because cobits can both build and be built from the natural resources  $[qq]$ ,  $[c \rightarrow c]$  and  $[q \rightarrow q]$ , and coherent versions of teleportation and superdense coding are exactly dual to each other:

$$[q \rightarrow q] + [qq] \geq 2[q \rightarrow qq]$$

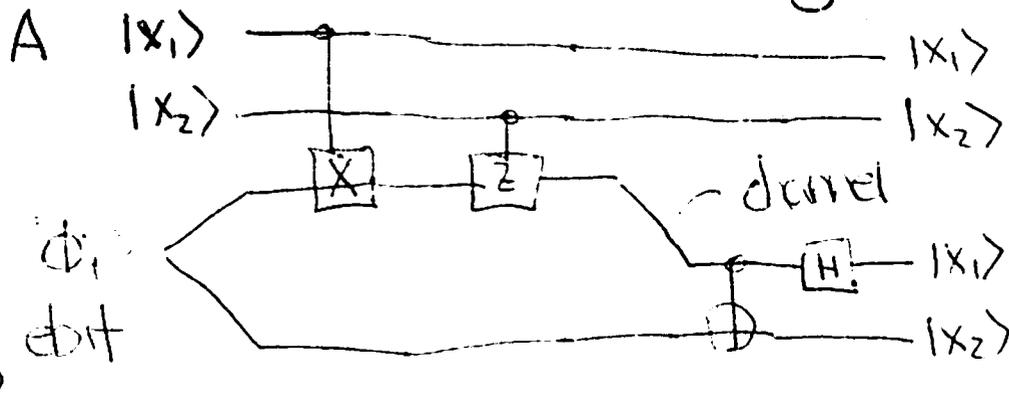
$$2[q \rightarrow qq] \geq [q \rightarrow q] + [qq]$$

$$\Rightarrow 2[q \rightarrow qq] = [q \rightarrow q] + [qq]$$

In the usual teleportation & superdense coding protocols, entanglement is always consumed; but ~~in~~ in the coherent versions we get back the resources we consume.

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### Coherent superdense coding



### Coherent teleportation

