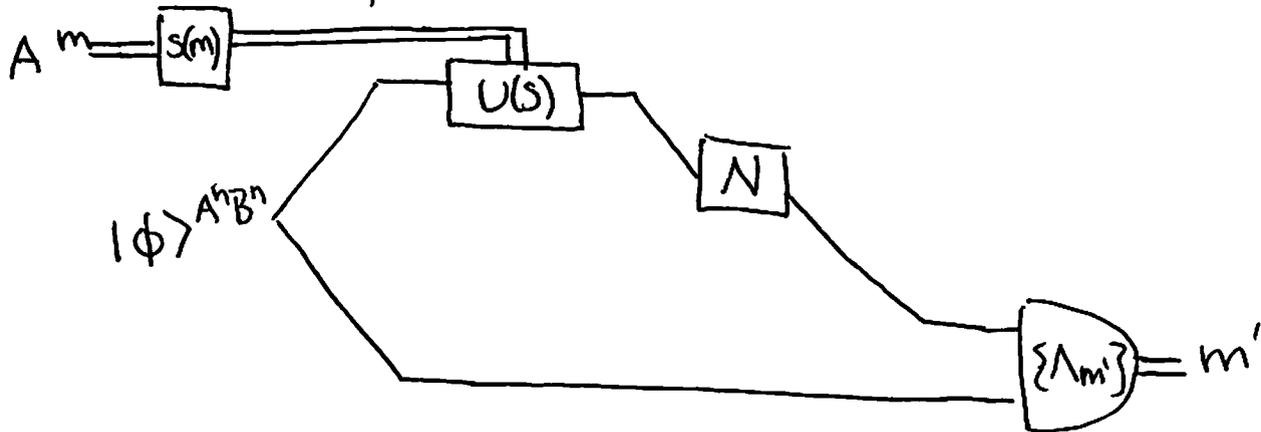


Lecture 21

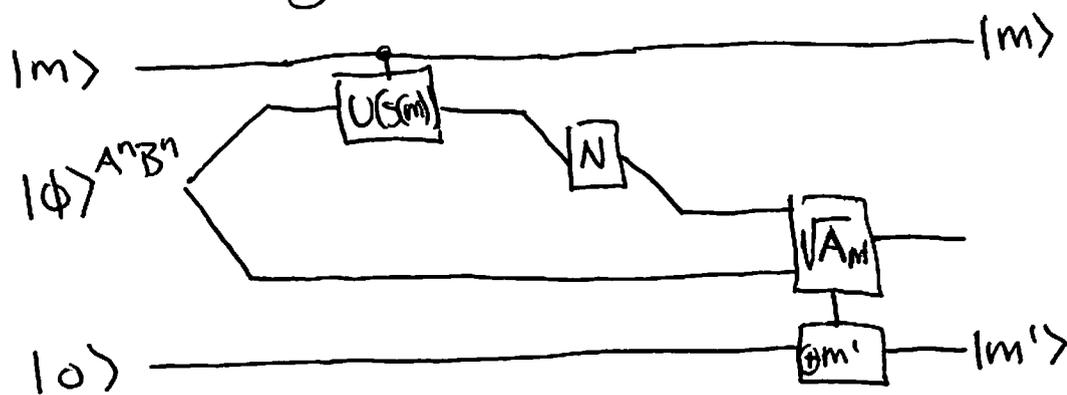
①

The point of introducing coherent bit channels is that we can turn the theorem about EA classical comm. into a theorem about EA coherent comm just by "coherifying" the classical protocol.

Classical protocol:



We replace the classically controlled encoding circuit $U(S(m))$ by a Q -controlled unitary $\sum_m |m\rangle\langle m| \otimes U(S(m))$, make the input m a quantum state, and replace the POVM by a coherent (gentle) measurement:



Thm (Entanglement-assisted coherent communication). The resource inequality ②

$$\langle N \rangle + H(A)_\rho [qq] \geq I(A; B)_\rho [q \rightarrow qq]$$

is an achievable rate for an EA coherent comm protocol over a noisy channel N in the limit $n \rightarrow \infty$, where

$$\rho^{AB} \equiv (\mathcal{I}^A \otimes N^{A' \rightarrow B})(\phi^{AA'})$$

\uparrow pure

Proof If we just go through the steps of the "coherified" protocol outlined above, ~~we get~~ ~~the state~~ with the purified input

$$|\psi\rangle^{RA_1} \equiv \sum_{\ell, m=1}^{d_1} \alpha_{\ell m} |\ell\rangle^R |m\rangle^{A_1}$$

and using the isometric extension

$$U_N^{A^n \rightarrow B^n E^n}$$

and the shared entangled state $|\phi\rangle^{A^n B^n}$,

we get

$$\sum_{\ell, m} \alpha_{\ell m} |\ell\rangle |m\rangle |\phi\rangle^{A^n B^n} \xrightarrow{\text{enc.}} \sum_{\ell, m} \alpha_{\ell m} |\ell\rangle |m\rangle (U(\ell, m) \otimes I) |\phi\rangle$$

$$= \sum_{\ell, m} \alpha_{\ell m} |\ell\rangle |m\rangle (I \otimes U^T) |\phi\rangle^{A^n B^n}$$

$$\rightarrow \sum_{\ell, m} \alpha_{\ell m} |\ell\rangle |m\rangle (U_N \otimes U^T) |\phi\rangle$$

$$= \sum_{\ell m} \alpha_{\ell m} |\ell\rangle |m\rangle (I \otimes U^T(s(m))) |\phi\rangle_{B^n E^n B^n}$$

coherent meas $\rightarrow \sum_{\ell m} \alpha_{\ell m} |\ell\rangle |m\rangle \sum_{m'} [I \otimes \sqrt{\lambda_{m'}} U^T(s(m))] |\phi\rangle_{B^n E^n B^n} |m'\rangle$

This state is $2\sqrt{\epsilon}$ close to

$$\sum_{\ell m} \alpha_{\ell m} |\ell\rangle |m\rangle (I \otimes U^T(s(m))) |\phi\rangle_{B^n E^n B^n} |m\rangle,$$

so Bob can coherently decouple his output register by the unitary $\sum_m U^*(s(m)) \otimes |m\rangle\langle m|$.

So the decoupled systems are in the state

$$\sum_{\ell m} \alpha_{\ell m} |\ell\rangle |m\rangle_{A_1} |m\rangle_{B_1} \quad \square$$

Combining this with coherent superdense coding yields

$$2[q \rightarrow qq] \geq [q \rightarrow q] + [qq]$$

$$\langle N \rangle + H(A)_\rho [qq] \geq \frac{1}{2} I(A; B)_\rho ([q \rightarrow q] + [qq])$$

$$\Rightarrow \langle N \rangle + \underbrace{(H(A)_\rho - \frac{1}{2} I(A; B)_\rho)}_{\geq \frac{1}{2} I(A; B)_\rho} [qq] \geq \frac{1}{2} I(A; B)_\rho [q \rightarrow q]$$

$$= \frac{1}{2} I(A; E)_\rho.$$

So this significantly reduces the net use of entanglement (though it uses entanglement catalytically).

We can now combine this EAQ capacity result with entanglement distribution

$$[q \rightarrow q] \geq [qq]$$

to get an achievable rate for Q comm without entanglement:

$$\langle N \rangle + \frac{1}{2} I(A; E)_\rho \geq \frac{1}{2} (I(A; B) - I(A; E)) [q \rightarrow q] + \frac{1}{2} I(A; E) [qq]$$

$$\Rightarrow \langle N \rangle \geq \underbrace{\frac{1}{2} (I(A; B) - I(A; E))}_{I(A; B)_\rho} [q \rightarrow q]$$

By maximizing over all (pure) inputs ϕ we get the Direct Coding Thm for Q Comm

$$\langle N \rangle \geq Q(N) [q \rightarrow q]$$

where $Q(N) \equiv \max_{\phi} I(A; B)_\rho$, $\rho^{AB} \equiv (I \otimes N) \phi^{AA'}$.

Note that while the EAQ capacity formula is single letter, the Q capacity thm is not; so we would have to regularize

$$Q_{\text{reg}}(N) \equiv \lim_{n \rightarrow \infty} \frac{1}{n} Q(N^{\otimes n})$$

There are many more topics in QIT that we won't have time to cover. For instance: ⑤

- The Q comm converse thm
- Private classical comm and its connection to Q comm
- Protocols with noisy static resources (e.g., noisy superdense coding, entanglement distillation)
- Other coherent protocols (e.g., coherent state transfer)
- Joint encoding of multiple resources, such as Q comm, classical comm, and entanglement. E.g., "triple trade-offs"

$$\langle N \rangle \geq C[C \rightarrow C] + Q[Q \rightarrow Q] + E[QQ]$$

where the triples (C, Q, E) can be positive or negative, and lie inside an achievable region.

And of course, there are many open questions:

1. Are there single letter formulas for Q capacity and classical capacity?
2. Can we find examples of nonadditive channels?
3. And soon and on.

It's a dynamic and fascinating field! Thank you for your interest!